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16. Applications

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Exact relations for composites: towards a complete solution

This is joint work with Yury Grabovsky and Dan Sage. Typically, the elastic and electrical properties of composite materials are strongly microstructure dependent. So it comes as a nice surprise to come across exact formulae for (or linking) effective moduli that are universally valid no matter what the microstructure. Such exact formulae provide useful benchmarks for testing numerical and actual experimental data, and for evaluating the merit of various approximation schemes.
For example, consider the effective thermal expansion coefficient and effective bulk modulus of a composite of two isotropic phases. When one heats the composite the phases expand at different rates and the microgeometry is distorted. Similarly when one pressurizes the composite the microgeometry is distorted because of the differences in the bulk moduli of the phases. However one can imagine applying a judicious combination of pressure and temperature increase so the two phases, and hence the composite, expand at exactly the same rate. By this argument one obtains Levin's exact formula linking the effective bulk modulus and effective thermal expansion coefficient. As a second example, consider electrical conduction in a two dimensional medium. The electric field is curl free while the current field is divergence free, and they are linked by a constitutive relation which incorporates the conductivity of the medium. Now in two dimensions rotation of a divergence free field at each point by 90 degrees produces a curl-free field and vice-versa. Such a rotation of the electric field and current field produces, respectively, a new current field and new electric field which solve the conductivity equations in a new "dual" medium with a different conductivity. One easily obtains the formulae of Keller, Dykhne and Mendelson linking the effective conductivity tensors of the original medium and its dual. These formalae lead to exact expressions for the effective conductivity of self-dual media such as checkerboards and two-dimensional polycrystals. As a third example consider a two-phase thermoelectric medium where the constitutive law gives the heat current and electrical current as linear combinations of the temperature gradient and electric field. Now one is free to take a new pair of potentials which are linear combinations of the temperature and electric potential and one is free to take a new pair of divergence free fields which are linear combinations of the heat current and electrical current. By choosing approriate combinations one eliminates the cross-coupling terms.

In this way one obtains the formulae of Straley and Milgrom and Shtrikman linking the effective theroelectric coefficient with the effective electrical and thermal conductivities.

There are many more known exact relations, not all of which are so simple. Traditionally they have been derived by a variety of different arguments, as the above examples suggest. One might wonder if there is a more general approach. Indeed there is! Here we present a systematic theory of exact relations embracing the known exact relations and establishing new ones. The search for exact relations is reduced to a search for tensor subspaces satisfying certain simple algebraic conditions. Previously Grabovsky and Sage found an algebraic condition which was necessary for an exact relation to hold. It derives from the observation that an exact relation must hold at least for laminate microgeometries. Here we present a sufficient condition which guarantees that an exact relation will hold for all composites. It is derived from an analysis of certain convergent series expansions for the effective tensor. One of many new exact relations we derive is for the effective shear modulus of a class of three-dimensional polycrystalline materials.

Although the necessary conditions and sufficient conditions look quite similar, a gap remains between them. It is an open and challenging question as to what condition is both necessary and sufficient.