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Abstracts

of

Plenary

and

Invited Lectures

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The abstracts of the plenary lectures are given in section 0, ordered alphabetically by authors' names, the abstracts of the invited lectures are ordered by sections and, within each section, alphabetically by the authors' names, except for sections 16 and 18, where a special arrangement was made.

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ICM 1998
Abstracts of Plenary and Invited Lectures
Contents

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0. Plenary Lectures

1991 MS Classification: 32L10,57R20,58G10

Bismut, Jean-Michel, Université Paris-Sud 91405 Orsay, France

Local index theory and higher analytic torsion

Our talk will explain refinements of the Atiyah-Singer index theorem and of the Riemann-Roch-Grothendieck theorem, which have been obtained using local index theoretic techniques. These methods permit the construction of secondary invariants, the analytic torsion forms, or analytic torsion currents, whose functorial properties will be described.
Here is a short example. Let $\pi : X \to S$ be a holomorphic submersion of complex manifolds with compact fibres Z. Let E be a holomorphic vector bundle on X. Let $R\pi_*(E)$ be the higher direct image of E. By Riemann-Roch-Grothendieck,

$$\mathrm{ch}(R\pi_*(E)) = \pi_*(\mathrm{Td}(TX/S)\mathrm{ch}(E)) \text{ in } CH(S). \tag{1}$$

Recall that a metric g^E on a holomorphic vector bundle E determines a holomorphic hermitian connection ∇^E. Let ω^X be a Kähler metric on TX. It induces a metric g^{TZ} on $TZ = TX/S$. Let g^E be a metric on E. Assume that $R\pi_*(E)$ is locally free. Let $g^{R\pi_*(E)}$ be the L_2 metric on $R\pi_*(E)$ one obtains via fibrewise Hodge theory. Using the superconnection formalism of Quillen, one contructs higher analytic torsion forms $T(\omega^X, g^E)$, such that

$$\frac{\overline{\partial}\partial}{2i\pi}T(\omega^X, g^E) = \mathrm{ch}(R\pi_*(E), g^{R\pi_*(E)})\pi_*[\mathrm{Td}(TZ, g^{TZ})\mathrm{ch}(E, g^{R\pi_*(E)})]. \tag{2}$$

The analytic torsion forms are secondary objects attached to the submersion $\pi : X \to S$ and to the given metrics. Adiabatic limit techniques are used to establish the functorial properties verified by the above objects.
This leads in particular to a refinement of the theorem Riemann-Roch-Grothendieck, the Gillet-Soulé Riemann-Roch theorem in Arakelov geometry.
In the context of de Rham theory, one can derive a form of Riemann-Roch-Grothendieck for flat vector bundles, and construct still mysterious secondary invariants.

1991 MS Classification: 11, 14, 58F

Deninger, Christopher, Westf. Wilhelms-Universität Münster, Germany

Some analogies between number theory and dynamical systems on foliated spaces

For the arithmetic study of varieties over finite fields powerful cohomological methods are available which among other applications have led to a proof of analogs of the Riemann conjecture. In the talk we first explain the shape that a corresponding cohomological formalism for varieties over $\mathbb{Q}$ should take. We then discuss how it would relate to the many conjectures on L-series and indicate a couple of consequences that can be proved by standard methods. Finally we explain how the quest for this elusive cohomology theory leads naturally into the theory of dynamical systems on foliated spaces.

1991 MS Classification:

Persi Diaconis, Department of Mathematics and ORIE, Cornell University; Departments of Mathematics and Statistics, Stanford University, USA

From Shuffling Cards to Walking Around the Building

There has been spectacular progress in recent years in deriving sharp rates of convergence of Markov chains to their stationary distribution. A typical problem: How many times should a deck of cards be shuffled to mix it? The answer depends on how the cards are shuffled. For the usual method of riffle shuffling 52 cards, the answer is "about seven". This is work of Gilbert, Shannon, Reeds, Aldous, Bayer and Diaconis. The analysis shows a sharp threshold in convergence; for n cards $\frac{3}{2}\log_2 n$ shuffles are necessary and suffice. The analysis rests on an algebraic miracle which has ramifications in algebraic combinatorics (descent theory for Coxeter groups) and in giving Hodge type decompositions for Hochschild homology.
In very recent work, Bidigare, Hanlon and Rockmore have found an elegant generalization of shuffling to random walk on the chambers of Euclidean hyperplane arrangements. This has been abstracted further to natural walks on buildings in joint work with Ken Brown. These generalizations illuminate the algebraic miracle and permit sharp analysis of a host of natural new Markov chains. Perhaps the most interesting of these is chains arising in random tiling models for quasi crystals (work with Billera and Brown).

1991 MS Classification:

Gallavotti, Giovanni, Universitá La Sapienza, Roma, Italy

The Chaotic hypothesis and universal large deviations properties

The chaotic hypothesis will be discussed and its meaning and consequences will be exemplified via some applications to Anosov systems.

1991 MS Classification: 65,65F,65N,65R

Hackbusch, Wolfgang, Universität zu Kiel, Germany

From Classical Numerical Mathematics to Scientific Computing

Many of the fundamental numerical methods trace back to the time of Newton and Gauss. However, the discipline of Numerical Mathematics as a field of its own did not appear before the electronic computer came into use. The computer not only marks the beginning of Numerical Mathematics. For the development of this field it is quite essential that the computer technology is steadily improving. This leads to an increasing complexity of

the problems to be solved and to several new questions. As a consequence, Numerical Mathematics has changed a lot over the last 50 years. One sign of these changes is the fact that often the name Scientific Computing is used instead of Numerical Mathematics.

It should be noted that Numerical Mathematics or Scientific Computing combine the following parts which mutually dependent: modelling, algorithms, analysis. Modelling is (a) the appropriate mathematical formulation of a problem from a field outside Mathematics and (b) the discretisation process with translates infinitely dimensional problems into finitely dimensional ones. The heart of Numerical Mathematics are the algorithms enabling the solution process. Numerical Analysis comes into play to judge the consequences of the discretisation process, to control the algorithms (reliability, quality, costs, etc.). As pointed out in the end of the lecture, it now seems to be necessary to add a further subfield: the implementation of numerical methods on the computer.

The lecture will, in particular, illustrate the consequences for the treatment of partial differential equations. Since after the discretisation process, the arising systems of equations can be as large as the computer memory allows, the computer technology obviously has an important influence. It may be self-evident that we would like the algorithms to be as efficient as possible, i. e., they should yield the desired results for lowest computational costs. This vague request can be made more precise. We explain why the development of the computer technology directly leads to the need of algorithms with linear complexity, i. e., the computational work for performing an algorithm with n input data must be proportional to n.

The algorithms satisfying this requirement often show a hierarchical structure. An early example is the Fast Fourier Transform (FFT) which is almost of linear complexity and has a typical recursive structure. Wavelets and algorithms exploiting their features are a more recent example. For the solution of discrete elliptic partial differential equations, multi-grid methods have been developed and turned out to be very flexible. They essentially make use of a hierarchy of discrete problems.

The search for efficient methods has led to a further principle, the adaptivity. Since memory is the limiting factor, one tries to adapt the discretisation in such a way to the problem that the best accuracy can be obtained. There are cases, where the adaptation to the problem can be designed a prior, but, usually, the adaptation process is done a posteriori, more precisely, during the computational process. Examples will be given, where the refinement and coarsening of finite elements is governed by this principle. Adaptation gives again rise to intertwine the design of algorithms with Numerical Analysis. Furthermore, the discretisation process and the solution process are often combined.

It is not only the speed and capacity of the computer which is improving but also its architecture. We discuss the effects to numerical algorithms. Mathematical structures which have been developed by this reason are various decomposition methods (domain decomposition and subspace decomposi-

tion methods). However, it seems that decomposition is more helpful as an analysis tool than for the design of efficient algorithms.
We discuss how the principles "hierarchy", "adaptation" and "decomposition" interact. It can be observed that, because of the increased complexity, a new difficulty arises: The implementation process becomes more and more time consuming and needs better scientific support.
The request of linear complexity is very restrictive and seems to exclude, e. g., the treatment of linear systems with a full matrix, since simple operations like a matrix-vector multiplication then are of quadratic complexity. The lecture is concluded by a discussion of this problem. A positive answer are given in the case of integral equations.

1991 MS Classification: 32, 34, 35, 49, 49, 58, 70
Hofer, Helmut, Courant Institute, New York, USA
Dynamics, Topology and Holomorphic Curves

There is an intimate relationship between a certain class of dynamical systems, the topology of the underlying spaces and a suitable holomorphic curve theory. The class of dynamical systems for which there is this relationship contains in particular all geodesic flows (viewed as autonomous Hamiltonian flows confined to the unit sphere bundle).
As a consequence there are very deep and interesting connections between symplectic rigidity theory, symplectic homology, quantum cohomology, Gromov-Witten invariants and Hamiltonian dynamics.
Starting with the classical Seifert conjecture the fine line between "guaranteed existence" and "possible non-existence" of periodic orbits in dynamical systems will be exposed, leading to the important Weinstein conjecture. As it turns out, there is generally (however depending on the topology of the underlying spaces) an abundance of periodic orbits in Hamiltonian systems. Though periodic orbits seem to be only a very special aspect of the dynamics of a Hamiltonian system, it is meanwhile an established fact, that they carry an enormous amount of information. This manifests itself for example in a symplectic homology theory, which associates to an open subset U in phase space, an integer k, and two real numbers $a \leq b$ a group $H_k^{(a,b]}(U)$, which is a symplectic invariant of the set U. These invariants are build for sufficiently nice sets using the combinatorics of periodic orbits on ∂U viewed as a Hamiltonian energy surface. These homology groups provide new symplectic invariants (besides volume) for open sets in phase space.
Reeb flows constitute a special family of non-singular flows containing in particular all autonomous Hamiltonian flows confined to a regular energy surface, as long as the Hamiltonian has the form "kinetic energy" + "potential energy". In particular geodesic flows lead to Reeb flows.
Reeb flows are tightly connected with contact geometry, the odd-dimensional analogue of symplectic geometry. The dynamics of Reeb vector fields on an odd-dimensional manifold M is closely related to a holomorphic curve theory in $\mathbf{R} \times M$ and periodic orbits have to be viewed as the trace of holomorphic cylinders to $M = \{0\} \times M \subset \mathbf{R} \times M$. This leads to the

proof of the Weinstein conjecture for many three-manifolds. There are also interesting connections of the Weinstein conjecture with Gromov-Witten invariants.
Holomorphic curves are most rigid in real dimension 4 due to the "positivity of intersection". In this case the holomorphic curve approach gives a novel tool to construct global surfaces of section or generalisations thereof for Reeb vector fields, under minimal assumptions on the flow. As a consequence one can for example show that the Hamiltonian flow on a strictly convex energy surface in $\mathbf{R}^4$ admits a global disk-like surface of section. In particular there are precisely two geometrically distinct periodic orbits or infinitely many. More general constructions are possible for the geodesic flow of every non-degenerate Riemannian metric on S^2 (and conjecturally on any closed surface).
There is also an interesting topological aspect if M is a three-manifold equipped with a so-called tight contact form. In this case the holomorphic curve theory in $\mathbf{R} \times M$ is closely connected to the topology of the three-manifold M.

1991 MS Classification: 03C45, 03C60
Hrushovski, Ehud, Hebrew University, Jerusalem, Israel
Geometric model theory

A confluence of two streams of model theory has led in recent years to some interesting connections with other subjects, including finite group theory and algebraic geometry. I will attempt to survey these two streams and explain the nature of the contact. The main ingredients are:
1) Algebraic model theory: model completeness theorems .
2) A general theory of finite (and infinite) dimensional geometries, of a certain type. (Stability and its generalizations.)
3) Characterization theorems for classical geometries within the abstract class considered in (2).
The prototype for model completeness theorems is Tarski's theorem relating to real algebraic geometry. It asserts that the class of "semi-algebraic sets" - subsets of $\mathbb{R}^n$ defined by polynomial equalities and inequalities - is closed under projections from $\mathbb{R}^n$ to $\mathbb{R}^m$. it follows that all basic elementary operations (including, for instance, taking the closure of a set) can be carried out without leaving this class. As a corollary, an algorithm exists to determine in advance the outcome of any elementary geometric construction; and one is guaranteed that such a construction will never lead to topological or set theoretic pathologies.
Similar results have been proved (i. a.) for p-adic algebraic geometry, for geometries incorporating analytic functions (on bounded domains, and some others), for the ring of algebraic integers. The present applications use the model completeness results for fields with differential and difference operators. (These results are due to Robinson, Ax, Kochen, Macintyre, Van den Dries, Denef, Wilkie, and others.)
The theory of finite dimensional geometries referred to in (2) begins with the

conclusion of the model completeness results, without assuming however any particular algebraic structure. One is given abstractly a class of "definable" sets (corresponding to the semi-algebraic sets in Tarski's theorem) and a dimension theory on the definable sets.
The theory generalizes basic features of algebraic geometry, including algebraic groups and homogeneous spaces; in this it generalizes the differential Galois theory of Lie and Kolchin. It allows also some phenomena that do not occur classically, in particular for the mixing of geometries of essentially different types. Shelah's theory shows how to analyze a given structure in terms of certain simple geometries within it, and describes their possible interactions.
(3) Ordinary algebraic geometry over algebraically closed fields is entirely characterized, within this class of abstract geometries, by a nondegeneracy and nonlinearity condition and a condition on the dimensions of intersections (generally, intersecting a k-dimensional set with one of codimension 1 should give a set of pure dimension $k-1$.) This result of Zil'ber and the author appeared initially as a foundational result; curiously it turned out to be a main component in the applications.
The connection with geometry is made by considering algebraic varieties with additional structure; here we will take it to be a vector field. Similar theorems hold for several commuting fields, or for correspondences (the latter required an extension of stability theory.)
We will illustrate the theory using a theorem placing a limit on the number of integral subvarieties of the flow. It is generally felt that algebraic integral varieties of sufficiently general flows should be rare, but their analysis is difficult and general results are few. (Except in codimension one, where the qualitative situation is understood thanks to results of Jouanolou). The approach here is to consider these varieties as the closed sets of an abstract geometry, and to use model theoretic analysis of such geometries, The model theory then identifies the exceptional cases, where many integral subvarieties do exist, and describes dividing lines that must be taken into account in any general description of algebraic ODE's and their transformations. Related results have been used to draw consequences in diophantine geometry.
Let V be a smooth complex algebraic variety, with an algebraic vector field ξ. We will be interested in algebraic families $\{U\}$ of algebraic subvarieties V, that are left invariant by the flow corresponding to ξ. We do not demand that U itself be left invariant; only that the deformations of U remain in the same family. Call a subvariety U belonging to such a family, ξcoherent. Thus every point is ξ-coherent, as well as every integral subvariety of ξ. One defines a "geometry" on V, roughly speaking by taking the "closed sets" of X (or of X^n) to be the coherent subvarieties of X or X^n.
By a "curve", let us mean an infinite ξ-coherent subvariety of V, with no proper subvarieties of the same kind. It may well have higher dimension than 1, as an algebraic variety. Through each point there passes at least one "curve"; there may be infinitely or uncountably many. If there are more

than $n = \dim(V)$ "curves" through every (sufficiently general) point of V, we will say that the Kolchin geometry is nondegenerate.
If $\xi = 0$, every subvariety is integral, and we simply have the usual algebraic geometry on V. More generally if ξ admits an algebraic first integral, at least a part of the geometry is classical. The conclusion of (3) will help identify such cases.
Theorem Assume the Kolchin geometry on V is nondegenerate. After removing from V a finite number of lower dimensional integral subvarieties, one of the following occurs:
a. ξ has an algebraic first integral; i. e. there exists a map $f : V \to W$, W an algebraic variety of dimension ≥ 1, such that the vector field ξ is parallel to the fibers of f.
b. There exists an equivariant map $f : V \to V'$, V' an algebraic variety of smaller dimension carrying a vector field ξ', such that the fibers of f are principal homogeneous spaces for algebraic groups; and the action respects the vector field.
c. There exists a map f as in (3), such that the fibers are rational images of Abelian varieties under a finite map; the vector field corresponds to the Manin equation.
The equations occurring in (3) have an interesting structure that is precisely described by the model theory; it is this description that has been used in applications to diophantine geometry.

The proof has the following stages. Quantifier elimination for differential fields (and some additional model theory for these fields) shows that the abstract geometry corresponding to the coherent subvarieties enjoys the basic properties necessary to apply (2). The consequent dimension theory will be bounded by, but will not in general coincide with, the original dimension on the variety. Shelah's theory will be used to find a canonical filtration by maps $f : V \to W$, such that the fibers at each stage are coordinatized by one-dimensional geometries. The situation then splits according to the the structure theorems for these simple geometries, and one obtains an abstract version of the case division. At this point one recalls the differential geometric provenance of the geometry, and deduces (a-c).
Shelah's theory is actually more general than what is indicated here, and does not assume *finite* dimensionality. This greater generality should be of use in future applications, in particular to PDEs.
On the other hand, restrictive assumptions are currently made on the nature of the dimension theory. These are valid in complex algebraic geometry, but rule out e. g. p-adic geometry. Indeed the theory did not initially apply to difference equations; stability needed to be generalized to a wider context, simple theories (Shelah, Kim, Pillay, Hart, Wagner, Buechler, Shami, and others). A separate theory (O-minimality) exists for geometries resembling the reals, and in particular the analog of (3) has been proved by Peterzil and Starchenko. But a satisfactory common framework remains a challenge.
Future applications will depend on expansions of the current frontier of model completeness results, perhaps covering for example fields with global

aspects such as theories of heights, and on extensions of the pure model theory of geometric structures.
(References in the article text.)

1991 MS Classification:

Macdonald, I. G., Queen Mary and Westfield College, University of London, England

Constant Term Identities, Orthogonal Polynomials and Affine Hecke Algebras

Constant term indentities

The archetype of the constant term identities to be considered is the following statement, first conjectured in 1961 by F. J. Dyson but proved soon afterwards. Let $x_1, \ldots, x_n$ be independent indeterminates and k a positive integer. Then the constant term in the Laurent polynomial

$$\Delta_k(x) = \prod_{\substack{i \neq j \\ i,j=1}}^{n} \left(1\frac{x_i}{x_j}\right)^k \tag{1}$$

is equal to $(nk)!/(k!)^n$.
Later, in 1975, G. E. Andrews proposed an extrapolation of (1) involving an extra parameter q. We shall use the notion

$$(x;q)_k = \prod_{i=0}^{k-1} \left(1 - q^i x\right).$$

Clearly, as $q \to 1$ we have $(x;q)_k \to (1x)^k$, and $(q;q)_k/(1-q)^k \to k!$
Let

$$\Delta_k(x;q) = \prod_{\substack{i \neq j \\ i,j=1}}^{n} \left(\varepsilon_{ij}\frac{x_i}{x_j};q\right)_k \tag{2}$$

where $\varepsilon_{ij} = 1$ or q according as $i < j$ or $i > j$. Then Andrews conjectured that the constant term of (2) (i. e. , involving q but none of the x_i) should be

$$(q;q)_{nk}/(q;q)_k^n. \tag{3}$$

Clearly (2) reduces to (1) as $q \to 1$, so that Andrews' conjecture includes Dyson's.
From our point of view, the polynomial $\Delta_k(x;q)$ is to be regarded as attached to the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ consisting of the $n \times n$ complex matrices with trace zero. In fact each finite-dimensional complex simple Lie algebra $\mathfrak{g}$ gives rise to an analogous Laurent polynomial and constant term identity.

Orthogonal polynomials
Let Λ_n denote the ring of symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in the field $F = \mathbb{Q}(q)$. For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$, let m_λ denote the monomial symmetric function indexed by λ, that is to say the sum of all distinct monomials obtainable from $x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}$ by permuting the x_i, Clearly the m_λ form an F-basis of Λ_n.
We shall use the polynomial $\Delta_k = \Delta_k(x; q)$ to define a scalar product on Λ_n, as follows: if $f, g \in \Lambda_n$ then

$$\langle f, g \rangle_k = \text{ constant term in } f\bar{g}\Delta_k$$

where $\bar{g}$ is obtained from g by replacing each x_i by x_i^{-1}. One shows then that there is a unique basis (P_λ) of Λ_n, indexed by partitions of length $\leq n$, such that

(a) $\qquad P_\lambda = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu$

with coefficients $a_{\lambda\mu} \in F$, where $\mu < \lambda$ means that μ precedes λ in the lexicographic ordering; and

(b) $\qquad \langle P_\lambda, P_\mu \rangle = 0 \quad$ whenever $\lambda \neq \mu$.

Moreover, there is a closed formula for the squared norm of P_λ, namely

$$\langle P_\lambda, P_\lambda \rangle_k = \frac{(t;t)_n}{(1t)^n} \prod_{\substack{i<j \\ i,j=1}}^{n} \frac{(q^{\lambda_i - \lambda_j} t^{j-i}; q)_k}{(q^{\lambda_i \lambda_j} t^{j-i}; q^{-1})_k} \tag{4}$$

where $t = q^k$. In particular, when $\lambda = 0$ we have $P_\lambda = 1$, and (4) then reduces to the constant term (3) of Δ_k.
Again, all this can be done in the context of an arbitrary finite-dimensional simple Lie algebra $\mathfrak{g}$ in place of $\mathfrak{sl}_n$, and thus for each such $\mathfrak{g}$ we have a family of orthogonal polynomials P_λ, symmetric under the Weyl group of $\mathfrak{g}$ and indexed by the dominant weights.
Affine Hecke algebras
In the case of $\mathfrak{sl}_n$ the norm formula (4) can be proved directly. For arbitrary $\mathfrak{g}$ there is an analogous formula which was first proved in full generality by Cherednik. Cherednik's proof uses the affine Hecke algebra $\mathcal{H}$ attached to $\mathfrak{g}$ and its action on the space spanned by the polynomials P_λ; this leads to a family of commuting operators on this space whose simultaneous eigenfunctions are precisely the P_λ. The norm formula (4) – or rather its counterpart for arbitrary $\mathfrak{g}$ – can then be established by induction on k.

1991 MS Classification: Primary 94A12; Secondary 41A25, 46E35, 62G05
Stéphane Mallat, Ecole Polytechnique, Paris, France
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Applied Mathematics Meets Signal Processing

1. Beyond Fourier
The Fourier transform has long ruled over signal processing, leaving little

room for new challenging mathematics. Until the 70's, signals were mostly speech and sounds, which are well modeled by Gaussian processes. As a result, linear algorithms could be considered optimal among all non-linear procedures. Adding a hypothesis of stationarity narrows down to the exclusive class of convolution operators, and these are diagonalized by the Fourier transform.

The situation has completely changed with the development of image processing. Images are poorly modeled by Gaussian processes, and non-linear algorithms can considerably outperform linear procedures. This eye opener showed the importance of non-linear and non-stationary operators. Signal processing has become a new application area of modern mathematics, which takes its roots in approximation theory, harmonic analysis [5], non-linear PDE [3], probability and statistics [2]. It is the source of new developments in all these domains, and also an entry to the more complex issues of computational vision and perception [6].

2. Compressed Representations

The solution of many signal processing problems requires finding a representation that efficiently compresses the signals and operators involved. Besides data compression, this holds for estimation and inverse problems, as well as pattern recognition. This lecture considers particular examples that are closely related to the recent development of the wavelet theory [1,4,5], while emphasizing a general approach to non-linear problems.

2.1 Image Compression

Let us decompose an image $f \in \mathbf{L^2}(\mathbf{R}^2)$ in an orthonormal basis $\mathcal{B} = \{g_m\}_m$

$$f = \sum_m \langle f, g_m \rangle \, g_m \, . \tag{1}$$

A compressed code is constructed by approximating the inner-product $\langle f, g_m \rangle$ by quantized values $Q(\langle f, g_m \rangle)$ that are coded with R bits. The restored signal $\tilde{f} = \sum_m Q(\langle f, g_m \rangle)\, g_m$ has a distortion $d(f, R) = \|f - \tilde{f}\|_{L^2}$ that must be minimized. This distortion is closely related to the non-linear approximation error $\epsilon(f, T) = \|f - f_T\|_{L^2}$ when thresholding at T the inner products of f:

$$f_T = \sum_{|\langle f, g_m \rangle| > T} \langle f, g_m \rangle \, g_m .$$

For f in a ball of a Banach space $\mathbf{H}$, one can prove [4] that the decay rate of $\epsilon(f, T)$ for $T \to \infty$ is maximized by choosing an *unconditional* basis of $\mathbf{H}$. The basis $\mathcal{B}$ is an unconditional basis of $\mathbf{H}$ if the norm $\|f\|_H$ for $f \in \mathbf{H}$ is equivalent to a quasi-norm on the modulus of inner products $\{|\langle f, g_m \rangle|\}_m$ [5].

Little prior information is known on most images, but they often belong to the $\mathbf{BV}$ space of bounded variation functions:

$$\|f\|_{BV} = \int |\vec{\nabla} f(x)| \, dx < +\infty.$$

This motivates the use of wavelet bases. There exists no unconditional basis in **BV** but wavelet bases are unconditional bases of Besov spaces that give a close embedding of **BV**. Wavelet image coders are currently the most efficient, but could be improved by explicitly taking into account geometric image properties.

2.2 Signal Restoration and Identification

Recent techniques to compress vectors and operators allow one to reduce many non-linear estimation and restoration problems to simple thresholdings in appropriate bases. Suppose that a signal $f \in \mathbf{H}$ is contaminated by an additive noise which is a Gaussian white process W of variance σ^2. The measured signal is

$$Z = f + W. \tag{2}$$

Removing the noise consists in finding an operator L such that the estimation $F = L(Z)$ of f reduces the expected risk $r(f, \sigma) = E\{\|f - F\|^2_{L^2}\}$. One can prove that $r(f, \sigma)$ is closely related to the non-linear approximation error $\epsilon(f, T)$ [2]. The maximum risk $r(f, \sigma)$ for f in a ball of $\mathbf{H}$ is thus nearly minimized by thresholding the decomposition coefficients of Z in an unconditional basis of $\mathbf{H}$. Applications to bounded variation images are shown.

The next level of difficulty occurs when the measured signal is also degraded by an operator K

$$Z = Kf + W, \tag{3}$$

with an inverse K^{-1} than is not bounded on the range of K. Examples on blurred satellite images illustrate this case. We must now find a representation that compresses both f and K. If $K\,K^* = D\,T$ where D is diagonal in an unconditional basis of $\mathbf{H}$ and T is bounded with a bounded inverse then a nearly optimal estimator $F = L(Z)$ is also calculated with a thresholding strategy in this basis. When f has bounded variation and K is a blurring convolution operator, such a basis is constructed with wavelet packets [4].

More difficult estimation problems occur when the observed signal is only a realization of an underlying non-stationary and non-parameterized process that must be identified. Image textures are one such example. One can then estimate the covariance operator of this process, in which case the signal is represented by an operator as opposed to a vector. Estimating the covariance operator from a single realization requires finding a basis where it's matrix is nearly diagonal, thus reducing the number of coefficients to be estimated. Locally stationary processes are examples whose covariances are pseudo-differential operators, which are estimated by searching for bases in where the matrix is nearly diagonal. However, let us emphasize that image texture discrimination is still an open problem.

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1991 MS Classification: 53C15, 58D99
Dusa McDuff, State University of New York at Stony Brook, USA
Fibrations in Symplectic Topology

Every symplectic form on a $2n$-dimensional manifold is locally the exterior product of n area forms. This local product structure has global implications in symplectic topology. The talk will discuss several different situations in which one can see this, for example, the use of fibered mappings in the construction of efficient symplectic embeddings of fat ellipsoids into small balls, and the theory of Hamiltonian fibrations (work of Lalonde, Polterovich, Salamon and the speaker). The most spectacular example is the recent completion by Auroux of Donaldson's program, showing that every compact symplectic manifold admits a symplectic Lefschetz pencil.

1991 MS Classification: 81, 82
Miwa, Tetsuji, RIMS, Kyoto University, Japan
Algebraic Analysis of Solvable Lattice Models

The study of solvable lattice models in statistical mechanics started with the pioneering works by Bethe and Onsager more than 50 years ago. Solvability of the models means that we can compute physical quantities in closed form. Models in statistical mechanics are systems of infinite degrees of freedom. The basic question was why such systems can be solvable. People gradually realized that the infinite dimensional symmetries of the models answer this question. The birth of quantum groups brought an appropriate tool to describe such symmetries, and gave a connection between the solvable lattice models and the representation theory. In this talk, I will explain how to solve "solvable" models by using the representation theory of the quantum affine algebras.
References: Michio Jimbo and Tetsuji Miwa, Algebraic Analysis of Solvable Lattice Models, *CBMS Regional Conference Series in Mathematics,* **85** 1995, AMS, Providence, Rhode Island.

1991 MS Classification:
Moser, Jürgen, ETH Zürich, Switzerland
Dynamical Systems – Past and Present

This lecture presents, from a personal point of view, several important advances, solved and unsolved problems in dynamical systems, and shows their interconnection with other fields of mathematics.

1991 MS Classification: 60H15, 73D70, 35R60
Papanicolaou, George C., Stanford University, USA
Mathematical problems in geophysical wave propagation

The analysis of wave propagation in randomly inhomogeneous media leads to many difficult problems in stochastic processes and differential equations. Wave propagation in the earth's crust, seismic wave propagation, is particularly difficult to model, to analyze, to simulate and to use as a probing tool. In this lecture we will present a review of the mathematical problems that arise in seismic wave propagation and the role that modern applied mathematics can play in articulating them and in providing methods and tools for their solution.
After a brief review of some basic notions in waves in random media, such as the identification of the relevant length and time scales and properties of the coherent wave, we will describe the mathematical modeling of energy propagation in two important regimes:

- **Long wave, long distance wave propagation and the use of Radiative Transport Theory**
- **Local, exploration seismology, anisotropy and wave localization**

The main issues in the use of Radiative Transport Theory in seismic wave propagation are the following.

- When is it a valid description of the observed phenomena?
- When can it be replaced by a much simpler Diffusion Theory?
- What do we know about reflection and transmission at interfaces or boundaries when Radiative Transport is valid?
- What do we know about wave mode energy conversion in the Radiative Transport regime?

These questions have precise mathematical formulations and considerable progress in their analysis has been made in the last five to ten years. But many important problems remain open. The behavior of random waves near boundaries and interfaces is particularly difficult to analyze. New analytical methods from microlocal and stochastic analysis are needed for the more basic questions. Even methodologically simpler problems, like the passage

of Radiative Transport to Diffusion, are difficult to analyze near boundaries and many problems there are still open. It is interesting to note that the use of Radiative Transport in seismology is relatively recent, mostly in the last fifteen years, while Radiative Transport has been used to model the passage of light through inhomogeneous atmospheres for nearly a century.
The earth's crust is, however, a very complex medium and Radiative Transport need not be a suitable analytical tool when the inhomogeneities are strong and highly anisotropic. This is the case frequently in exploration seismology where propagation is confined to a few kilometers and wavelengths are relatively short (about one hundred meters). We are now close to the regime of **Wave Localization** where the random inhomogeneities act to limit spatially the propagation of wave energy. We know a lot about the mathematics of Wave Localization in randomly layered media and we will review briefly several results obtained in the last few years. But there are many difficult open problems when it comes to understanding the localization-delocalization (or transport) transition as we go from perfectly layered random media to highly anisotropic ones and then to more isotropic random inhomogeneities. We know very little here and numerical simulations do not help because the kind of resolution required to observe localization phenomena is impossible to achieve at present.
We will conclude the lecture with a brief review of computational issues, parameter estimation and imaging in the presence of random inhomogeneities.

1991 MS Classification:
Gilles Pisier, Université Paris VI, and Texas A&M University
Operator Spaces and Similarity Problems

We will present an overview of the theory of "Operator Spaces" (sometimes called "non-commutative Banach spaces"), recently developed by Effros, Ruan, Blecher, Paulsen and others. An operator space is simply a linear subspace of the space $B(H)$ of all bounded operators on a Hilbert space H, but the morphisms acting between two such objects are taken to be the "completely bounded" maps which have played an important rôle in C^*-algebra theory in the 80's and 90's, following important earlier seminal work (Stinespring, Arveson, Lance, Effros, Choi, Haagerup, ...) on "completely positive" mappings. Central to this new theory is a specific new duality which parallels the usual Banach space one and which has proved to be very fruitful. The talk will present several applications of this ideology to operator algebras and to various similarity problems. We will discuss our recent example of a polynomially bounded operator in l_2 which is not similar to a contraction (answering negatively a question of Halmos), as well as the notion of "similarity degree", which is an operator algebraic analog of the maximal word-length in a finitely generated group.

1991 MS Classification:
Sarnak, Peter, Princeton University, USA
Zeta and L-functions

The theory of zeta and L-functions is at the center of a number of recent developments in number theory. We will review some of these developments ranging from the Riemann Zeta function to standard L-functions for automorphic forms on the general linear group. Applications to classical problems in number theory and related areas will be highlighted.

1991 MS Classification: 11Y05, 68Q05, 81P99
Shor, Peter W., AT&T Labs, Florham Park, USA
Quantum Computing

Church's thesis says that any computable function is computable on a Turing machine. This thesis arose in the 1930's, and was motivated by the realization that three quite distinct definitions of computable functions were all equivalent. It is well-known that this is not a theorem, as Church's thesis does not specify a rigorous mathematical definition of "computable"; specifying such a definition would lead to a provable theorem (and in many cases has), but it detracts from the generality of the thesis. What is somewhat less commonly realized is that this thesis can be viewed as a statement about the laws of physics, by interpreting computable as meaning computable in the physical world. For this interpretation, if the laws of physics are computable by a Turing machine, then Church's thesis is true.
The development of digital computers rendered the distinction between computable and uncomputable functions too coarse in practice, as it does not take into account the time required for computation. What was needed was some characterization of efficiently computable functions. Theoretical computer scientists reached a good compromise between theory and practice with the definition of polynomial-time computable functions as those functions whose value can be computed in a number of steps polynomial in the input size. This definition is abstract enough to have good theoretical properties for use in proofs, while for functions arising in practice it includes most of the efficiently computable ones and excludes most of those not efficiently computable. This definition naturally gave rise to a "folk thesis, " the polynomial Church's thesis, which says that any function physically computable in time t can be computed on a Turing machine in time p(t), where p is some polynomial.
Is this thesis valid? One good place to start looking for counterexamples would be physical systems which require large amounts of computer time to simulate. Two obvious such candidates are turbulence and quantum mechanics. In 1982, Feynman considered the case of quantum mechanics, and concluded that quantum mechanical systems seem to inherently require an exponential overhead to simulate on digital computers. In a "side remark, " he proposed using quantum computers, operating on quantum mechanical principles, to circumvent this problem. Deutsch (1985) gave a definition of

a quantum Turing machine and suggested that quantum computers might solve classical problems more quickly than digital computers. It currently appears that this is indeed the case. One of the pieces of evidence for this is that quantum computers can factor integers in polynomial time, something which it is not known how to do on classical computers despite many years of study.

A nice abstract model of a quantum computer is the quantum circuit. This machine computes using n 2-state quantum systems; each of these 2-state systems is called a qubit. The quantum state space of the joint system is a complex vector space of dimension 2^n, formed from the tensor product of n two-dimensional complex vector spaces, one associated with each qubit. The state of this system is described by unit vector in this 2^n-dimensional space. Because of the Heisenberg uncertainty principle, the output state cannot be obtained exactly. Rather, if the end result of the computation is $\sum_{i=0}^{2^n-1} \alpha_i v_i$, the observed output will be some integer $0 \le i < 2^n$, and i will be observed with probability $|\alpha_i|^2$ (here the v_i are basis vectors of the quantum state space). Possible physical transitions are unitary transformations of this vector space. We cannot assume that we can perform any unitary transformation in one step, as this is not only physically unrealistic, but also would result in any function being computable in unit time. Rather, we must assume that in one step we can make a unitary transformation on one or two of the qubits, which via the tensor product naturally induces a transformation on the whole vector space. In fact, it suffices to be able to perform a small number of different one- and two-qubit unitary transformations to perform any quantum computation (Barenco *et al* 1995).

For factoring an L-bit number N, the best classical algorithm known is the number field sieve, which asymptotically takes time $O(\exp(cL^{1/3}\log^{2/3} L))$. On a quantum computer, the quantum factoring algorithm takes asymptotically $O(L^2 \log L \log\log L)$ steps. The key idea is to use a Fourier transform to find the period of the sequence $a_i = x^i (\text{mod N})$, from which period a factorization of N can be obtained. The period of this sequence is exponential in L, so this approach is not practical on a digital computer. On a quantum computer, however, we can find the period in polynomial time by exploiting the 2^{6L}-dimensional state space of $6L$ qubits, and taking the Fourier transform over this space.

Quantum computers will be very hard to realize physically. We need quantum systems which are relatively stable, and which interact strongly with each other but weakly with everything else. Without error correction, it would probably be an impossible engineering task to build quantum computers large enough to factor 100-digit numbers—factoring such a number requires billions of steps on a quantum computer, so each step would need to be accurate to better than one part in a billion. Fortunately, it is possible to design fault-tolerant circuits for quantum computers, which allow computations of arbitrary length to be performed with gates accurate only to some constant c. Current estimates put this constant larger than 10^{-4} (Preskill 1997).

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D. Deutsch, Quantum theory, the Church–Turing principle and the universal quantum computer, *Proc. Roy. Soc. London Ser. A* **400** (1985), 96–117.
R. Feynman, Simulating physics with computers, *Internat. J. Theoret. Phys.* **21** (1982), 467–488.
J. Preskill, Fault-tolerant quantum computation, LANL e-print quant-ph/9712048, available at URL `http://xxx.lanl.gov/` (1997).
P. W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, *SIAM J. Computing* **26** (1997), 1484–1509.

1991 MS Classification:
Sigmund, Karl, University of Vienna, Austria
The Population Dynamics of Conflict and Cooperation

Among the wide-ranging applications of mathematical modelling to biological problems, those dealing with ecology and evolution have proved particularly challenging for stochastic processes and dynamical systems. This lecture attempts a survey, emphasising the major feedback mechanisms acting on the densities of the interacting populations and keeping mostly to the framework of ordinary differential equations. (Time delays, spatial heterogeneities and stochastic fluctuations have a major impact in many applications.)
Even the simplest Lotka-Volterra models of three interacting species display a strong propensity for instability, for instance chaotic oscillations (in the one-predator, two-prey case) or heteroclinic attractors (for three competing species). Stability notions in the sense of Lagrange are often more appropriate than Lyapunov stability for population ecology. Basic questions concern permanence (can small perturbations lead to the extinction of some populations?) and invadability (will a population grow if introduced in small numbers?). Contingency and history-dependence play an essential role in the assembly of ecological communitities (keystone predators mediate the diversity of the ecosystem, species can invade but alter the community in such a way that they become extinct and cannot be re-introduced, a set of stably co-existing species cannot be obtained through sequential invasion by these species alone etc.)
Competition among conspecifics drives natural selection. Game theory allows to investigate the relative success of different behavioural types in conflicts within and between species (when fighting for resources, for instance, or investing in offspring). The interactions of individuals lead to collective dynamics of populations, regulating key parameters like sex ratio, clutch size or dispersal rate. The simplest dynamics, the replicator equation, is equivalent to the Lotka-Volterra dynamics. It is based on the assumption that strategies 'breed true' and are transmitted to offspring proportionally

to the payoff currently achieved. Other dynamics take account of the intricacies of Mendelian inheritance, and still others are based on transmission via imitation and learning. The latter approach – adaptation by cultural means – allows to apply evolutionary game theory to economics and social sciences, to analyse its relation with the equilibria concepts of classical game theory and to stress the importance of non-equilibrium outcomes. Adaptive dynamics (based on assuming that replication is only almost exact) provides a tool for describing long term evolution by mutation and selection.

Of biomedical relevance are applications to epidemiology, immunology and virology. Mathematical models help to understand the course of infectious diseases both within organisms and within populations, and suggest guidelines for treatment and vaccination. Within individual organisms, pathogens elicit immune responses which then attack the pathogens in much the same way as prey and predators interact. Within the population, the interactions of infected, susceptible and immune organisms lead to endemic or epidemic forms of the disease.

The evolution of virulence (i. e. the parasite-induced mortality of host organisms) has attracted particular attention. Pathogens use the bodies of their hosts both as resource and as vehicle. Mathematical models show that by having to spread within and between organisms, parasites often face conflicting demands on transmission rate and long-term exploitation. In the case of co-infection, i. e. when several strains compete, selection on parasites need not optimise the exploitation of the host (or the basic reproductive rate, i. e. the number of infected hosts produced by an infected individual in a population of susceptibles). Game theoretical arguments help in analysing such situations.

The parasite ecology is further complicated by countermeasures of the hosts. Due to their short generation times, parasites are usually at an advantage – witness the rapidly evolving resistance to antibiotics – but sexual reproduction allows host organisms to recombine immunotypes and thus to present shifting targets to the pathogens trying to enter the cells. Mathematical models for the resulting arms races display a profusion of limit cycles, irregular oscillations and heteroclinic cycles and make evolution look like a treadmill rather than a ladder.

Nevertheless, evolution has been punctuated by some major transitions to new levels of organisation, for instance self-replicating molecules, chromosomes, cells, multicellular organisms, colonies and societies. In most cases, this emergence of nested hierarchies was due to the fusion of formerly independent units into entities of higher order (which remain threatened by exploitation through mutinies of 'selfish' units). Evolutionary game theory provides a tool for the analysis of such feats of cooperation: catalytic loops of RNA molecules exhibit limit cycles, the suppression of 'outlaw genes' within the genome leads to heteroclinic repellors, etc.

In particular, direct and indirect reciprocity, which play a basic role in human societies, can be understood by means of repeated interactions. Mutual aid emerges through learning rules, moralistic emotions, social pressure and reputation. Thus biological evolution leads game theory back to its original economic motivation.

1991 MS Classification:
Michel Talagrand, C. N. R. S., Université Paris VI, France
Huge random structures and mean field models for spin glasses

Consider a large population, the individuals being labeled 1 to N. Assume that any pair $(\epsilon_{ij}), i < j$ of them know each other, and let g_{ij} measure their respective feelings. In particular $g_{ij} > 0$ when they are friends, $g_{ij} < 0$ when they are enemies. Assume that (as a consequence of brilliant minds and strong personalities) the numbers (g_{ij}) are independent standard normal random variables. Then for many triples i, j, k, i is a friend of both j and k, but j and k are enemies. This creates a rather tense situation. To alleviate the problem, one can try to split the population in two parts, trying, as far as possible to put friends together and enemies apart. To determine to which group individual i belongs we assign to it a number $\sigma_i \in \{-1, 1\}$, using the fact that there is a one to one correspondence between the sequences $\boldsymbol{\sigma} = (\sigma_i)_{i \leq N}$ and the partitions of $\{1, \cdots, N\}$. The function

$$H(\boldsymbol{\sigma}) = \sum_{i<j} \sigma_i \sigma_j g_{ij}$$

that adds the interaction numbers g_{ij} of pairs in each group and subtract them for pairs in different groups is a natural measure of how successful the given partition is. How large can $H(\boldsymbol{\sigma})$ be? Finding $\boldsymbol{\sigma}$ that maximizes $H(\boldsymbol{\sigma})$ is an extremely difficult optimization problem. Numerical evidence (and extraordinarily clever speculation from the physicists) suggests that, for N large,

$$\max_{\boldsymbol{\sigma}} H(\boldsymbol{\sigma}) \simeq .7633 N^{3/2}$$

(Comparing to the natural scaling factor N^2, the philosophy of this is that splitting the population does not really improve matters). Suppose now that, instead of individuals, we are dealing with magnetic impurities, and that σ_i represents the spin of impurity i. As a first approximation, we need not be concerned about the precise location of the impurities, and we can assume that each of them interacts equally with all the other impurities of the sample (a so called mean field model). Then a very natural candidate for the energy of configuration $\boldsymbol{\sigma}$ is

$$H(\boldsymbol{\sigma}) = -\tfrac{1}{\sqrt{N}} \textstyle\sum_{i<j} \sigma_i \sigma_j g_{ij} - \sum_{i \leq N} h \sigma_i \tag{1}$$

There the minus sign follows physic's convention; the factor $1/\sqrt{N}$ is an appropriate normalization (to ensure that $H(\boldsymbol{\sigma})/N$ remains bounded) and h is an external field, that favors the plus spins over the minus ones. Formula (1) is the Hamiltonian of the famous Sherrington-Kirkpatrick (SK) model for spin glasses. Rather than just being concerned with the minimum of $H(\boldsymbol{\sigma})$, (i. e. the ground state energy), statistical mechanics brings in an inverse temperature β and considers Gibb's measure, a probability on the set of all configurations given by

$$G(\boldsymbol{\sigma}) = Z^{-1} \exp(-\beta H(\boldsymbol{\sigma}))$$

where $Z = \sum_{\boldsymbol{\sigma}} \exp(-\beta H(\boldsymbol{\sigma}))$ is a normalization factor. Thus G gives large weight to the configurations with low energy, and all the more so when β is large (low temperature). Gibbs' measure is a random object (through the g_{ij}) and the problem is to understand its typical structure for N large. The physicists have put much effort into this question, and have obtained a very precise picture (although mathematicians might object to some of their methods, that involve concepts such as functions of a negative number of variables, diagonalization of matrices of dimension 0×0, and more). It is believed that at given h, if $\beta \leq \beta(h)$ the system "is in a pure state" (in a sense that will be made mathematically precise). This behavior will be called the high temperature region. On the other hand, if $\beta > \beta(h)$ (the low temperature region) it is believed that the system exhibits a very complex structure (the Parisi solution) involving in particular a spontaneous breaking of the system into different states (which means that when we observe the system at very long intervals we might see what appears to be different objects).

Besides the fascinating fact that the simple Hamiltonian (1) gives rise to extreme subtlety, the importance of this subject is that a number of (apparently unrelated) random combinatorial problems of considerable importance exhibit similar behavior. For example, let us consider the discrete cube $\Sigma_N = \{-1, 1\}^N$ in dimension N, and half spaces at a given distance from the origin, oriented in random directions. What is the typical size of their intersection with Σ_N? (a problem motivated by the theory of neural networks). To bring in statistical mechanics, one introduces the Hamiltonian on Σ_N defined at $\boldsymbol{\sigma}$ by counting the number of random half spaces to which $\boldsymbol{\sigma}$ belongs. Or consider the random K-sat problem, which is to know what proportion of the truth assignments of N Boolean variables will simultaneously satisfy M random clauses of length K. For all these models and others (among which the famous Hopfield model of associative memory) we have been able to prove that the physicist's magical formulas are indeed true at high enough temperatures, and for several of these models we obtained a very precise picture (capturing all effects of size $1/\sqrt{N}$). High temperature results are motivated by the fact that for several models, the "high temperature" region is thought to extend all the way to temperature zero. But of course, the main challenge is the low temperature region. There also has been progress there. For the p-spin interaction model (closely related to the SK model), we did succeed in rigorously proving the most striking prediction of physics: the spontaneous appearance of different "states", any two of them very far apart.

1991 MS Classification:
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Geometric Physics

Over the past two decades there has been growing interaction between theoretical physics and pure mathematics. Many of these connections have led to profound improvement in our understanding of physics as well as of mathematics. The aim of my talk is to give a non-technical review of some of these developments connected with string theory.

The central phenomenon in many of these links involves the notion of *duality*, which in some sense is a non-linear infinite dimensional generalization of Fourier transform. It suggests that two physical systems with completely different looking properties are nevertheless isomorphic if one takes into account "quantum geometry" on both sides. For many questions one side is simple (quantum geometry is isomorphic to classical one) and the other is hard (quantum geometry deforms the classical one). The equivalence of the systems gives rise to a rich set of mathematical identities. One of the best known examples of duality is known as "mirror symmetry" which relates topologically distinct pairs of Calabi-Yau manifolds and has applications in enumerative geometry. Other examples involve highly non-trivial "S-dualities" which among other things have found application to the study of smooth four manifold invariants. There have also been applications to questions of quantum gravity. In particular certain properties (the area of the horizon) of black hole solutions to Einstein equations have been related to growth of the cohomology of moduli space of certain minimal submanifold in a Calabi-Yau threefold.

A central theme in applications of dualities is a physical interpretation of singularities of manifolds. The most well known example is the $A-D-E$ singularities of $K3$ manifold which lead to $A-D-E$ gauge symmetry in the physical setup. The geometry of vanishing cycles is a key ingredient in the physical interpretation of singularities. More generally, singularities of manifolds encode universality classes of quantum field theories. This leads not only to a deeper understanding of the singularities of manifolds but can also be used to "geometrically engineer" new quantum field theories for physics. Through this method many non-trivial properties of quantum field theories get related to classical symmetries of geometry. An example of this application (in the context of Montonen-Olive duality) is a simple geometric interpretation of the modular properties of the generating function of the Euler class of moduli space of instantons on four manifolds.

1991 MS Classification: 58, 34, 60
Viana, Marcelo, IMPA - Rio de Janeiro, Brazil

Dynamics: a probabilistic and geometric perspective

The probabilistic viewpoint has, over the last decades, become a full player in the very realm of deterministic dynamical systems. A clear reason,

though not unique, was the realization that the long term evolution of many systems depends sensitively on the initial state, so that the behaviour of individual trajectories is in some sense unpredictable. Then one turns to *statistical properties of large sets of orbits* – positive probability in phase space -as a more suitable probe into the system's behaviour.

The 1960's had seen the introduction by Smale of the notion of uniform hyperbolicity – uniform expansion and uniform contraction along complementary directions –, heir to a geometric tradition going back to Hadamard, Perron, Poincaré, Birkhoff, Andronov, and immediately influencing the work of Anosov on the geodesic flow of manifolds with negative curvature. Through the work of several mathematicians, the theory of uniformly hyperbolic systems came to provide a remarkably detailed picture of a large class of dynamical systems, including cases of sensitive behaviour.

An achievement was the proof that uniformly hyperbolic systems are structurally stable systems, and the unique ones in the C^1 topology (completed by Mañé and, more recently, Hayashi). Another, was the conclusion, by Sinai, Ruelle, Bowen, that their statistical behaviour can be described in terms of a *finite* number of objects : there exist invariant probability measures $\mu_1, \ldots, \mu_N$ such that the time average

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)} \quad \text{(for maps } f\text{)} \quad \text{or} \quad \lim_{n\to+\infty} \frac{1}{T} \int_0^T \delta_{f^t(z)}\, dt \quad \text{(for flows } f^t\text{)}$$

exists and coincides with some of the *SRB measures* μ_i, for a full Lebesgue probability set of points z in phase space.

But it was soon realized that systems can be persistently non-hyperbolic – persistently unstable – and so uniform hyperbolicity falls short of a general theory of dynamical systems. This included systems with infinitely many periodic attractors (Newhouse), as well as pioneer examples of "chaotic" systems (Lorenz, Hénon, Feigenbaum, Coullet-Tresser), which often had some sort of physical motivation. A comprehension of geometric and probabilistic aspects of these examples helped set a way towards a comprehensive theory of dynamical systems, that is currently on the make.

A central problem going back to Sinai, Eckmann, Ruelle, in the seventies: For "most" systems, do time averages exist at Lebesgue almost every orbit ? Palis conjectures that the answer is *Yes*, and that *the time averages are given by a finite number of SRB measures*, for a C^k dense subset of systems, $k \geq 1$. This is the cornerstone of a global program that also predicts that statistical properties of such systems, restricted to the basin of each attractor μ_i, are stable, namely under small random perturbations of the system: stochastic stability.

Research on this program and these problems has been efervescent in recent years. The following is but a sample of some latest developments.

Concerning one-dimensional maps, Lyubich proved that Lebesgue almost every quadratic map $x \mapsto 1 - ax^2$ admits a unique SRB measure, either a Dirac measure on a periodic orbit or an absolutely continuous measure.

Jakobson had first proved that this latter case occurs with positive probability in parameter space.

Hénon-like attractors of surface diffeomorphisms, have the same form of probabilistic persistence, and are the first class of genuinely non-hyperbolic attractors to be reasonably well-understood, specially from an ergodic point of view (Benedicks, Carleson, Mora, Viana, Young). They support a unique SRB measure, with exponential mixing properties, and they are stochastically stable. And they have the *no-holes property*: the set of points whose time average is given by the SRB measure has full Lebesgue probability in the topological basin of the attractor.

At the core of the problem of understanding general non-hyperbolic maps is the phenomenon of *homoclinic tangencies*, that is, non-transverse intersections between stable and unstable manifolds of a same periodic point. It was conjectured by Palis that every surface diffeomorphism can be C^k approximated by another which is either hyperbolic or has a homoclinic tangency. For $k = 1$ a proof was recently given by Pujals, Sambarino. This also means that one expects the set of non-hyperbolic maps to have some sort of homogeneity, indeed, several partial results relate homoclinic tangencies with other forms of non-hyperbolic behaviour. A recent result of Moreira, Yoccoz exposes to a new level of depth the role played by fractional dimensions in homoclinic bifurcations (explored before by Newhouse, Palis, Takens).

A main novelty in the context of flows is the phenomenon of invariant sets containing robustly – meaning, for a whole C^1 open set of systems – singularities together with regular dense orbits. A theory of these *singular* (or Lorenz-like) *sets*, specially in dimension three, is being developed by Morales, Pacifico, Pujals, where this phenomenon is characterized in terms of a notion of *singular hyperbolicity*. The higher dimensional case is still very much open, but it is now known that multidimensional Lorenz-like attractors do exist (Bonatti, Pumariño, Viana).

For high dimensional maps and flows, new phenomena enter the scene, and problems and conjectures are reformulated accordingly. More generally than homoclinic tangencies, one must take into account heteroclinic cycles: intersections between invariant manifolds of possibly different periodic points.

The main part of the dynamics of a uniformly hyperbolic system admits a decomposition into a finite number of pieces which are transitive – dense orbits – and remain so under any small perturbation of the system. What could replace these basic hyperbolic blocks in a general context ?

For diffeomorphisms in dimension three, invariant sets that are robustly transitive must be *partially hyperbolic* (Díaz, Pujals, Ures): uniform hyperbolicity in some direction – invariant subbundle of the tangent space – *dominating* what happens in a complementary invariant direction. In higher dimensions uniformly hyperbolic subbundles need not exist (Bonatti, Viana), but robustly transitive sets always admit a dominated splitting into two invariant subbundles (Bonatti, Díaz, Pujals).

Can we describe the ergodic properties of robustly transitive systems ? The volume preserving case is very much studied by Pugh, Shub, and others. For dissipative diffeomorphisms partial results on the existence and finitude of SRB measures are being provided by Alves, Bonatti, Viana. The methods developed in this setting suggest that non-uniform hyperbolicity – positive Lyapunov exponents – on positive Lebesgue measure subsets, may suffice for the existence of SRB measures. In particular, non-uniformly expanding smooth maps without critical points admit ergodic invariant measures absolutely continuous with respect to Lebesgue measure.

The reader will have noticed that I kept my focus on general (non-conservative) dynamical systems, without mention to the remarkable progress attained also for Hamiltonian and symplectic systems. In a Congress that counts among its contributors some of the greatest experts in conservative dynamics, I hope I will be excused for doing so.

1991 MS Classification: 14XX
Voevodsky, Vladimir, Northwestern University, USA
Homotopy Theory of Algebraic Varieties

Algebraic varieties over a field form a category which is both similar to and very different from the category of topological spaces. Formalization of different aspects of this elusive similarity was always among main sources of new proofs and constructions both in algebraic geometry and in topology. In the last several years it became clear that there is a theory which provides common framework for many diffrent constructions of this type. This theory is called the *homotopy theory of algebraic varieties*. It is based on simple observation that the category of algebraic varieties over any field has an object which is analogous to the unit interval in the topological category namely the affine line $\mathbf{A}^1$.
Earlier attempts to use this analogy to construct algebro-geometrical versions of of different topological (co-)homology theories go back to Karoubi, Villamayor, Jardine, Weibel and Suslin. In this talk I will outline the main features of the homotopy theory of algebraic varieties over fields of characteristic zero as we see it now. In its present form this theory originated in the work of Fabien Morel and myself. It provides foundations for most of the new constructions used in the recent proof of the Milnor conjecture relating K-theory, etale cohomology and quadratic forms over fields.
The naive notion of $\mathbf{A}^1$-homotopy classes of morphisms in the category of algebraic varieties over k is replaced in this theory by a more technical one of morphisms in an appropriately defined homotopy category of algebraic varieties. This category is the homotopy category associated with a complete and cocomplete, proper, simplicial closed model category in the sense of Quillen which we call the category of spaces $SPC(k)$ over k. There is a functor from the category Var/k of algebraic varieties over k to $SPC(k)$ which is almost a full embedding and almost a surjection on isomorphism classes of objects.
The main distinctive feature of the homotopy theory in $SPC(k)$ is the

existence of two different circles - one corresponding to the affine line with two points glued together and another one to the affine line without a point. This leads to two different suspension functors on the category of pointed spaces over k which both have to be inverted to get the stable homotopy category $SHot(k)$.

Any object in $SHot(k)$ defines in the usual way a cohomology theory on the category of algebraic varieties. Since there are two suspensions all the theories obtained this way are *bi-indexed*. I will describe briefly three fundamental examples of such theories. Homotopy *algebraic K-theory* (an $\mathbf{A}^1$-homotopy invariant version of algebraic K-theory first introduced by C. Weibel) turns out to be representable by a spectrum BGL directly analogous to the usual BU spectrum in topology. *Algebraic cobordism* is the theory representable by a direct analog MGL of the Thom spectrum MU. It plays fundamental role of the universal theory with direct image homomorphisms for all smooth proper morphisms. *Motivic cohomology* is the theory representable by Eilenberg-MacLane spectrum $H_{\mathbf{Z}}$ whose terms are defined as infinite symmetric powers of spheres (analogous to the definition of Eilenberg-MacLane spaces through the Dold-Thom theorem).

We strongly believe that all the usual operations connecting the topological prototypes of these theories have analogs in algebro-geometrical setting. Some of them such that characteristic classes are relatively straightforward. Others like Steenrod operations in motivic cohomology are more difficult to construct. Yet others like "motivic" analogs of Atiyah-Hirzebruch spectral sequences for algebraic K-theory and algebraic cobordisms are still unknown.

1. Logic

1991 MS Classification: 3,4,5,28
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Generic Large Cardinals: New Axioms for Mathematics?

While the standard axiomatization of mathematics, (*Zermelo-Fraenkel Set Theory with the Axiom of Choice*, ZFC) has been extremely successful in resolving the foundational issues that arose at the turn of the century, it has some glaring shortcomings. These shortcomings are largely due to their inability to settle certain natural problems.
Most of these problems are of an infinitary character (though many can be coded as questions about the solutions to various diophantine equations). Most prominent among these problems are Hilbert's 1^{st} problem (the Continuum Hypothesis), and issues having to do with the use of the Axiom of Choice. The development of Forcing, in the early 1960's, led to independence results in most areas of mathematics that have a strong infinitary content, particularly including measure theory and other parts of analysis, infinite group theory, topology and combinatorics.
In a program encouraged by Gödel, attempts were made to find natural extensions of ZFC that were strong enough to settle these questions. (Of course, Gödel's incompleteness theorems prevent the development of a *complete* axiomatization.) In these efforts, two competing axiom systems arose: *Determinacy Axioms* and *Large Cardinal Axioms*. Each system has its merits: Determinacy Axioms answer most questions about definable, or constructible sets, and Large Cardinal Axioms are natural closure and symmetry properties of the mathematical universe. However, on the surface the two systems have very little to do with each other.
Much work was done exploring the connections between these two axiom systems, culminating in a series of theorems in the late 1980's showing that they are inextricably intertwined. An apparently technical device that arose in the course of this work was the notion of a *Generic Large Cardinal* or generic elementary embedding. These assumptions eventually turned out to be natural generalizations of various Large Cardinal Axioms and have the virtue that they settle the Continuum Hypothesis, imply the Determinacy axioms and can be viewed as strong symmetry principles.
Because of the close link between Forcing and Boolean Algebras, these axioms are most conveniently expressed as properties of ideals on Boolean Algebras. Ulam, in 1930, proposed the axiom of *Measurable Cardinals*, i. e. the existence of a set X and a countably complete, non-atomic, two valued measure μ defined on all subsets of X. This can be rephrased as the existence of a countably complete prime ideal on the Boolean Algebra of subsets of X that contains all of the singletons. Ulam showed that such a set X must have cardinality much larger than the real numbers or the collection of subsets of the real numbers.
Gitik and Shelah extended this by showing that there cannot be a countably complete ideal I on the collection of subsets of the real numbers such that the

quotient Boolean Algebra has a dense countable subset (and similar results for larger sets than the real numbers). Thus the strongest ideal property one can hope is consistent is an ideal such that the quotient Boolean algebra has a dense set of cardinality $\aleph_1$ (the first uncountable cardinal). Such ideals are called $\aleph_1$-*dense*.
Some examples of theorems of the author discussed in the talk include:

Theorem *If there is a normal, fine* $\aleph_1$*-dense ideal on the subsets of* $P(\mathbb{R})$ *of cardinality* $\mathbb{R}$ *then*:

-The Continuum Hypothesis holds
-There is a Suslin Line
-Kurepa's Hypothesis Fails
-$\Box_{\omega_1}$ fails.
-$\omega_2 \to (\omega_1^2, \alpha)$ for all $\alpha < \omega_2$ (joint with Hajnal)
-A. D. holds in $L(\mathbb{R})$ (Heavily using results of Woodin)

(The hypothesis of this theorem are chosen for clarity; it is known they are far from optimal. Woodin has improved the CH result.) To transfer some of these properties to bigger cardinals it is possible to show, for example:

Theorem Suppose for all regular λ, and all natural numbers n, there is a normal fine ideal on $[\lambda^{+n}]^{\aleph_{n+1}}$ and the Continuum Hypothesis holds, then the Generalized Continuum Hypothesis holds.

The exact relationship of these ideal properties to large cardinals still retains many challenging open problems. However, it is possible to show that they form a hierarchy of consistency strength that sometimes directly meshes with the consistency strength of large cardinals. Typical theorems include:

Theorem Suppose that there is a normal, fine ideal I on $[\omega_3]^{\omega_2}$ and a set $X \in \breve{I}$ and $Y, Z \in V$ such that whenever $j : V \to M$ is the generic embedding of V into its ultrapower, we know $j``\omega_3 = Y$ and $j(X) = Z$, then there is an inner model with a huge cardinal

The hypothesis of the this theorem can be sharpened to a form of Chang's Conjecture with an additional combinatorial property. In the other direction the best known result for $\aleph_1$-dense ideals is:

Theorem Suppose that there is a pair of almost huge cardinals κ_0, κ_1 and an almost huge embedding j with $j(\kappa_0) = \kappa_1$. Then it is consistent that there is a uniform, countably complete $\aleph_1$-dense ideal on $\aleph_2$.

Finally, the talk will discuss the role played by generic large cardinal embeddings in showing reflection properties and their relationship to various squares. An important example of this is the following joint theorem of the author, Cummings and Magidor:

Theorem Suppose it is consistent with ZFC that there are infinitely many supercompact cardinals. Then it is consistent with ZFC that every stationary subset of $\aleph_{\omega+1}$ reflects and the property $\Box_{\aleph_{\omega+1},\omega}$ holds.

1991 MS Classification: 3, 4
Hjorth, Greg, University of California at Los Angeles, USA
When is an equivalence relation classifiable?

In the works of Mackey [7], Glimm [2], and Effros [1], one finds the following notion of classifiability: An equivalence relation E is *classifiable* if the quotient object, X/E, consisting of all the equivalence classes, has a *nice* Borel structure.
More precisely, given an equivalence relation E on a standard Borel space X, let us follow terminology from analysis and say that E is *smooth* if the set of all equivalence classes $\{[x]_E : x \in X\}$ equipped with the σ-algebra generated by the projections of the E-invariant Borel sets may be realized as a subspace of a standard Borel space – or, equivalently, E is reducible to the equality relation on the reals in the sense that there is a Borel $f : X \to \mathbb{R}$ such that $x_1 E x_2$ if and only $f(x_1) = f(x_2)$.
This is the paradigm of classifiability that one finds implicit in much of the theory of infinite dimensional group representations and Ornstein's celebrated work in [8] on entropy in the context of Bernoulli shifts. It has been subjected to intense investigation following papers of Glimm, Effros, and then much later Harrington, Kechris, and Louveau, and is to a large degree well understood.
Yet despite the seductive charm of this wonderfully simple idea, it also true that this notion of classification does *not* match a great variety of examples from mathematical practise. For instance, [6] presents the Ulm invariants for countable abelian p-groups as a scheme of classification, and yet the equivalence relation of isomorphism on countable abelian p-groups is not smooth in the above sense. Similarly the analysis from [3] of discrete spectrum measure preserving transformations provides complete invariants for an equivalence relation that is not reducible to the equality relation on the reals. Finally, close to the very concerns that gave rise to the original definition, one finds in the spectral theorem – and its classification of unitary operators by measures considered up to absolute continuity – a classification theorem for a quotient object that *does not* admit such a "nice Borel structure".
In my talk I will describe some recent work in descriptive set theory which relates to a notion of classification far more generous than the smooth/non-smooth dichotomy. In the last few years it has been possible to develop an understanding of when an equivalence relation admits classification by countable discrete structures (considered up to isomorphism) – or, equivalently, is reducible to the equality relation on the hereditarily countable sets.

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1991 MS Classification: 03C45
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Meager forking and m-independence

Assume $\mathfrak{C}$ is a large saturated model of a countable complete first-order theory T and A, B are countable subsets of $\mathfrak{C}$. For any $a \in \mathfrak{C}$ we define the type $tp(a/A)$ of a over A as the set of all A-definable subsets of $\mathfrak{C}$ containing a. $S(A)$ denotes the topological space of all such types over A. Basic open sets are of the form $\{p \in S(A) : X \in p\}$, where X is an A-definable subset of $\mathfrak{C}$. $S(A)$ is homeomorphic to a closed subset of Cantor set.
We say that a is algebraic over A if a belongs to a finite A-definable subset of $\mathfrak{C}$. We define the algebraic closure $acl(A)$ of A as the set of all elements of $\mathfrak{C}$ algebraic over A. $acl(A)$ is countable. For $p \in S(B)$, $p(\mathfrak{C}) = \{a \in \mathfrak{C} : tp(a/B) = p\}$ and we define the trace of p over A as the set $Tr_A(p) = \{tp(a/acl(A)) : a \in p(\mathfrak{C})\}$, a closed subset of $S(acl(A))$. Similarly we define $Tr_A(X)$ for any subset X of $\mathfrak{C}$.
Assuming T is stable, Shelah defined on $\mathfrak{C}$ the notion of forking independence, generalizing those of linear independence in a vector space and algebraic independence in an algebraically closed field. He used this notion in his classification theory, dealing mainly with uncountable structures. The geometric aspects of forking independence were crucial for the work of Zilber, Hrushovski and others on the fine structure of models of stable theories.
One of the main open problems of model theory is Vaught's conjecture, saying that if T has $< 2^{\aleph_0}$ (that is, *few*) countable models, then T has countably many of them. Shelah proved this conjecture for ω-stable theories (1984), and Buechler for superstable theories of finite rank (1993).

For a proof of Vaught's conjecture it is important to investigate the ways, in which a type $p \in S(A)$ may be non-isolated. This is closely related to the size of $Tr_A(p)$. When T is superstable, then it may happen that for some finite sets $A, B \subset \mathfrak{C}$, for some non-isolated type $p \in S(B)$, the set $Tr_A(p)$ is infinite. In order to approach Vaught's conjecture for arbitrary superstable theories it seems necessary to investigate such traces of types. In this lecture I will speak of this research.

Namely, I introduced meager forking, m-independence and other notions, designed to measure traces of types. Using these notions I proved that if T is superstable with few countable models, then traces of types can not be arbitrary closed subsets of the space of types, but rather are subject to some strong restrictions. For instance, they are traces of some definable subsets of $\mathfrak{C}$. Developing the theory of m-independence enabled me to prove Vaught's conjecture for some superstable theories of infinite rank.

These notions may be useful not only for Vaught's conjecture, but also for description of arbitrary models of stable theories. I proved that in a small stable theory m-independence is the strongest notion of independence on finite tuples of elements of $\mathfrak{C}$, which has some natural properties (like e. g. symmetry, transitivity) and strengthens forking independence. Then I developed the theory of m-independence parallel to the theory of forking independence.

In particular, the notions of meager forking, meager types, $\mathcal{M}$-rank, $*$-finite and $*$-algebraic tuples, m-stable and m-normal theories correspond to forking, regular types, U-rank, imaginary elements, superstable and 1-based theories in the theory of forking independence.

I proved that if T is superstable with few countable models, then T is m-stable, m-normal and the $\mathcal{M}$-rank of any (trace of a) complete type is finite and not larger than the U-rank of this type.

We say that a type $p \in S(A)$ is regular if forking dependence is a combinatorial pregeometry on $p(\mathfrak{C})$. Moreover p is called locally modular if this pregeometry is locally modular. The existence of locally modular types is fundamental for geometric model theory. It turns out that in a small superstable theory the phenomenon of local modularity of some regular types is closely related to the topological structure of the space of types, via the notions of meager forking and meager type.

A surprising consequence of m-normality is that for any group G interpretable in a superstable theory T with few countable models, the quotient group G/G^0 is abelian-by-finite (G^0 is the connected component of G).

Introducing $*$-algebraic tuples allowed me also to extend the definition of m-independence to an arbitrary small theory (where forking independence is usually not applicable). This may allow us to apply (suitably modified) methods of geometric model theory in a new context.

1991 MS Classification: 04
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Basis Problems in Combinatorial Set Theory

Basis problems in combinatorial set theory are questions that ask whether a given class of structures is in some sense determined by a finite (or countable) list of its "critical" elements. Problems of this sort have some value even in cases where a given class of structures does not have much of an intrinsic interest in itself (for example, the class of uncountable graphs or the class of uncountable linear orderings) because positive answers to this kind of questions tend to be highly applicable on a much wider scale.
This will be an overview of some of the best known Basis problems in this area, many of which have a strong Ramsey-theoretic flavour which is related to, and in some sense explains, the wide applicability. We shall also try to explain a place from which many of the "critical" structures come from, the concept of a special Aronszajn tress, that has been a constant source of inspiration in this area for more than 60 years.

1991 MS Classification: 3, 26, 51
Wilkie, Alex J., Oxford University, UK

O-minimality

The study of 0-minimal structures, also known as geometric categories, is a branch of model theory but it has potential geometrical interest. The aim of the pure theory, which I shall review briefly, is to generalise many of hereditary, geometrical and finiteness properties enjoyed by the category of subanalytic sets and maps (eg Whitney stratification, triangulation, Lojasevic inequality) to a general axiomatic setting. The second half of the talk will focus on examples. In particular I shall indicate why (real) functions satisfying Pfaffian differential equations live in O-minimal structures.

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2. Algebra

1991 MS Classification: 14L15, 17B50
Friedlander, Eric M., Northwestern University, USA
Geometry of Infinitesimal Group Schemes

The cohomology and representation theory of infinitesimal group schemes over a field of positive characteristic have fascinating relationships and geometric structures. Earlier studies, undertaken jointly with Brian Parshall, primarily concerned infinitesimal group schemes of height 1 (which have a more familiar interpretation as finite dimensional restricted Lie algebras). Many of the results of E. Friedlander and B. Parshall for restricted Lie algebras are modeled on analogous results for finite groups, especially algebro-geometric structures introduced by Daniel Quillen and Jon Carlson. More recent joint work with Andrei Suslin and Christopher Bendel has achieved a dramatic extension of these results to infinitesimal group schemes of arbitrary height and provided sharper results in the height 1 case.
Let k be a field of characteristic $p > 0$. A finite group scheme G over k is a finite dimensional cocommutative Hopf algebra A_G with (commutative) dual Hopf algebra $k[G]$. Such a finite group scheme G is said to be infinitesimal if $k[G]$ is a local algebra. The height of an infinitesimal group scheme G is the minimum integer r such that the maximal ideal of $k[G]$ is generated by elements whose p^r-power equals 0. If G is an infinitesimal group scheme of height 1, then A_G equals the restricted enveloping algebra of $\mathcal{G} = \mathcal{L}\rangle\rceil(\mathcal{G})$.
Infinitesimal group schemes arise in the study of the (rational) representation theory of affine algebraic groups K over k. The r-th Frobenius kernel of K, $K_{(r)}$, is defined as the scheme-theoretic kernel of the the r-th iterate of Frobenius $\phi^r : K \to K^{(r)}$ (where $K^{(r)}$ is the Frobenius-twisted form of K, isomorphic to K whenever K is defined over the finite field of p^r elements). So defined, $K_{(r)}$ is an infinitesimal group scheme of height r. Unlike the situation in characteristic 0, the representation theory of the algebraic group K is not faithfully reflected in the representation theory of $Lie(K)$; on the other hand, the study of all the "infinitesimal neighborhoods of the identity" of K (i. e., the study of the representation theory of all $K_{(r)}$) often does faithfully reflect the representation theory of K.
(Rational) representations of an affine group scheme G are defined to be comodules for the Hopf algebra $k[G]$; if G is a finite group scheme, then such representations are equivalent to modules for A_G. One central aspect of the representation theory of G is the study of the Ext-groups $Ext^i_G(M, N)$ where M, N are rational G-modules. In particular, the cohomology algebra $H^*(G, k) = Ext^*_G(k, k)$ contains significant information about G and its representations.
The geometric nature of the representation theory of height 1 infinitesimal group schemes is reflected in the following theorem of E. Friedlander and B. Parshall, with bound on p improved by Henning Andersen and Jens Jantzen. If K is a semi-simple algebraic group over k with Coxeter number of each factor less than p, then the cohomology algebra $H^*(K_{(1)}, k)$ is

concentrated in even dimensions and equals the coordinate algebra of the variety of nilpotent matrices in $Lie(K)$ [1], [2].
A key theorem of E. Friedlander and A. Suslin is the following: if G is a finite group scheme over k, then $H^*(G,k)$ is finitely generated [5]. If G is either discrete (i. e., equivalent to the group algebra of a finite group) or infinitesimal of height 1, then finite generation is well known; the proof of this result even for height 2 infinitesimal groups schemes has required new techniques and considerable effort. For G an infinitesimal group scheme of height r, E. Friedlander and A. Suslin in collaboration with C. Bendel identified $H^*(G,k)$ modulo p^r-powers. Stated geometrically, the spectrum of $H^*(G,k)$ is the variety of all infinitesimal 1-parameter subgroups of G (i. e., morphisms $G_{a(r)} \to G$) [6], [7]. In the special case $r=1$, this is a sharper form of a theorem of J. Jantzen.
Following work of Jon Carlson, George Avrunin, and Leonard Scott in the context of representation of finite groups, E. Friedlander and B. Parshall (using J. Jantzen's theorem mentioned above) proved the following result. Let G be an infinitesimal group scheme of height 1 and let M be a finite dimensional (rational) G-module. Then the variety of the annihilator ideal inside the symmetric algebra on $\mathcal{G}^*$, $S^*(\mathcal{G}^*)$, for a natural action of $S^*(\mathcal{G}^*)$ via $H^{ev}(G,k)$ on $Ext^*(M,M)$ can be identified with the subvariety of $\mathcal{G} = \mathcal{L}\rangle\rceil(\mathcal{G})$ consisting of elements $X \in \mathcal{G}$ whose p-th power is 0 and for which M restricted to $\langle X\rangle$ is injective, where $\langle X\rangle$ is the cyclic Lie algebra generated by X [3], [4]. Once again, A. Suslin, E. Friedlander, and C. Bendel sharpened and generalized this result to an infinitesimal group scheme G of arbitrary height r; a description of the variety of the annihilator ideal of the commutative algebra $H^{ev}(G,k)$ acting on $Ext^*(M,M)$ is given in terms of the restrictions of the rational G-module M to infinitesimal 1-parameter subgroups of G [7].
Infinitesimal 1-parameter subgroups of G play a role very analogous to (shifted) elementary abelian subgroups of a finite group. One challenging problem for future study is to develop methods which apply to all finite group schemes and which specialize to give known results for finite groups and infinitesimal group schemes. Another less well defined challenge is to usefully employ more subtle invariants of rational representations which reflect finer local information than merely the injectivity of the representation.

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1991 MS Classification: 20
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On the Burnside problem for groups of even exponent

Recall that the notorious Burnside problem (posed in 1902, see [B]) asks whether any finitely generated group that satisfies the law $x^n \equiv 1$ (n is a fixed positive integer called the *exponent* of G) is necessarily finite. This is proven only for exponents $n \leq 4$ and $n = 6$ (the case $n = 3$ is due to Burnside, $n = 4$ to Sanov, and $n = 6$ to Hall, see [MKS]). A negative solution to the Burnside problem (for odd exponents) was given by Novikov and Adian [NA] in 1968 who proved the existence of infinite m-generator groups with $m \geq 2$ of any odd exponent $n \geq 4381$ (a simpler geometric solution to this problem for odd $n > 10^{10}$ was later given by Ol'shanskii [Ol]; note that Adian [Ad] later improved on the original estimate to odd $n \geq 665$). We remark that attempts to approach the Burnside problem via finite groups gave rise to a restricted version of the Burnside problem [M] that asks whether there exists a number $f(m,n)$ so that the order of any finite m-generator group of exponent n is less than $f(m,n)$. The existence of such a bound $f(m,n)$ was proven for prime n by Kostrikin [K] and for $n = p^\ell$ with prime p by Zelmanov [Z1]-[Z2].

However, the Burnside problem for even exponents n without odd divisor ≥ 665, especially interesting for $n = 2^\ell \gg 1$, remained open. The principal difference between odd and even exponents in the Burnside problem can be illustrated by pointing out that, on the one hand, for every odd $n \gg 1$ there are infinite 2-generator groups of exponent n all of whose proper subgroups are cyclic and, on the other hand, any 2-group the orders of whose abelian (or finite) subgroups are bounded is itself finite.

A negative solution to the Burnside problem for even exponents $n \gg 1$ given in speaker's article [Iv] is based on the following inductive construction (which is analogous to Ol'shanskii's construction [Ol] for odd $n > 10^{10}$). Let F_m be a free group over an alphabet $\mathcal{A} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$, $m > 1$, $n \geq 2^{48}$ and n be divisible by 2^9 (from now on we impose these restrictions on m and n unless otherwise stated; note this estimate $n \geq 2^{48}$ has been improved on by Lysenok [L] to $n \geq 2^{13}$). By induction on i, let $B(m,n,0) = F_m$ and, assuming that the group $B(m,n,i-1)$ with $i \geq 1$ is already constructed as a quotient group of F_m, define A_i to be a shortest element of F_m (if any) the order of whose image in $B(m,n,i-1)$ (under the natural epimorphism $\psi_{i-1} : F_m \to B(m,n,i-1)$) is infinite. Then $B(m,n,i)$ is constructed as a quotient group of $B(m,n,i-1)$ by the normal closure of $\psi_{i-1}(A_i^n)$. Clearly, $B(m,n,i)$ has a presentation of the form $B(m,n,i) = \langle\, a_1, \dots, a_m \parallel A_1^n, \dots, A_{i-1}^n, A_i^n \,\rangle$, where $A_1^n, \dots, A_{i-1}^n, A_i^n$ are the defining relators of $B(m,n,i)$. It is proven in [Iv] that for every i the word A_i does exist. Furthermore, it is shown that the direct limit $B(m,n,\infty)$ of the groups $B(m,n,i)$ as $i \to \infty$ (obtained by imposing on F_m of relators A_i^n for all $i = 1, 2, \dots$) is exactly the free m-generator Burnside group $B(m,n)$ of exponent n ($B(m,n)$ is the quotient group F_n/F_m^n, where F_m^n is the subgroup generated by all nth powers of elements of F_m). The infiniteness of the group $B(m,n)$ already follows from the existence of the word A_i for every $i \geq 1$, since, otherwise, $B(m,n)$ could be given by finitely many relators and so A_i would fail to exist for sufficiently large i. It is also shown that the word and conjugacy problems for the group $B(m,n)$ are solvable (for odd $n \geq 665$ it is known due to Novikov and Adian [Ad]). As a matter of fact, these decision problems are effectively reduced to the word problem for groups $B(m,n,i)$ and it is shown that each $B(m,n,i)$ satisfies a linear isoperimetric inequality and hence $B(m,n,i)$ is a hyperbolic group in the sense of Gromov [G].

It is worth noting that the structure of finite subgroups of the groups $B(m,n,i)$, $B(m,n)$ is very complex when the exponent n is even and, in fact, finite subgroups of $B(m,n,i)$, $B(m,n)$ turn out to be so important in proofs of [Iv] that at least a third of article [Iv] is an investigation of their various properties and another third is a preparation of necessary techniques to conduct this investigation. The central result related to finite subgroups of the groups $B(m,n,i)$, $B(m,n)$ is the following: Let $n = n_1 n_2$, where n_1 is the maximal odd divisor of n. Then any finite subgroup G of $B(m,n,i)$, $B(m,n)$ is isomorphic to a subgroup of the direct product $D(2n_1) \times D(2n_2)^\ell$ for some ℓ, where $D(2k)$ denotes a dihedral group of order $2k$ (for odd $n \geq 665$ it is known due to Novikov and Adian that all finite and abelian subgroups of $B(m,n)$ are cyclic [Ad]). The natural question on whether this description is complete (i. e. every subgroup of $D(2n_1) \times D(2n_2)^\ell$ can actually be found in $B(m,n)$) is solved in the affirmative in a joint work with Ol'shanskii [IvOl1] by proving the following theorem: Let G be a finite 2-subgroup of $B(m,n)$. Then the centralizer $C_{B(m,n)}(G)$ of G in $B(m,n)$ contains a subgroup B isomorphic to a free Burnside group $B(\infty,n)$ of infinite countable rank such that $G \cap B = \{1\}$, whence $\langle G,\ B\rangle = G \times B$. (Since $B(\infty,n)$ obviously contains subgroups isomorphic to both $D(2n_1)$ and $D(2n_2)$ an embedding of $D(2n_1) \times D(2n_2)^\ell$ in $B(m,n)$ becomes trivial.) Among other results on subgroups of $B(m,n)$ proven in [IvOl1] are the following: The centralizer $C_{B(m,n)}(S)$ of a subgroup S is infinite if and only if S is a locally finite 2-group. Any infinite locally finite subgroup

L is contained in a unique maximal locally finite subgroup while any finite 2-subgroup is contained in continuously many pairwise nonisomorphic maximal locally finite subgroups. A clear complete description of infinite (maximal) locally finite subgroups of $B(m,n)$ is also obtained.

The machinery developed in [Iv] for solving the Burnside problem for even exponents has turned out to be useful for other problems. For example, Gromov [G] conjectured that for every nonelementary (= non-cyclic-by-finite) hyperbolic group G there exists an $n = n(G)$ such that the quotient G/G^n is infinite. This Gromov conjecture (along with other results on such quotients G/G^n) is proven in [IvOl2] by carrying out the machinery of [Iv] to nonelementary hyperbolic groups.

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1991 MS Classification:
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Simple Groups in Computational Group Theory

Given an arbitrary subset S of the symmetric group S_n, can one efficiently find properties of $G = \langle S \rangle$, such as $|G|$, the derived series, a composition series, Sylow subgroups ... ? I will briefly survey recent approaches to such questions, emphasizing the use of simple groups and of probabilistic methods for devising algorithms.

1991 MS Classification: 20G
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Spetses

Weyl groups are finite groups acting as a reflection group on a rational vector space. Starting with the classification of finite-dimensional semisimple complex Lie algebras in terms of their Weyl groups, these rational reflection groups made their appearance as the skeleton of many more algebraic objects, like algebraic groups, finite groups of Lie type, quantum groups, Iwahori-Hecke algebras, braid groups, ... governing much of the structure and of the properties of these objects.
By extension of base field, Weyl groups are in particular finite complex reflection groups. In this talk we report on recent results and conjectures about analogues of some of the above structures with 'Weyl group' a complex reflection group. It was first noted by Springer that non-real reflection groups naturally appear inside Weyl groups as what is now called relative Weyl groups, that is, normalizers modulo centralizers of subspaces. The importance of this construction was revealed in the work of Broué, Michel and the author on the ℓ-blocks of characters of finite groups of Lie type. There it was shown that the unipotent characters inside an ℓ-block are parametrized by the irreducible characters of a relative Weyl group, which in general is a non-real reflection group. A possible interpretation of this result was subsequently proposed by Broué and the author in terms of the so-called cyclotomic Hecke algebra attached to a finite complex reflection group. This generalizes the usual Iwahori-Hecke algebra of real reflection groups.
The results of Lusztig show that the set of unipotent characters of a finite group of Lie type only depends on the Weyl group of the associated algebraic group, together with the action of the Frobenius endomorphism on it. Moreover, this set carries a rich combinatorial structure. In the course of this classification Lusztig observed that similar sets can formally also be attached to those finite real reflection groups which are not Weyl groups, just as if there existed an algebraic group whose Weyl group is non-crystallographic. On a conference on the Greek island Spetses it was first proposed by Michel Broué that a similar construction might also exist for non-real reflection groups. This was shown to be true for all but finitely

many types of irreducible reflection groups by constructing sets of 'unipotent characters', Fourier transform matrices, and eigenvalues of Frobenius, which satisfy all the combinatorial properties of unipotent characters of actual finite groups of Lie type.
The central object in this construction is the cyclotomic Hecke algebra of a complex reflection group W. This can be defined by generators and relations, as in the case of real reflection groups. Broué, Rouquier and the author gave a more intrinsic construction as a quotient of the braid group of W, i. e., of the fundamental group of the space of regular orbits of W on its natural vector space.
It is tempting to speculate about an underlying algebraic structure giving rise to the unipotent characters attached to complex reflection groups. We would like to call such a conjectural object 'spets', because it was first dreamed of in Spetses and should possess such 'spetsial' properties. Until now, such an object has not been found, but a lot of intriguing evidence for its existence has been given.

1991 MS Classification: 14, 12
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Birational automorphisms of algebraic varieties

A *Fano fibration* is a projective algebraic variety V, equipped with a surjective morphism $\pi\colon V \to S$, the generic fiber F_η of which is a Fano variety over the (non-closed) field $\mathbf{C}(S)$, i. e. the anticanonical class $(-K_{F_\eta})$ is ample. Fano fibrations form the principal subject of birational geometry. A Fano fibration $\pi\colon V \to S$ is said to be *birationally rigid*, if for any birational map $\chi\colon V - - \to V'$ and any linear system $|D'|$ on V', free in codimension 1, there exists a fiber-wise birational automorphism $\chi^* \in \operatorname{Bir} F_\eta \subset \operatorname{Bir} V$ such that for $|D| = (\chi \circ \chi^*)^{-1}_*|D'|$ (the proper inverse image of $|D'|$ on V') the condition $|aD' + bK_{V'}| = \emptyset$ implies that $|aD + bK_V| = \emptyset$ for any $a, b \in \mathbf{Z}_+$. This property is very strong: it implies, for instance, that any birational map $\chi\colon V - - \to V'$ onto another Fano fibration is fiber-wise. In particular, birationally rigid V is non-rational and its group of birational automorphisms $\operatorname{Bir} V$ reduces to the group of birational automorphisms of the generic fiber, $\operatorname{Bir} F_\eta$. A Fano fibration is said to be *birationally superrigid*, if in the definition above, we can always take $\chi^* = \mathrm{id}_V$. For a superrigid Fano variety V with $\operatorname{Pic} V \cong \mathbf{Z}$ we have $\operatorname{Bir} V = \operatorname{Aut} V$. The first example of a birationally rigid threefold was given by V. A. Iskovskikh and Yu. I. Manin (smooth quartics in $\mathbf{P}^4$; 1971).
Recently, new series of birationally rigid Fano fibrations were discovered. Let $\pi\colon V \to \mathbf{P}^1$ be a smooth threefold, fibered into del Pezzo surfaces of degree $d = 1, 2$ or 3, $\operatorname{Pic} V = \mathbf{Z}K_V \oplus \mathbf{Z}F$, $F = \pi^{-1}(t)$ is a fiber, satisfying the following condition: $MK_V^2 - f$ is a non-effective 1-class for any $M \in \mathbf{Z}$, where $f \in A^2(V)$ is the class of a line in F.
Theorem 1. *For $d = 1$ the Fano fibration $\pi\colon V \to \mathbf{P}^1$ is birationally superrigid, for $d = 2, 3$ it is rigid.*
Let $V = V_d \subset \mathbf{P}^d$, $d \geq 5$, be a smooth hypersurface of degree d, satisfying

the following condition: for any point $x \in V$ and any affine coordinate system with the origin at x the hypersurface V is given by the equation $f = f_1 + \cdots + f_d$, where the set of homogeneous polynomials $(f_1, \ldots, f_{d-1})$ make a regular sequence in $\mathcal{O}_{x,\mathbf{P}^d}$ (hypersurfaces satisfying this property form a non-empty Zariski open subset of the space of hypersurfaces of the given degree d).

Theorem 2. *The hypersurface V is birationally superrigid.*

Let $V = V_d$ be a sufficiently general quasi-smooth Fano hypersurface in the weighted projective space $\mathbf{P}^4(1, a_1, a_2, a_3, a_4)$, where $d = a_1 + a_2 + a_3 + a_4$. There are 95 families of these hypersurfaces.

Theorem 3. *V is birationally rigid.*

The last result is joint work with A. Corti and M. Reid.

Other examples of birationally rigid Fano fibrations will be discussed, together with some open problems and natural conjectures.

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1991 MS Classification: 16, 18

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Tilting Theory and Quasitilted algebras

Tilting theory is a central topic in the representation theory of artin algebras, with origins in work of Bernstein - Gelfand - Ponomarev in the early seventies. There has been extensive interaction with various research directions in representation theory, as well as in other branches of algebra. In this lecture we shall survey the historical development of tilting theory, and in particular we discuss quasitilted algebras, a recent outgrowth of tilting theory.

Let Λ be (for simplicity) a finite dimensional algebra over a field k, and let T be a finitely-generated (left) Λ-module. In general we cannot say much about the connection between the module theories for Λ and for the associated endomorphism algebra $\Gamma := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. We would like to impose conditions on T which are strong enough so that information can be transferred from Λ to Γ, and at the same time weak enough so that Γ need not be too closely related to Λ. A tilting module provides some balance between these two requirements.

An object T in the category of finitely generated Λ-modules $\mathrm{mod}\,\Lambda$ is a (1-)tilting module if T is of projective dimension at most one, $\mathrm{Ext}^1_\Lambda(T, T) = 0$, and there is an exact sequence of Λ-modules $0 \to \Lambda \to T_0 \to T_1 \to 0$, where T_0 and T_1 are summands of a finite direct sum of copies of T. Associated with a (1-)tilting module T is a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\,\Lambda$, where $\mathcal{T} = \{C \mid \mathrm{Ext}^1_\Lambda(T, C) = 0\}$, and there are equivalences of $\mathcal{T}$ and $\mathcal{F}$ with subcategories of $\operatorname{mod}\Gamma$. Especially important is the case when Λ is hereditary, in which case $\operatorname{mod}\Lambda$ has been thoroughly investigated. The corresponding algebras Γ are then called *tilted* algebras. They have played an important role in the representation theory of algebras since many general questions can be reduced to problems about tilted algebras, which are well understood because of their close connection with hereditary algebras. This level of generality was reached in work of Brenner - Butler and Happel - Ringel in the late seventies. Their results generalized the module theoretic interpretation, given in joint work with Auslander and Platzeck, of the reflection functors of Bernstein - Gelfand - Ponomarev.

We discuss three main lines of further developments, where the first two go via a generalization to (n-)tilting modules (where the projective dimension of T is n). One direction is concerned with the correspondence, established in work with Auslander, between (n-)tilting modules and a certain type of subcategories of $\operatorname{mod}\Lambda$, given by

$$T \mapsto \mathcal{T} = \{C \mid \operatorname{Ext}^1_\Lambda(T, C) = 0 \text{ for all } i > 0\}$$

Here there are applications to algebraic groups, via work of Ringel on quasi-hereditary algebras, and connections with dualizing modules in commutative ring theory. The second line of development goes via Happel's discovery of the connection with derived categories. In the third direction we developed, with Happel and Smalø, a theory of tilting with respect to torsion pairs which are not necessarily induced by a (1-)tilting module. Here we were naturally led to deal with abelian categories rather than just module categories.

As a by-product of our general theory we introduced a new class of algebras, the *quasitilted* algebras, as the endomorphism algebras of what we call tilting objects in a hereditary (i. e., $\operatorname{Ext}^2(-,-) = 0$) k-category $\mathcal{H}$ (with finite dimensional homomorphism and extension spaces). Even though it is not clear what the categories $\mathcal{H}$ may be, quasitilted algebras can be described as the algebras of global dimension at most two such that each indecomposable module is of projective or injective dimension at most one. Two important classes of quasitilted algebras are the tilted algebras and the canonical algebras of Ringel. The categories $\mathcal{H} := \operatorname{mod} H$, where H is a finite dimensional hereditary algebra, give rise to the former class, and the categories $\mathcal{H}$ of coherent sheaves on weighted projective lines, introduced by Geigle - Lenzing, give rise to the latter class. It is an important problem to classify all possible categories $\mathcal{H}$. It is conjectured that the two types of $\mathcal{H}$ mentioned above exhaust all possibilities up to derived equivalence. In work with Happel we have shown that this is true if $\mathcal{H}$ is connected and contains either a directing object X (that is, X does not lie on a cycle of non-zero non-isomorphisms between indecomposable modules) or a non-zero object of finite length (in the case k is algebraically closed).

1991 MS Classification: 18, 20

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The abelian defect group conjecture

Let G be a finite group, k an algebraically closed field of prime characteristic p, and B a block of the group algebra kG with positive defect (i. e., a block that is not just a matrix ring over k). Over 10 years ago, Alperin formulated his famous 'Weight Conjecture', which states that the number of isomorphism classes of B-modules can be calculated 'locally' (i. e., in terms of the representation theory of normalizers of non-trivial p-subgroups of G) using an explicit formula. This conjecture subsumed several earlier conjectures on the number of simple modules for particular classes of blocks. It is still one of the major unsolved problems in the representation theory of general finite groups.

In the case of a block with abelian defect group, Alperin's Weight Conjecture simply claims that B and its Brauer correspondent b have the same number of simple modules (up to isomorphism). For this case, Broué formulated in 1989 a much more precise conjecture on the relationship between the two blocks. His Abelian Defect Group Conjecture claims that the derived categories $D^b(\mathrm{mod} - B)$ and $D^b(\mathrm{mod} - b)$ of the module categories of B and b are equivalent as triangulated categories. Since the number of simple B-modules is the rank of the Grothendieck group of $D^b(\mathrm{mod} - B)$, and hence an invariant of the derived category, Broué's conjecture implies that of Alperin for blocks with abelian defect group, but also says far more.

Character-theoretic and other circumstantial evidence for the Abelian Defect Group Conjecture accumulated rapidly, but verification of the conjecture for particular groups proved very difficult except in very simple cases. However, in the last two years there has been rapid progress in verifying special cases of the conjecture.

In my lecture, I will survey current knowledge on the Abelian Defect Group Conjecture, its consequences, and related questions, and describe some of the more recent work on special cases of the conjecture.

Reference: M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Astérisque* **181–182** (1990), 61–92.

1991 MS Classification: 20P05, 20D06, 20B15

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Simple groups, permutation groups, and probability

In recent years probabilistic methods have proved useful in the solution of several difficult problems concerning finite groups. In some cases the probabilistic nature of the problem is apparent from its very formulation; but in other cases the use of probability, or counting, seems surprising, and cannot be anticipated by the nature of the problem.

In some branches of mathematics it is quite common to use probabilistic methods in order to prove existence theorems; however, this is less common in group theory; indeed, it is our hope that the probabilistic approach will have sufficiently many group-theoretic applications so as to become a standard tool in group theory.

The roots of the subject lie in a series of papers by Erdős and Turán in which they study the properties of random permutations. In 1969 Dixon used the Erdős-Turán theory to settle an old conjecture of Netto, proving that two randomly chosen elements of the alternating group A_n generate A_n with probability $\to 1$ as $n \to \infty$. He proceeded to conjecture that a similar result holds for arbitrary (nonabelian) finite simple groups. Our first result settles Dixon's conjecture in the affirmative.

Theorem 1 (Liebeck-Shalev, 1995): *Two random elements of a finite simple group G generate G with probability $\to 1$ as $|G| \to \infty$.*

Our work follows important results of Kantor and Lubotzky. Like other results presented below, it applies the Classification of Finite Simple Groups. It is interesting that Theorem 1 can be applied in the study of residual properties of free groups.

We also obtain results on random generation by special pairs of elements.

Theorem 2: (i) (Liebeck-Shalev, 1996) *A finite simple group is almost surely generated by a random involution and a random additional element.*
(ii) (Guralnick-Liebeck-Saxl-Shalev, 1998[1]) *A finite simple group is almost surely generated by a random element and a random conjugate of it.*

Part (i) above settles a conjecture of Kantor and Lubotzky (1990), and part (ii) solves a question of G. Robinson.

We now turn to some recent applications of the probabilistic approach. The first concerns the longstanding problem of finding the finite simple quotients of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, namely the finite simple groups that can be generated by two elements of orders 2 and 3 respectively. Partial answers were provided throughout this century using explicit constructions. Focusing on classical groups we show that (with some exceptions) almost all pairs of elements of orders 2 and 3 are generating pairs. As a result we deduce the following.

Theorem 3 (Liebeck-Shalev, 1996): *Except* $\mathrm{PSp}_4(2^k)$, $\mathrm{PSp}_4(3^k)$ *and finitely many other groups, all finite simple classical groups can be obtained as a quotient of the modular group.*

The groups $\mathrm{PSp}_4(2^k)$ and $\mathrm{PSp}_4(3^k)$ are genuine exceptions.

Let us now turn to applications which involve permutation groups. Recall that a base for a permutation group is a subset of the permutation domain whose pointwise stabilizer is trivial. Bases play an important role in computational group theory and in estimating orders of primitive permutation groups. In 1981 Babai conjectured that primitive permutation groups with no large alternating sections have small bases. We settle this as follows.

Theorem 4 (Gluck-Seress-Shalev, 1998): *A primitive permutation group not involving A_d has a base of size bounded in terms of d.*

[1] work still in preprint form

This provides a structural explanation for the celebrated Babai-Cameron-Pálfy theorem, stating that the order of the groups above is polynomial in n. The core of the proof of Theorem 4 is probabilistic.
We also settle another base conjecture, posed by Cameron in 1990.

Theorem 5 (Liebeck-Shalev, 1998[1]) *There is an absolute constant c such that if G is an almost simple primitive permutation group, then G has a base of size at most c, with known exceptions.*

The exceptions are alternating or symmetric groups acting on subsets or partitions, and classical groups in subspace actions. We actually find c such that almost all c-tuples from the permutation domain form a base. The proof involves improvements of previous bounds on minimal degrees of permutation groups, which in turn have some other applications.

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3. Number Theory and Arithmetic Algebraic Geometry

1991 MS Classification: 14G20, 14F20, 11G25, 32P05, 32C37
Vladimir G. Berkovich
***P*-adic analytic spaces**

A valuation on a field k is a function $|\ |: k \to \mathbf{R}_+$ with the properties: $|x| = 0$ if and only if $x = 0$, $|xy| = |x| \cdot |y|$, and (the triangle axiom) $|x+y| \le |x| + |y|$. It is well known that either the completion of k with respect to a valuation is $\mathbf{R}$ or $\mathbf{C}$, the fields of real and of complex numbers, or the valuation on k satisfies the more strong (non-Archimedean) triangle axiom $|x+y| \le \max(|x|, |y|)$. The main (and the first discovered) example of a non-Archimedean valuation is the p-adic one on the field of rational numbers $\mathbf{Q}$. Here p is a prime number, and the p-adic valuation of a non-zero number of the form $p^n \frac{a}{b}$, where a and b are prime to p, is equal to $\frac{1}{p^n}$. The field of p-adic numbers $\mathbf{Q}_p$, the completion of $\mathbf{Q}$ with respect to the p-adic valuation, is of great importance in number theory. Is there geometry over $\mathbf{Q}_p$ as it is over $\mathbf{R}$ and $\mathbf{C}$? At the moment one is able to consider only analogs of complex analytic geometry since convergent power series make sense over $\mathbf{Q}_p$ as well, but analogs of functions of other types, e. g., smooth ones, are not yet known.

In comparison with $\mathbf{R}$ and $\mathbf{C}$, any field k complete with respect to a non-Archimedean valuation is totally disconnected and, therefore, the n-dimensional vector space k^n, considered with the functions which are locally convergent power series, is not an appropriate analog of the complex analytic space $\mathbf{C}^n$. In this connection, in rigid analytic geometry introduced by Tate in 1961 the analog of $\mathbf{C}^n$ is, roughly speaking, the space k^n provided with an extra structure. (More precisely, instead of k^n one considers the maximal spectrum of the ring of polynomials $k[T_1, \dots, T_n]$, and the extra structure is a Grothendieck topology on it.) The framework of rigid analytic geometry enables one to construct a full analog of the complex analytic theory of coherent sheaves and their cohomology, but does not allow a direct application of the intuitive idea of continuity and, in particular, of the homotopy and singular homology notions.

It turns out that there exist p-adic analytic spaces to which the above notions can be directly applied. They are in a sense a geometrization of rigid analytic spaces, their definition is much simpler than that of the latter, and it is in fact a natural generalization of that of complex analytic spaces. For example, the analog of $\mathbf{C}^n$ is the space $\mathbf{A}^n$ of the multiplicative semi-norms on the ring $k[T_1, \dots, T_n]$ that extend the valuation on k, and the analytic functions on it are local limits of rational functions. (The vector space k^n is naturally embedded in $\mathbf{A}^n$, and the topology of $\mathbf{A}^n$ induces the canonical topology on k^n.) Basic topological properties of the analytic spaces are as follows. Each point has a fundamental system of open neighborhoods which are locally compact, countable at infinity and arcwise connected, and the topological dimension of a paracompact analytic space is equal to its (algebraic) dimension.

Furthermore, recall that a formal scheme over the ring of integers k° of k has the closed fibre $\mathcal{X}_s$, which is a scheme of locally finite type over the residue field of k, and the generic fibre $\mathcal{X}_\eta$, which is an analytic space over k. It turns out that, if the canonical morphism from $\mathcal{X}$ to the formal spectrum of k° is a composition of semi-stable morphisms, then the homotopy type of $\mathcal{X}_\eta$ (and of any dense Zariski open subset of $\mathcal{X}_\eta$) coincides with that of the geometric realization of a simplicial set associated to $\mathcal{X}_s$. (For example, the analytification of a Zariski open subset of a proper scheme with good reduction is contractible, and the Tate elliptic curve is homotopy equivalent to the circle.) The above fact and results of J. de Jong on alterations are used to prove that any smooth analytic space over k is locally contractible. In particular, such a space has a universal covering which is a Galois one with the Galois group isomorphic to the fundamental group of the space and, if the space is paracompact , its cohomology groups with coefficients in a constant sheaf coincide with the singular cohomology groups. One also proves that, if X is the analytification of a proper scheme over k, then the cohomology groups $H^q(X, \mathbf{Z})$ are finitely generated, and there exists a finite separable extension k' of k such that for any non-Archimedean field K over k' one has $H^q(X\widehat{\otimes}k', \mathbf{Z}) \stackrel{\sim}{\to} H^q(X\widehat{\otimes}K, \mathbf{Z})$.
As it is clear from the above, the (topological) cohomology groups of analytic spaces over k don't play the same role as the similar cohomology groups of complex analytic spaces. The role of the latter is played by étale cohomology groups. Étale cohomology theory developed is parallel to that for schemes but, in comparison with the latter, is much simpler, and the reason for this is that the analytic spaces have good topology. Étale cohomology of the analytic spaces is used for the study of the vanishing cycles sheaves of schemes and for constructing smooth representations of p-adic reductive groups.

1991 MS Classification: 11
Colmez, Pierre, CNRS, École Normale Supérieure et Institut de Math. de Jussieu, Paris, France
(φ, Γ)-modules and Iwasawa theory of p-adic representations

The talk will try to report on the classification of p-adic represention of a local field in terms of (φ, Γ)-modules. This classification was introduced by Fontaine some ten years ago and brings a new perspective on local class field theory and its generalisations. It is still in progress but a number of interesting results (due to various people) have already been obtained especially in the applications to Iwasawa theory. Before describing a little bit more in detail what the content of the talk should be, let us introduce some notation.

Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. If K is a finite extension of $\mathbf{Q}_p$ and $n \in \mathbf{N}$, set $K_n = \mathbf{Q}_p(\mu_{p^n})$ and let K_∞ be the cyclotomic extension of K union of all the K_n. Let $\mathcal{G}_K$ be the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ and $\mathcal{H}_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$. Finally, set $\Gamma_K = \mathcal{G}_K/\mathcal{H}_K = \mathrm{Gal}(K_\infty/K)$.

The theory of (φ, Γ_K)-modules attaches to any p-adic representation V of $\mathcal{H}_K$ a module $D(V)$ over a ring of power series in one variable, endowed with an action of the Frobenius φ. The main point of the theory is that one can recover V from $D(V)$ which is a priori much simpler (Γ_K being in general procyclic, $D(V)$ is given by two operators (φ and a generator of Γ_K) satisfying a commutation relation). The theory of p-adic representations of $\mathcal{G}_K$ is thus equivalent to the study of (φ, Γ)-modules which naturally gives rise to three types of questions.

– Can one read the properties of the representation directly on its (φ, Γ)-module?

– Can one recover from $D(V)$ other classical invariants of V and use $D(V)$ to attach new invariants to V?

– Can one use (φ, Γ)-modules to construct interesting p-adic representations of $\mathcal{G}_K$?

Concerning the first question, the idea is that, the nicer the representation is, the nicer its (φ, Γ)-module must be. In this direction one can prove that representations of $\mathcal{G}_K$ are nicer than general representations of $\mathcal{H}_K$: they are *overconvergent*. If K is unramified over $\mathbf{Q}_p$, crystalline representations of $\mathcal{G}_K$ are as nice as possible: they are *of finite height*.

Concerning the second question, the most interesting result is a description of the Galois cohomology of V in terms of $D(V)$ from which one can recover Tate's local duality and describe the Iwasawa modules attached to V. One obtains in this way a vast generalisation of Coleman's isomorphism and, if the representation is assumed to be de Rham, of Coates-Wiles homomorphisms as well as an explicit reciprocity law relating these Coates-Wiles homomorphisms to Bloch-Kato exponential maps (these last results use the overconvergence of representations of $\mathcal{G}_K$). It is not unreasonable to expect that these results will share some new light on p-adic L-functions. One can also, using the infinitesimal action of Γ_K, attach to V an overconvergent differential module endowed with a Frobenius and use this module to recover the generalised Hodge-Tate decomposition of Sen and characterise de Rham representations.

Finally, one can construct (φ, Γ)-modules having good properties and recover the results of Fontaine and Laffaille on crystalline representations from this construction.

1991 MS Classification: 11
Duke, William, Rutgers University, USA
Bounds for arithmetic multiplicities

An important role in several areas of Number Theory is played by multiplicities, those of automorphic representations and of zeros of L-functions being prime examples. While one would ultimately like to understand such multiplicities exactly, in many cases we only have conjectures and, concerning estimates for sizes, only trivial ones. This is particularly true when the multiplicity cannot be effectively isolated using standard analytic tools such as the trace formula or the explicit formula for the zeros of L-functions. In some cases there has been much work constructing explicit examples and hence lower bounds for multiplicities. Upper bounds also present a serious challenge. The work of Phillips and Sarnak and later of Wolpert on the disappearance of cusp forms illustrates this in the case of the spectral multiplicity of congruence groups.

This lecture will describe some recent applications of techniques that allow one to give non-trivial upper bounds for multiplicities in certain arithmetic instances. These applications include in particular estimates for dimensions of spaces of cusp forms of weight one, sizes of bounded order subgroups of class groups, multiplicities of number fields with a given degree and discriminant and numbers of elliptic curves with various arithmetic conditions. The techniques share a common strategy, which combines spectral mean value theorems with special constraints only present in arithmetic cases. In the example of bounding the dimensions of spaces of cusp forms of weight one, this combination of ideas uses the Hilbert space structure of automorphic forms together with the rigidity of Galois representations, whose existence is guaranteed by the Deligne-Serre theorem.

In addition to giving particular applications we shall suggest conjectures and offer some general directions for further work on the estimation of arithmetic multiplicities.

1991 MS Classification: 11J (11J85, 11J89, 11J91) , 11F11

Gramain, François, Université de Saint-Étienne, France

Some results of algebraic independence

The most impressive result of the last years is due to Yu. V. Nesterenko in 1996 [N2] and concerns modular functions.

With Ramanujan's notation, let P, Q, R be the three first Eisenstein series $P(z) = 1 - 24\sum_{n\geq 1}\sigma_1(n)\, z^n$, $Q(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)\, z^n$, and $R(z) = 1 - 504\sum_{n\geq 1}\sigma_5(n)\, z^n$, where, for $n \in \mathbf{N}$, $\sigma_k(n) = \sum_{d|n} d^k$. The fundamental result is the following:

Theorem (Nesterenko). *If $q \in \mathbf{C}$ satisfies $0 < |q| < 1$, then the transcendence degree over $\mathbf{Q}$ of the field $\mathbf{Q}(q, P(q), Q(q), R(q))$ is at least 3.*

The links between Eisenstein series and elliptic curves or elliptic functions give, for example the following corollaries:

Corollary 1. *Let $\wp$ be an elliptic Weierstrass function with lattice of periods $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$, where $\omega_2/\omega_1 = \tau$. Let η_1 be the quasi-period associated to ω_1 and suppose that the invariants g_2 and g_3 are algebraic. Then the three numbers $e^{2i\pi\tau}$, ω_1/π and η_1/π are algebraically independent over $\mathbf{Q}$.*

Corollary 2. *With the same notation, in the case of complex multiplication, i. e. if τ is an algebraic number of degree 2, and if η is the quasi-period associated to the period $\omega \neq 0$ of $\wp$, then each set $\{e^{2i\pi\tau}, \omega, \eta\}$ and $\{e^{2i\pi\tau}, \omega, \pi\}$ consists of 3 algebraically independent (over $\mathbf{Q}$) numbers.*

In particular, for $\tau = i$, with the elliptic curve of equation $y^2 = 4x^3 - 4x$, one gets **the algebraic independence of the three numbers** π, e^{π} and $\Gamma(1/4)$.

This is the first result of algebraic independence about values of modular functions obtained by the use of modularity properties and not by elliptic properties. In the 70's, G. V. Chudnovsky obtained transcendence degree 2 by elliptic methods, for example the algebraic independence of ω_1/π and η_1/π under the hypothesis of corollary 1, and of ω and π under the hypothesis of corollary 2, in particular the algebraic independence of π and $\Gamma(1/4)$.

Another translation of the theorem in terms of Jacobi theta-functions gives, for example, the transcendence of $\sum_{n\geq 1} \ell^{-n^2}$ for any positive integer $\ell \geq 2$. This number was considered by Liouville (1844) who proved its irrationality.

Nesterenko's proof of the theorem has a classical form, using the construction (by the box principle) of an auxiliary function, a zero estimate and a criterion for algebraic independence (see the talk [N1] of Nesterenko at ICM90). Notice that this proof gives also quantitative results (measures of algebraic independence).

But, in the recent years, appeared new proofs of algebraic independence, which proceed from another approach. The new methods developped by M. Waldschmidt and D. Roy ([RW1] and [RW2]), M. Laurent and D. Roy ([LR]) and P. Philippon ([P1] and [P2], which present a systematic treatment for transcendence and algebraic independence) use different points of vue, but the three have two common points: they use interpolation determinants (introduced in the 80's by M. Laurent in transcendence proofs) instead of auxiliary functions and they use general properties of Diophantine approximation.

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1991 MS Classification: 11
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Determination of rational points using Dirichlet series

Study of points of finite order of elliptic curves over number fields translates into questions about rational points of modular curves. Many methods to study rational points of curves use an analysis of the group of rational points of the jacobian variety. In the case of modular curves, the results obtained in the direction of the Birch and Swinnerton-Dyer conjecture provide a particularly favorable background for such a method. In the talk, we will focus on abelian fields.

1991 MS Classification: 14F30
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The Intrinsic Hodge Theory of p-adic Hyperbolic Curves

A *hyperbolic curve* is an algebraic curve obtained by removing r points from a smooth, proper curve of genus g, where g and r are nonnegative integers such that $2g-2+r>0$. If X is a hyperbolic curve over the field of complex numbers $\mathbf{C}$, then X gives rise in a natural way to a Riemann surface $\mathcal{X}$. As one knows from complex analysis, the most fundamental fact concerning such a Riemann surface (due to Köbe) is that it may be *uniformized by the upper half-plane*, i. e., $\mathcal{X} \cong \mathfrak{H}/\Gamma$ where $\mathfrak{H} \stackrel{\text{def}}{=} \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$, and $\Gamma \cong \pi_1(\mathcal{X})$ (the topological fundamental group of $\mathcal{X}$) is a discontinuous group acting on $\mathfrak{H}$. Note that the action of Γ on $\mathfrak{H}$ defines a canonical representation $\rho_{\mathcal{X}} : \pi_1(\mathcal{X}) \to PSL_2(\mathbf{R}) = \text{Aut}_{\text{Holomorphic}}(\mathfrak{H})$. Now there are (at least) two ways to interpret the Köbe uniformization:

The Physical Approach: The Köbe uniformization provides a "physical, " geometric way to reconstruct the algebraic curve X from its topological fundamental group $\pi_1(\mathcal{X})$, equipped with some canonical *arithmetic* information, i. e. , $\rho_{\mathcal{X}}$.

The Modular Approach: The representation $\rho_{\mathcal{X}}$ gives rise to a special kind of $\mathbf{P}^1$-bundle with connection $(P \to X, \nabla_P)$ on the algebraic curve X called an *indigenous bundle*. More generally, the moduli space of indigenous bundles forms an algebraic torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ (in fact, defined over $\mathbf{Z}[\frac{1}{2}]$), called the *Schwarz torsor*, over the moduli stack of hyperbolic curves of type (g,r). Then Teichmüller theory (i. e., in particular, the uniformization theory of hyperbolic curves and their moduli) may be formulated as the study of the *canonical real analytic section of the Schwarz torsor* defined by the indigenous bundle arising from the Köbe uniformization.

The aim of the present talk is to discuss various generalizations to the p-adic case of these physical and modular aspects of Köbe's uniformization theorem.

Note that both the physical and modular approaches to the Köbe uniformization assert that there is a certain equivalence

$$\text{algebraic geometry} \Longleftrightarrow \text{topology endowed with an arithmetic structure}$$

That is, on the algebraic geometry side, we have the scheme (respectively, stack) given by the curve X itself in the physical approach (respectively, its moduli $\mathcal{M}_{g,r}$ in the modular approach), whereas on the "topology plus arithmetic structure" side, we have the theory of the canonical representation $\rho_{\mathcal{X}}$ of $\pi_1(\mathcal{X})$. This sort of equivalence is reminiscent of that given by classical or p-adic *Hodge theory* between the de Rham or Hodge cohomology of an algebraic variety (on the algebraic geometry side), and the singular or étale cohomology (equipped with Galois action) on the topology plus arithmetic side. In our case, however, instead of dealing with the cohomology of the curve, we are dealing with "the curve itself" and its moduli. It is for this reason that we refer to this theory as the *intrinsic Hodge theory* of the curve X.

Now let X be a hyperbolic curve over a finite extension K of the field $\mathbf{Q}_p$ of p-adic numbers. Let $\pi_1(X)$ be its *algebraic fundamental group*. Note that $\pi_1(X)$ comes equipped with a natural augmentation to Γ_K, the absolute Galois group of K. Then one has the following p-adic analogue (cf. [4]) of the *physical aspect* of the Köbe uniformization: *One may recover the scheme X (in the sense of the functor defined by X by considering dominant K-morphisms to X from K-smooth schemes S of finite type) as the functor of open, continuous Γ_K-compatible homomorphisms $\pi_1(S) \to \pi(X)$.* This sort of result was predicted by Grothendieck's *anabelian philosophy*, and builds on earlier work of H. Nakamura and A. Tamagawa. It may also be considered as the analogue for hyperbolic curves of the Tate conjecture on homomorphisms between abelian varieties.

Finally, we have a p-adic analogue of the *modular aspect* of the Köbe uniformization, i. e., of Teichmüller theory. Just as Teichmüller theory may be regarded as the theory of the canonical real analytic section of the Schwarz torsor, the p-adic analogue of Teichmüller theory concerns the issue of finding *canonical p-adic sections of the Schwarz torsor*. In characteristic p, this is achieved by considering the stack $\mathcal{N}_{g,r}$ of indigenous bundles with square nilpotent p-curvature. In our talk, we will discuss work on the fine structure of $\mathcal{N}_{g,r}$ (cf. [3]) in a "neighborhood of infinity" (i. e., near totally degenerate curves), as well as the theory of canonical p-adic liftings (cf. [1,2]) for various portions of $\mathcal{N}_{g,r}$.

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1991 MS Classification: 11 J 13, 11 J 68, 11 J 61, 11 D 61, 11 D 72
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The Absolute Subspace Theorem and Applications

1. The Subspace Theorem
The Subspace Theorem in Diophantine Approximation by W. M. Schmidt (1972) reads as follows:
Let $L_1, \dots, L_n$ *be linearly independent linear forms,* $L_i(\boldsymbol{X}) = \lambda_{i1}X_1 + \dots + \lambda_{in}X_n$ *with algebraic coefficients* λ_{ij}*. Suppose* $\varepsilon > 0$*. Consider the inequality*

$$|L_1(\boldsymbol{x}) \dots L_n(\boldsymbol{x})| < |\boldsymbol{x}|^{-\varepsilon} \tag{1}$$

in $\boldsymbol{x} \in \mathbb{Z}^n$*, where* $|\boldsymbol{x}| = (x_1^2 + \dots + x_n^2)^{1/2}$*. Then there exist finitely many proper linear subspaces* $T_1, \dots, T_t$ *of* $\mathbb{Q}^n$ *such that the set of solutions of* (1) *is contained in the union*

$$T_1 \cup \ldots \cup T_t. \tag{2}$$

Later on the Subspace Theorem has been extended to include a finite set S of p-adic valuations and also to deal with the situation where the variables are taken from the ring of integers of a number field K. It is in fact this generalization that is most interesting for applications.
In the last decade quite some activity was going on to give uniform upper bounds for the *number* t of subspaces $T_1, \dots T_t$ needed in (2), to cover the set of solutions of (1).
We will discuss in detail the most recent result in that context due to Evertse and Schlickewei, the *Quantitative Absolute Subspace Theorem* (to appear). Here the *variables are allowed to vary inside* $\overline{\mathbb{Q}}^n$ (the field of all algebraic numbers) *instead of* $\mathbb{Q}^n$ *or* K^n. Roughly speaking, if we disregard the "very small" solutions of (1), we obtain in this general situation an *upper bound for the number of subspaces in* (2) *that depends only upon the dimension* n *and upon* ε.
2. Applications
We will give several applications to Diophantine equations.
Let us mention here one of them: Let $\{u_n\}_{n\in\mathbb{Z}}$ be a linear recurrence sequence of order k with elements in a field of characteristic 0, i. e., a sequence satisfying a recurrence relation

$$u_{n+k} = a_{k-1}u_{n+k-1} + \dots + a_0 u_n \quad (n \in \mathbb{Z}).$$

A classical conjecture has been as follows:
If the sequence $\{u_n\}$ *is "nondegenerate" then the number of solutions* $n \in \mathbb{Z}$ *of the equation*

$$u_n = 0$$

is bounded in terms of the order k *of the sequence only.*
This conjecture now is a Theorem. It has been settled in the case when $\{u_n\}$ has only simple characteristic roots by Evertse, Schlickewei and W. M. Schmidt (to appear) and shortly afterwards by W. M. Schmidt (to appear) in general.

1991 MS Classification: 11, 14
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p-adic Hodge Theory in the semi-stable reduction case

I will give a brief survey of recent developments of p-adic Hodge theory in the semi-stable reduction case.
For a proper smooth algebraic variety X over the complex number field, Hodge theory relates the singular cohomology $H^*(X^{\mathrm{an}}, \mathbb{Q})$ of the analytic space X^{an} associated to X with the de Rham cohomology $H^*_{\mathrm{dR}}(X/\mathbb{C})$. p-adic Hodge theory is its analogue for a proper smooth algebraic variety X over a p-adic field (more precisely, a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field, e. g. a finite extension of $\mathbb{Q}_p$). The p-adic étale cohomology $H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, which is a $\mathbb{Q}_p$-vector space of finite dimension endowed with a continuous linear action of the absolute Galois group of K, plays the role of the singular cohomology. The de Rham cohomology $H^*_{\mathrm{dR}}(X/K)$ is still defined in the p-adic case and is a K-vector space of finite dimension.
The p-adic Hodge Theory dates back to the works of J. Tate and A. Grothendieck ([9], [8]) on an abelian variety with good reduction and the conjecture of Tate ([9]) (the Hodge-Tate conjecture=C_{HT}): an analogue of the Hodge decomposition. By establishing a functor connecting p-adic étale and de Rham cohomologies endowed with various additional structures (called a "mysterious functor" by A. Grothendieck) , J. -M. Fontaine (and U. Jannsen in the semi-stable case) formulated several refined conjectures: the de Rham conjecture=C_{dR}, the crystalline conjecture=C_{crys} and the semi-stable conjecture=C_{st} ([4], [5], [6]). These conjectures were studied by many mathematicians such as M. Raynaud, J. -M. Fontaine, G. Laffaille, W. Messing, S. Bloch, K. Kato, G. Faltings and O. Hyodo, and the conjectures C_{HT}, C_{dR} and C_{crys} were already solved without any restriction by G. Faltings in the end of 1980s. There were two approaches to the conjectures: the method of syntomic cohomology and p-adic vanishing cycles by Fontaine-Messing and Kato, and the method of almost étale extensions by Faltings. (Recently another method was found by W. Niziol). For C_{st}, we needed the assumption $\dim X < (p-1)/2$ (K. Kato by the first method) or that X has a model with at most a semi-stable curve singularity (G. Faltings by the second one) at that time.
In these four years, the restriction was removed first by myself by the Kato's method (for $p \geq 3$ in 1994 and for $p = 2$ in 1996) [10] and then by G. Faltings by his method (1996) [7] together with its generalization to non-constant coefficients and the relative case. Also a beautiful theory for p-torsion cohomologies in the semi-stable case was established by C. Breuil (1996 [1]) by overcoming the difficulty in treating the cohomologies degree-wise in a previous work of Faltings. Thanks to the alteration of de Jong [3], we can now derive easily all the other conjectures from C_{st} and further I proved, by the method of [2], that the p-adic étale cohomology of a proper

variety is always a potentially semi-stable representation even if it has a singularity. I will survey these developments in my talk.

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1991 MS Classification: 11, 14
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Small points, Heegner points, and Arakelov theory

In this talk, I will explain the recent applications of Arakelov theory to the Bogomolov conjecture on small points, and to the Gross-Zagier formula for Heegner points.

Theorem A (Bogomolov conjecture). *Let Y be a subvariety of an abelian variety A defined over $\bar{\mathbb{Q}}$. Let $h : A(\bar{\mathbb{Q}}) \to \mathbb{R}$ be a Néron-Tate height function (with respect to a symmetric projective embedding of A). Assume that Y is not a translation of an abelian subvariety by a torsion point. Then there is a positive number ϵ such that the set*

$$\{x \in Y(\bar{\mathbb{Q}}) : \quad h(x) < \epsilon\}$$

is not Zariski dense in Y.

Remarks:
1. As the above set contains all torsion points in Y, the above theorem implies a theorem of Raynaud on Lang's conjecture.
2. When Y is the image of a map $X \to A = \mathrm{Jac}(X)$ of a curve to its Jacobian defined by the canonical divisor of X, this theorem is due to E. Ullmo.

Actually one can prove the following stronger statement about probability measure rather than Zariski topology:

Theorem B. *Let $(x_n, n \in \mathbb{N})$ be a sequence of points in $A(\bar{\mathbb{Q}})$ such that the following conditions are verified:*
1. There is no subsequence of $(x_n, n \in \mathbb{N})$ contained in a translation of proper abelian subvariety by a torsion point;
2. $\lim_{n\to\infty} h(x_n) = 0$.
Then the probability measures of the Galois orbits of x_n converge to the Haar measure of $A(\mathbb{C})$.

One can use this to show that the set of torsion points on A over the maximal totally real fields is finite.

Theorem C (Gross-Zagier formula). *Let E be a modular elliptic curve defined over a totally real field F with the conductor $\mathfrak{n}$. Let K/F be an imaginary quadratic extension which is unramified over places dividing $\mathfrak{n}$ and such that the functional equation of $L(E/K, s)$ has the negative sign. Then*

$$L'(E/K, 1) = \text{(positive constant)} \cdot h(y_K)$$

where $y_K \in E(K)$ is the Heegner point, and h is the Néron-Tate height on E.

Remarks:
1. Here we say E/F is modular if $L(E, s) = L(f, s)$ with f a Hilbert modular form over F. In our case, E has a uniformization by certain Shimura curve (by the Jacquet-Langlands theory and the Eisenstein divisor class). The points y_K is the image of certain CM-points on Shimura curves.
2. When $F = \mathbb{Q}$ and every factor of $\mathfrak{n}$ is split in K, this theorem is due to Gross and Zagier. Modulo the construction of Eisenstein divisor class, the general statement was first conjectured by Gross.

Assume that $\mathrm{ord}_{s=1} L(E, s) \leq 1$, and that either g is odd or $\mathfrak{n}$ is not a square. Then one may show that there is a K as in the theorem such that $L'(E/K, 1) \neq 0$. This implies that y_K is nontorsion. By Kolyvagin's theory of Euler system, this in turn implies the following statement concerning the Birch and Swinnerton-Dyer conjecture:

Theorem D. *Assume that $\mathrm{ord}_{s=1} L(E, s) \leq 1$, and that either g is odd or $\mathfrak{n}$ is not a square. Then*
1. The Mordell-Weil group $E(F)$ has the rank equal to $\mathrm{ord}_{s=1} L(E, s)$.
2. The Shafarevich group of E over F is finite.

In the proofs of both Theorem A and Theorem C, Arakelov theory (developed by Deligne, Faltings, Gillet, Soulé, Szpiro, etc.) plays the crucial rule in the computation of Néron-Tate heights. By Hodge index theorem in this theory, one can either express the Néron-Tate heights as the intersections of arithmetic line bundles on the integral models of abelian varieties, or on the integral models (i. e., arithmetic surfaces) of curves.

4. Algebraic Geometry

1991 MS Classification: 14J32, 14J28, 14J60, 81T30

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Geometry and String Duality

Superstring theory may be said to have six manifestations:

1. Type I open superstring
2. Type IIA superstring
3. Type IIB superstring
4. $E_8 \times E_8$ heterotic string
5. $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string
6. Eleven-dimensional supergravity (or "M-theory")

Of these, the first 5 theories naturally live in ten dimensions while the last one lives in eleven dimensions.

In order to describe the world in a smaller number of dimensions one compactifies space-time. That is, one replaces the ten-dimensional space-time of string theory by a product $X \times M$ where X is some (small) compact space of d-dimensions and M is a flat Minkowski space of $10 - d$ dimensions. (In the case of M-theory we let X be $(d+1)$-dimensional.) One then imagines M to be the flat space-time we observe in the universe.

The notion of duality is that one may take two types of string theory from our list above and compactify them in two different ways and achieve isomorphic physics in the space M. As an example, one may take the type IIA superstring and compactify on some Calabi–Yau threefold X (and so $d = 6$) to describe four-dimensional space-time. One may also take the type IIB superstring and compactify it on some other Calabi–Yau threefold Y and hope to produce exactly the same physics. Since the physics in M is determined by the geometry of X and Y, this statement must imply some relationship between the geometry of X and Y.

This example is the string duality version of "mirror symmetry". That is, X and Y are said to be a mirror pair of Calabi–Yau threefolds if and only if the type IIA string compactified on X is "isomorphic to" the type IIB string compactified on Y. Although this was not the original statement of mirror symmetry it is now regarded as its most succinct definition.

The key to understanding such duality statements is to analyze the relationship between the geometry of X and the physics in M. The first step is to consider the moduli space of a class of theories. That is, one enumerates the number of free parameters in the physics and tries to associate each one to some data for X. One of the keys to the rich structure of string duality is that one typically uses more data for X than one would normally consider from classical geometric constructions. For example, in the case of mirror

symmetry, as well as the complex structure and the Kähler form on X, one is also forced to consider an extra parameter $B \in H^2(X, \mathbb{R}/\mathbb{Z})$. This allows the Kähler form to be "complexified".
When one has identified the parameters which make up the moduli space one can attempt to map the moduli space of theories to the moduli space of its dual partner. This introduces a highly non-trivial geometrical correspondence between two families of compactifications. In the case of mirror symmetry this acts roughly by exchanging the complex structure with the complexified Kähler form. Consideration of the details of this map led to the celebrated enumeration of rational curves on X by Candelas et al.
Clearly we may analyze many more "mirror-symmetry-like" dualities other than the type IIA/IIB duality above. Indeed, the other dualities one would select from our list above should be expected to yield potentially more interesting correspondences since one is typically identifying more diverse objects.
One case which has been studied to a large extent in the past few years is that of heterotic/type IIA duality (particularly in the case of the $E_8 \times E_8$ heterotic string). That is, a type IIA string on X is dual to a $E_8 \times E_8$ heterotic string on Z. In this case, the data on the heterotic side must include the choice of a vector bundle on Z whose structure group is a subgroup of $E_8 \times E_8$ (or some sheaf-theoretic generalization of this).
In the case that X is a Calabi–Yau threefold, Z will be a product of a K3 surface and an elliptic curve. This duality therefore suggests some mapping from the moduli space of Calabi–Yau threefolds to the moduli space of K3 surfaces and elliptic curves together with some bundle data.
In this case, the elliptic curve part of the moduli space is fairly well-understood but that part associated to the K3 surface is somewhat more troublesome. The K3 part can be understood well in one special limit of the moduli space where X undergoes a rather drastic degeneration. Here the three-dimensional variation of Hodge structure on X decomposes as a variation of Hodge structure of a two-dimensional object and the variation of Hodge structure of a one-dimensional object. Here, the two-dimensional object can be identified as the required K3 surface and the one-dimensional object is a spectral curve specifying the vector bundle.

1991 MS Classification: 14, 52, 81
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Mirror symmetry and toric geometry

A *toric variety* of dimension n over $\mathbf{C}$ is a normal algebraic variety V containing $(\mathbf{C} \setminus \{0\})^n$ as a Zariski open subset in such a way that the algebraic group action of $(\mathbf{C} \setminus \{0\})^n$ on itself extends to a regular action on V. A projective toric variety V is called a *Gorenstein toric Fano variety* if V has at worst Gorenstein singularities and the anticanonical sheaf on V is ample. A convex n-dimensional polyhedron $\Delta \subset \mathbf{R}^n$ is called a *reflexive polyhedron* if the following conditions are satisfied: 1) all vertices of Δ belong to $\mathbf{Z}^n$; 2) the point $(0, \ldots, 0)$ belongs to the interior of Δ; 3) all vertices of the polar polyhedron

$$\Delta^* := \{(x_1, \dots, x_n) \in \mathbf{R}^n : \sum_{i=1}^{n} x_i y_i \geq -1 \ \ \forall (y_1, \dots, y_n) \in \Delta\}$$

belong to $\mathbf{Z}^n$. There exists a 1-to-1 correspondence between the isomorphism classes of n-dimensional Gorenstein toric Fano varieties V and n-dimensional reflexive polyhedra $\Delta \subset \mathbf{R}^n$ considered up to a linear transformation from $GL(n, \mathbf{Z})$. If Δ is a reflexive polyhedron, then the polar polyhedron Δ^* is again reflexive. The duality between reflexive polyhedra $\Delta \leftrightarrow \Delta^*$ induces a duality $V \leftrightarrow V^*$ between the corresponding Gorenstein toric Fano varieties. It turns out that Calabi-Yau hypersurfaces X in V are *mirror symmetric* to Calabi-Yau hypersurfaces X^* in V^*. Recently this observation became a powerful tool in the investigation of the mirror symmetry phenomenon discovered by physicists.
Reference: V. V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geometry* **3** (1994)

1991 MS Classification: 14H10, 14D22
Cornalba, Maurizio, Università di Pavia, Italia
Cohomology of moduli spaces of stable curves

I will illustrate some recent progress in the calculation of the cohomology of $\overline{\mathcal{M}}_{g,n}$, the Deligne-Mumford-Knudsen compactification of the moduli space of genus g algebraic curves with n numbered punctures. In one direction, an elementary algebro-geometric calculation of the first, second, third, and fifth rational cohomology groups of $\overline{\mathcal{M}}_{g,n}$, for all g and n, has been given by Arbarello and myself. The proof uses a Lefschetz-type argument to do induction on g and n, starting from a few cases that can be disposed of directly. In a different direction, considerable progress has been made, mostly by Getzler, in understanding the cohomology ring of $\overline{\mathcal{M}}_{g,n}$ for $g = 1$ and for $g = 2$ and low n. In particular, remarkable new relations satisfied by the tautological classes in the fourth cohomology of $\overline{\mathcal{M}}_{g,n}$ have been found by Getzler for $g = 1, n = 4$, and by Belorusski and Pandharipande for $g = 2, n = 3$.

1991 MS Classification: 14
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Barsotti-Tate groups and crystals

Let G and H be Barsotti-Tate groups (also called p-divisible groups) over a discrete valuation ring R with field of fractions K. In 1966 J. Tate proved that, if the characteristic of K is zero, then any homomorphism between the generic fibres $G \otimes K \to H \otimes K$ extends to a homomorphism $G \to H$ over R. In the introduction of exposé IX in SGA 7 (by A. Grothendieck, M. Raynaud and D. Rim) it was mentioned as a problem whether the same holds when the characteristic of K is p.

As a first step in the solution to this problem, A. Grothendieck, W. Messing, P. Berthelot and others developed a crystalline Dieudonné module theory. This theory classifies Barsotti-Tate groups over a scheme of characteristic p in terms of Dieudonné crystals. The final step is to solve the resulting extension problem for morphisms of Dieudonné crystals and more generally for F-crystals (to appear in Inventiones Mathematicae).

In the talk we will indicate what kind of algebraic objects come up in the study of these Dieudonné crystals and F-crystals. The proof of the extension result involves a comparison of various types of convergence conditions on formal power series over rings of Witt-vectors. We will try to explain this.

Consider the following statement: Let F be a field finitely generated over its prime field $\mathbf{F}_p$, let X and Y be abelian varieties over F. Then the natural map

$$\mathrm{Hom}(X,Y) \otimes_{\mathbf{Z}} \mathbf{Z}_p \longrightarrow \mathrm{Hom}(X[p^\infty], Y[p^\infty])$$

is an isomorphism. When F is finite, this was proved by J. Tate (1966). For general F this result can be derived from the extension theorem on homomorphisms of p-divisible groups. This derivation is completely similar to J. Zarhin's proof of the ℓ-adic case. (It is amusing to note that G. Faltings proof of the same statement in the number field case used the extension theorem on homomorphisms for p-divisible groups proved by Tate.)

Finally, we will mention a conjecture on semi-stable reduction for so-called overconvergent F-crystals; such a conjecture occurs in work of R. Crew, N. Tsuzuki and others. The conjecture is a p-adic analogue of the strong results (quasi-unipotency of monodromy and the nilpotent orbit theorem) for variations of Hodge structure. It is an optimistic conjecture which implies among other things that certain rigid cohomology groups are finite dimensional, thereby tying in to recent results in this direction (by P. Berthelot, G. Christol, Z. Mebkhout and others).

1991 MS Classification: 81T30
Dijkgraaf, Robbert, University of Amsterdam
The Mathematics of Fivebranes

(See section 11)

1991 MS Classification: 14C25, 14C30, 14D07, 32G20, 32J25
Green, Mark, University of California, Los Angeles, California (USA)
Higher Abel-Jacobi Maps

For a smooth projective variety X, the structure of the Chow group $CH^p(X)$ representing codimension p algebraic cycles modulo rational equivalence, is still basically a mystery when $p > 1$. There is a conjectural filtration proposed by Bloch-Beilinson-Deligne-Murre $F^m CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ whose graded pieces ought to have higher Abel-Jacobi maps $\mathrm{Gr}^m CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to J^p_m(X)$. The cases $0 \le p, m \le 1$ are known (divisors, Hodge class, Abel-Jacobi map). The case $p = 2$, $m = 2$ (kernel of the Albanese map for 0-cycles on surfaces) is an interesting first case to consider. A "toy model" for this is the relative

Chow group $CH^2(\mathbf{P}^2, T)$, where $T \subset \mathbf{P}^2$ is the triangle $z_0 z_1 z_2 = 0$, roughly described as 0-cycles on $\mathbf{P}^2 - T = \mathbf{C}^* \times \mathbf{C}^*$, modulo divisors of meromorphic functions f on curves $C \subset \mathbf{P}^2$ such that $f = 1$ on $C \cap T$. There is a series of maps

$$\psi_0\colon CH^2(\mathbf{P}^2, T) \to \mathbf{Z};$$
$$\psi_1\colon \ker(\psi_0) \to \mathbf{C}^* \oplus \mathbf{C}^*;$$
$$\psi_2\colon \ker(\psi_1) \to K_2(\mathbf{C}).$$

It is known (Bloch, Suslin) that these are all surjective and ψ_2 is injective. These all have simple algebraic descriptions—ψ_0 is degree; $\psi_1(a, b) = a \oplus b$; $\psi_2(a, b) = \{a, b\}$. The essential tool in proving this is the Suslin reciprocity theorem.
This example provides a model for the general case—one should think of $(\mathbf{P}^2, T)$ as analogous to a complete surface with $h^{2,0} = 1$. $K_2(\mathbf{C})$ is infinite-dimensional, as Mumford's theorem asserts for surfaces with $h^{2,0} \neq 0$. For a general X, one has

$$\psi_0\colon CH^2(X) \to \operatorname{Hdg}^2(X);$$
$$\psi_1\colon \ker(\psi_1) \to J^2(X);$$

where ψ_0 is the cycle class map to the Hodge classes on X, and ψ_1 is the Abel-Jacobi map. We have constructed part of the missing map ψ_2 in the case of 0-cycles on a surface.
The regulator map for a curve

$$X - D \xrightarrow{(f,g)} \mathbf{C}^* \times \mathbf{C}^*$$

is a homomorphism $r\colon \pi_1(X - D) \to \mathbf{C}/\mathbf{Z}(2)$ given by

$$r(\gamma) = \int_\gamma \log(f)\frac{dg}{g}\log(g(p))\int_\gamma \frac{df}{f},$$

where p is a base-point on γ; the answer does not depend on p. If $\gamma = \partial U$ for U a disc in $\mathbf{C}^* \times \mathbf{C}^*$, then

$$r(\gamma) = \int_U \frac{df}{f} \wedge \frac{dg}{g}.$$

This formula generalizes to a definition in the more general situation of a non-singular curve C and a map $f\colon C \to X$ to a smooth projective surface X. If $\mu \in \ker(H^2(X, \mathbf{Z}) \xrightarrow{f^*} H^2(C, \mathbf{Z}))$, then representing μ by a real d and d^c-closed 2-form, $f^*\mu = dd^c g$ for $g \in A^0(C)$, unique up to adding a constant. If $\gamma \in \ker(H_1(C, \mathbf{Z}) \to H_1(X, \mathbf{Z}))$, so that $\gamma = \partial\Gamma$ in X, then we define

$$e_{X,C}(\mu, \gamma) = \int_\Gamma \mu - \int_\gamma f^*(d^c g) \in \mathbf{R}/\mathbf{Z},$$

which does not depend on any of the choices. These quantities are known as **membrane integrals**. More intrinsically, $e_{X,C}$ is the extension class of the extension of mixed Hodge structures (MHS)

$$0 \to \operatorname{coker}(H^1(X) \to H^1(C)) \to H^2(X,C) \to \ker(H^2(X) \to H^2(C)) \to 0.$$

Denote the term on the left $H^1(C)_{\text{new}}$ and the term on the right $H^2(X)_C$; now by standard identifications

$$e_{X,C} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(H^2(X)_C, H^1(C)_{\text{new}}).$$

The basic idea is that $e_{X,C}$ should describe the map $f_*\colon J(C)_{\text{new}} \to CH^2(X)$. If Z is a 0-cycle on X in the kernel of the Albanese map, then if we choose C so that Z lies on $f(C)$ and lifts to $\tilde{Z} \in Z^1(C)$ such that $f_*\tilde{Z} = Z$ and $\deg(Z) = 0$ on each component of C, then $AJ_C(\tilde{Z})$ is represented by the extension class $e_{C,\tilde{Z}}$ of the extension of mixed Hodge structures of $H^1(C)_{\text{new}}$ by 1, and by standard identifications

$$e_{C,\tilde{Z}} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(H^1(C)_{\text{new}}, 1).$$

The two extensions of MHS fit together to give a 2-step extension of MHS of $H^2(X)_C$ by 1, which unfortunately cannot be used directly, since Ext^2's all are 0 for MHS. We combine the transcendental part of $e_{X,C}$ with $e_{C,\tilde{Z}}$. The tensor product followed by contraction gives an element

$$e_{X,C,\tilde{Z}} = e_{C,\tilde{Z}} \circ (e_{X,C})_{\text{tr}} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(H^2(X)_{\text{tr}}, 1).$$

If we let $U_2^2(X) = \{e_{X,C,\tilde{Z}} \mid f_*\tilde{Z} = 0 \text{ as a } 0\text{−cycle on } X\}$ and

$$J_2^2(X) = \frac{(\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(H^2(X)_{\text{tr}}, 1)}{U_2^2(X)},$$

then $Z \mapsto [e_{X,C,\tilde{Z}}]$ gives a well-defined invariant

$$\psi_2\colon \ker(\psi_1) \to J_2^2(X)$$

that is independent of the choices of C and $\tilde{Z}$, and which depends only on the rational equivalence class of Z on X. It is necessary to allow reducible curves C. Claire Voisin has shown that for surfaces with $h^{2,0} \neq 0$, the map ψ_2 has infinite-dimensional image, and also that it need not be injective.

An invariant which complements the one above was obtained in joint work with Phillip Griffiths. By work of Katz and Grothendieck, there is for any smooth projective variety X defined over $\mathbf{C}$ the arithmetic Gauss-Manin connection

$$\nabla_{X/\mathbf{Q}}\colon H^k(X,\mathbf{C}) \to \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} H^k(X,\mathbf{C}).$$

We define an **arithmetic Hodge structure (AHS)** to be a complex vector space V with a finite descending filtration $F^\bullet V$ and a connection $\nabla\colon V \to \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} V$ satisfying $\nabla^2 = 0$ and $\nabla F^p V \subseteq \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{p-1}V$ (Griffiths transversality) for all p. A short exact sequence $0 \to A \to B \to C \to 0$ of AHS has extension class

$$e \in \mathrm{Ext}^1_{\mathrm{AHS}}(C, A) = H^1(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(C, A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(C,A)}).$$

A 2-step exact sequence $0 \to A \to B \to C \to D \to 0$ of AHS has a well-defined map from the Yoneda Ext

$$\mathrm{Ext}^2_{\mathrm{AHS}}(D, A) \to H^2(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(D, A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(D,A)})$$

obtained by composing the extension class of the two 1-step extensions $0 \to A \to B \to E \to 0$, $0 \to E \to C \to D \to 0$ it breaks into. This is exactly the obstruction to finding an AHS V with B as a sub-AHS and C as a quotient AHS in a compatible way. In contrast to MHS, AHS have non-trivial Ext^2's, and these have geometric meaning.

This extension class theory fits in well with the pre-existing **arithmetic cycle class map**

$$\eta\colon CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathbf{H}^{2p}(\Omega^{\geq p}_{X/\mathbf{Q}})$$

whose graded pieces are $\eta_m\colon \ker(\eta_{m-1}) \to H^m(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{p-\bullet}H^{2p-m}(X, \mathbf{C}), \nabla_{X/\mathbf{Q}})$; one expects these to be consistent with the conjectural Bloch-Beilinson-Deligne-Murre filtration on $CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. For 0-cycles on a surface, we are able to show that η_2 is the element of $\mathrm{Ext}^2_{\mathrm{AHS}}(H^2(X), 1)$ coming from the 2-step extension of AHS using $Z \subset C \subset X$ analogous to the construction above in the mixed Hodge structure case.

1991 MS Classification:
Kapranov, M., Department of Mathematics Northwestern University, Evanston, USA
Operads and algebraic geometry

The study (motivated by mathematical physics) of algebraic varieties related to the moduli spaces of curves, helped to uncover important connections with the abstract algebraic theory of operads. This interaction led to new developments in both theories, and the purpose of the talk is to discuss some of them.

1. Operads. To explain why operads are interesting, it may be useful to recall a well known problem of Hilbert (as it came to be understood later). Let P be some class of functions considered in analysis (e. g. , continuous, smooth, algebraic etc.). Is it possible to express any function $f : \mathbf{R}^n \to \mathbf{R}$ of class P, depending of $n \geq 3$ variables, as a superposition of functions of class P, depending on 1 or 2 variables only?

For a set X, maps $X^n \to X$ are also called n-ary operations on X. A collection P of such maps (with possibly varying n) is called an *operad*, if it is closed under arbitrary superpositions as well as under permutations of

variables. Thus $P(n)$, the n-ary part of P, is acted upon by the symmetric group S_n. One can view P as a multivariable analog of a (semi)group of transformations of X.
As with groups, the actual working definition (due to J. P. May) splits the above naive one into two. An abstract operad is a collection $P = \{P(n), n \geq 0\}$ of sets equipped with S_n-actions and with certain maps mimicking the functional suporpositions. Then, an action of P on a set X is a rule which associated to any $\mu \in P(n)$ an n-ary operation in a compatible way. Instead of sets one can consider vector spaces (resp. cochain complexes), requiring all multi-variable maps to be multilinear. In this way we get linear (resp. $dg-$) operads.

2. Main examples. First, many familiar algebraic structures can be described via operad actions. For example, we have linear operads $\mathcal{A}s$, $\mathcal{C}om$, $\mathcal{L}ie$ governing associative, commutative and Lie algebras.
Second, the theory of moduli spaces provides examples of operads in the category of algebraic varieties or topological spaces. The most important are $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}$, where $\overline{\mathcal{M}}_0(n)$ is the moduli space of stable $n+1$-pointed curves of genus 0, and $\mathcal{M}_0 = \{\mathcal{M}_0(n)\}$ where $\mathcal{M}_0(n)$ is the moduli space of spheres with $n+1$ parametrized holes. One can consider curves/Riemann surfaces of arbitrary genus, thus obtaining operads $\overline{\mathcal{M}}, \mathcal{M}$. By taking the (topological) homology of the above moduli spaces, one gets graded linear operads $H_\bullet(\overline{\mathcal{M}}_0)$ etc. The theory of Gromov-Witten invariants [6] defines an action of $H_\bullet(\overline{\mathcal{M}}_0)$ on the space $H^\bullet(V, \mathbf{C})$ for any smooth projective variety V over $\mathbf{C}$.
Third, the graph complexes of Kontsevich (and their analogs) lead to dg-operads relating the first two groups of examples.

3. Operad duality. To get a conceptual understanding of the (tree part of the) graph complexes, V. Ginzburg and the author [3] introduced a certain duality among linear (and more generally, dg-) operads.
A linear operad P is called quadratic, if every $P(n)$ is spanned by superposition of elements of $P(2)$ and also all the relations among elements of $P(2)$ follow from those holding already in $P(3)$. For such P, its Koszul dual $P^!$ is defined to have the dual space of generators and orthogonal space of relations. For example, $\mathcal{C}om^! = \mathcal{L}ie$ while $\mathcal{A}s^! = \mathcal{A}s$. More generally, for any dg-operad P a version of the graph complex construction gives the cobar-dual operad $D(P)$. For a quadratic P there is a natural map $D(P) \to P^!$, and P is called Koszul if it is a quasi-isomorphism. For example, $\mathcal{A}s, \mathcal{C}om, \mathcal{L}ie$ are Koszul. E. Getzler generalized the concept of Koszul duality to operads with generators in any degree and proved that $H_\bullet(\overline{\mathcal{M}}_0)$ is Koszul.

4. Cyclic and modular operads. Operads coming from moduli spaces possess some extra structure. This structure was axiomatized by E. Getzler and the author [1-2] in the concepts of cyclic and modular operads. Cyclic operads posess the action of S_{n+1}, not just S_n, on the space of n-ary operations. This structure is present on the operads related to genus 0 moduli spaces. Modular operads are, in addition, graded by an extra parameter ("genus") and possess contraction maps lowering the arity and raising the genus (with moduli spaces of curves and their homology giving prime examples). The cobar-duality can be generalized to cyclic and modular cases. The generating function of the dual operad is for ordinary operads given by the composition inverse, for cyclic operads by the Legendre transform and for modular operads by the Fourier transform (in an infinite-dimensional setting).

5. Geometry based on an operad. An affine algebraic variety can be described in terms of its commutative algebra of functions. So if P is a linear operad related to $\mathcal{C}om$ by a morphism of operads, it becomes possible to speak about (semiformal) P-manifolds as ordinary smooth algebraic varieties equipped with some extra structure reflecting P. For $P = \mathcal{A}s$ the theory of such objects ("noncommutative manifolds") was worked out in [5]. The general feature of such P-geometry is that the curvature data satisfy the relations of the dual operad $P^!$. For the case $P = \mathcal{C}om$ (i. e., for ordinary manifolds) this means that the Atiyah class of any complex manifold satisfies the Jacobi identity [4].

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5. Differential Geometry and Global Analysis

1991 MS Classification:

Burago, Dima, Pennsylvania State University, USA

Hard Balls Gas and Alexandrov Spaces of Curvature Bounded Above

Consider the hard balls model of statistical physics, that is a system of several hard balls moving freely and colliding elastically. One asks whether the number of collisions that may occur in this system can be estimated from above by a bound depending only on the number of balls and their masses? If we consider the balls moving in unbounded Euclidean space, we count the total number of collisions in infinite time. For a system of balls in a box, we mean the number of collisions in unit time (for a fixed value of kinetic energy). A "physical" version of these problems goes back to Boltzman, while their first mathematical formulation is probably due to Ya. Sinai.

In early 60th V. Arnold "speculated that such systems can be considered as the limit case of geodesic flows on negatively curved manifolds (the curvature being concentrated on the collisions hypersurface)" (quoted from Arnold's lecture at the meeting in the Fields institute dedicated to his 60th birthday). Indeed, this is nowadays well known (due to the works of Sinai, Bunimovich, Chernov, Katok, Strelcyn, Szasz, Simanyi and many others) that a large portion of the results in the smooth theory of (semi-)hyperbolic systems can be generalized (with appropriate modifications) for (semi-)dispersing billiards. In spite of this, the construction suggested by Arnold has never been used. It also caused several serious objections; in particular, A. Katok pointed out that such approximations by geodesic flows on manifolds necessarily produce geodesics bending around collision hypersurfaces and therefore having no analogs in the billiard system.

In the talk we will discuss elementary constructions representing billiard trajectories as geodesics in appropriate spaces. Unfortunately, these spaces are not even topological manifolds: they are lengths spaces of curvature bounded above in the sense of A. D. Alexandrov. Nevertheless, applying these constructions transforms a certain type of problems about billiards into purely geometric statements; and although a problem may seem rather difficult in its billiard clothing, its geometric counterpart may turn out to be relatively easy by the standards of the modern metric geometry. In particular, this approach allowed to solve the problems formulated at the beginning of the abstract by means of applying rather standard arguments in the theory of length spaces of curvature bounded above. We will also discuss new geometric problems arising in connection with these constructions.

1991 MS Classification: 13, 10, 8

Colding, Tobias Holck, Courant Institute, New York University, USA

Manifolds and spaces with Ricci curvature bounds

Results about manifolds and more generally spaces with Ricci curvature bounds will be presented. This will include geometric, topological and function theoretical results.

1991 MS Classification: 58b

Donaldson, Simon, Stanford University, USA

Positive line bundles in symplectic geometry

In this lecture we will describe how certain familiar techniques in complex geometry can be applied to symplectic manifolds. The general setting is that we fix an almost-complex structure on a compact symplectic manifold (M, ω) and a unitary connection on a complex line bundle L over M whose curvature is $-i\omega$. In the classical case of a complex Kahler manifold V this would be a positive line bundle, and one knows that a high tensor power $L^{\otimes k}$ has many holomorphic sections. More precisely, these holomorphic sections yield a projective embedding of V, and generic projections give rational maps from V to other complex projective spaces. In the general case one studies "approximately" holomorphic sections of $L^{\otimes k}$, and applies these to produce objects with the same qualitative properties as the paradigms in complex geometry. The success of this technique hinges on certain refined transversality theorems, with estimates, to achieve uniform control with respect to the large parameter k. Results proved using this technique include existence theorems for symplectic submanifolds and Lefschetz fibrations . Combined with work of Gompf, we obtain, in principle, a complete topological characterisation of symplectic 4-manifolds.

These methods can also be applied to Lagrangian submanifolds and symplectomorphisms. Key roles are played by the generalised Dehn twists about Lagrangian spheres, which enter as the monodromy of a Lefschetz fibration around a single critical value. In the last part of the lecture we will describe some connections with recent work of P. Seidel , which involves the symplectic version of the Floer homology theory. This Floer theory, which has been developed by Fukaya and others, has an intricate algebraic structure—it turns the set of isotopy classes of Lagrangian submanifolds into a category—about which relatively little is known.

1991 MS Classification: 53B50, 32G34, 35Q15, 35Q53, 52B

Dubrovin, Boris, SISSA, Trieste, Italy, and Steklov Math. Institute, Moscow, Russia

Geometry and analytic theory of Frobenius manifolds

Frobenius manifolds were invented as a coordinate-free form of the following *WDVV equations of associativity*: to find a quasihomogeneous function $F(t)$ of the variables $(t^1, \ldots, t^n) = t$ and a constant nondegenerate symmetric matrix $\left(\eta^{\alpha\beta}\right)$ such that the following combinations of the triple derivatives of $F(t)$

$$c_{\alpha\beta}^{\gamma}(t) := \sum_{\epsilon=1}^{n} \eta^{\gamma\epsilon} \frac{\partial^3 F(t)}{\partial t^\epsilon \partial t^\alpha \partial t^\beta}$$

for any t are structure constants of an associative algebra

$$A_t = \mathrm{span}(e_1, \ldots, e_n), \ e_\alpha \cdot e_\beta = \sum_{\gamma=1}^{n} c_{\alpha\beta}^{\gamma}(t) e_\gamma, \ \alpha, \beta = 1, \ldots, n$$

with the unity e_1. These equations were discovered by physicists E. Witten, R. Dijkgraaf, E. Verlinde and H. Verlinde in the beginning of '90s.

Main mathematical applications of Frobenius manifolds are in the theory of Gromov - Witten invariants of smooth algebraic varieties, in singularity theory, in differential geometry of the orbit spaces of reflection groups and their extensions, in the hamiltonian theory of integrable hierarchies. The theory of Frobenius manifolds establishes remarkable relationships between these, sometimes rather distant, mathematical theories.

Frobenius manifolds have amazingly rich geometrical structure. This allows to classify *semisimple* Frobenius manifolds where the algebra A_t is assumed to be semisimple for generic values of t. The moduli of n-dimensional semisimple Frobenius manifolds are the monodromy data of the linear differential operator with rational coefficients of the form

$$L = \frac{d}{dz} - U - \frac{1}{z} V$$

where U, V are z-independent $n \times n$ diagonal and antisymmetric matrices respectively. The Frobenius manifold structure is realized on the space of isomonodromy deformations of the operator L.

The basic ideas of this classification will be explained in the talk. It gives rise to new problems of the theory of isomonodromy deformations, reflection groups, and integrable hierarchies. Some of them will also be discussed in the talk.

References [1] B. Dubrovin, Integrable systems in topological field theory, Nucl. Phys. **B 379** (1992) 627 - 689. [2] B. Dubrovin, Geometry of 2D topological field theories, In: *Integrable Systems and Quantum Groups*, Eds. M. Francaviglia, S. Greco, Springer Lect. Notes in Math. **1620** (1996) 120 - 348. [3] B. Dubrovin, Painlevé transcendents in two-dimensional topological feild theory, math. AG/9803107, to appear in Proceedings of the school "The Painlevé property, one century later", Cargèse, 1996.

1991 MS Classification: 53C15
Eliashberg, Yakov, Stanford University, USA

Invariants in Contact Topology

A contact manifold, i. e. a smooth manifold together with a completely non-integrable tangent hyperplane field, is an odd-dimensional counterpart of a symplectic manifold. Maximal integral submanifolds of contact manifolds, called Legendrian, are contact analogs of Lagrangian submanifolds of symplectic manifolds.

Despite the fact that contact geometry abides Gromov's C^0-rigidity, invariants of contact manifolds and their Legendrian submanifolds are much more difficult to define than in the symplectic case, due to a projective character of contact geometry.
In the talk there will be given a survey of some recent results in this area. In particular, the following topics will be discussed:

1) invariants of open contact manifolds, especially in the 3-dimensional case;

2) invariants of Legendrian isotopy via the theory of generating functions;

3) contact homology theory.

In Part 2 this is a joint work with M. Gromov, and in Part 3 with H. Hofer.

1991 MS Classification: Primary: 58F17, 53C20, 58E35, 58F11; Secondary: 53C23, 53C21, 58E11
Sylvestre Gallot, Université de Mathématiques Pures et Appliquées (U. M. P. A.), École Normale Supérieure de Lyon, France (on a joined work with G. Courtois and G. Besson)
Curvature-decreasing maps are volume-decreasing

The "Minimal Volume" of a differentiable manifold was introduced by M. Gromov in order to generalize (in any dimension) the Gauss-Bonnet-Chern-Weil formulas for characteristic classes and the induced estimates on volume and curvature. Giving a lower bound of the minimal volume in terms of the simplicial volume, M. Gromov obtained such a generalization. He thus conjectured that this inequality may be sharpened, i. e. that the minimal volume of a hyperbolic manifold is achieved by the hyperbolic metric.
We proved this conjecture by settling a real analogue of the Schwarz's lemma : if X, Y are two manifolds such that the curvature of X is negative and not greater than the one of Y, then any homotopy class of maps from Y to X contains a map which contracts volumes. We give a construction of this map which, under the assumptions of Mostow's rigidity theorems, turns to be an isometry, providing a unified proof of these rigidity theorems. It moreover proves that the set of all Einstein metrics, on any compact 4-dimensional hyperbolic manifold, reduces to a single point.
The entropy of a riemannian manifold is the limit of $\frac{1}{R}\, Log(Vol\, B(x,R))$, for big balls $B(x,R)$ in the riemannian universal covering of the manifold. The real Schwarz's lemma also gives a sharp inequality between the entropies of Y and X (provided that X is a compact locally symmetric negatively curved manifold and that there exists a map of non zero degree from Y to X, this inequality works for any compact riemannian manifold Y). This answers conjectures of A. Katok and M. Gromov about the minimal entropy of a manifold.
As this inequality is an equality iff Y is isometric to X, and as the volume and the entropy of a riemannian manifold are dynamical invariants, it implies

that Y and X have conjugate geodesic flows iff they are isometric.
This also ends the proof of the Lichnerowicz's conjecture : any negatively curved compact locally harmonic manifold is a quotient of a (noncompact) rank-one-symmetric space.
We recently generalized this method to cases where $dim(X) \neq dim(Y)$ and where X is no longer assumed to be compact nor regular; we use it in order to obtain rigidity results for representations of discrete groups into SO(N,1) and SU(N,1). The real Schwarz's lemma is also used in order to construct diffeomorphisms between negatively curved manifolds and hyperbolic ones (assuming almost maximal volume or pinched curvature).

1991 MS Classification: 35, 49, 53
Huisken, Gerhard, Universität Tübingen, Germany
Evolving hypersurfaces by their curvature

Let $F_0 : M^n \to (N^{n+1}, g)$ be a smooth immersion of a hypersurface in a Riemannian manifold (N^{n+1}, g). We study one–parameter families $F : M^n \times [0,T] \to (N^{n+1}, g)$ of hypersurfaces $M_t^n = F(\cdot,t)(M^n)$ satisfying a parabolic initial value problem

$$\begin{aligned} \frac{d}{dt}F(p,t) &= f\,\nu\,(p,t), \qquad\qquad p \in M^n, \quad t \in [0,T], \\ F(p,0) &= F_0(p), \end{aligned}$$

where ν is a choice of unit normal and f is some smooth function depending on the principal curvatures $\{\lambda_i\}_{1\le i\le n}$ of the surface at the point $F(p,t)$.
Typical examples are the mean curvature flow ($f = H = \lambda_1 + \cdots + \lambda_n$), the inverse mean curvature flow ($f = -1/H$), the Gauss–curvature flow ($f = G = \lambda_1 \cdots \lambda_n$), the affine curvature flow ($f = G^{1/(n+2)}$) and the harmonic mean curvature flow $\big(f = 1/(\lambda_1^{-1} + \cdots \lambda_n^{-1})\big)$. Based on the concept of a heat equation, the analytical structure of this system of evolution equations is closely related to the harmonic map heatflow and the Ricci flow of Riemannian metrics.
The last decade has seen major advances in the understanding of local and global properties of these parabolic deformations. The close interplay between analytical properties of the flow equation and geometrical assumptions on the underlying manifold has provided an elegant new approach to many classical results, e. g. to closed geodesics by curve–shortening, to sphere theorems by harmonic mean curvature flow and to the affine isoperimetric inequality via affine mean curvature flow.
Classical mean curvature flow ($f = H$) is the L^2–gradient flow for the area functional and can be considered as the parabolic analogue of the minimal surface equation. For this flow some aspects of short–term and long–term behaviour are well understood, and new insight into the structure of singularities was obtained in recent years. In certain cases it seems now possible to attack the problem of extending the flow in a controlled way beyond sin-

gularities. In a weak setting mean curvature flow plays an important role as a model problem for the theory of interfaces and phase transitions.
In a recent development the inverse mean curvature flow ($f = -1/H$) has been used in a generalised setting to give a proof of the Riemannian Penrose inequality in General Relativity, providing an optimal lower bound for the total mass of an isolated gravitating system modelled by an asymptotically flat three manifold of non–negative scalar curvature, in terms of the size of a contained black hole modelled by an outermost minimal surface (joint work of G. Huisken and T. Ilmanen, see *IMRN No.* **20** (1997), 1045–1058).
The lecture describes typical phenomena encountered in the geometric hypersurface evolutions above and explains some main techniques and results.

1991 MS Classification: 53, 55, 58
Joyce, Dominic, Lincoln College, Oxford, England
Compact Riemannian manifolds with exceptional holonomy

In the classification of Riemannian holonomy groups, the most mysterious cases are the *exceptional holonomy groups* G_2 in 7 dimensions, and $Spin(7)$ in 8 dimensions. We outline the construction of the first known examples of compact 7- and 8-manifolds with holonomy G_2 and $Spin(7)$.
In the case of G_2, we first choose a flat G_2-structure on the torus T^7, preserved by a finite group G of diffeomorphisms of T^7. Then T^7/G is an *orbifold*. Secondly, we resolve the singularities of T^7/G to get a compact 7-manifold M. Finally we use analysis, and an understanding of Calabi-Yau metrics, to construct a family of metrics with holonomy G_2 on M, which converge to the singular metric on T^7/G.

1991 MS Classification: 53C15, 57S20, 58F99, 58F15
Labourie, François, Université Paris-Sud, France
Large Group Actions on Manifolds

In dynamical systems, one studies flows (actions of $\mathbb{R}$) or diffeomorphisms (actions of $\mathbb{Z}$). It is of course more difficult to construct an action of a general group on a manifold, with a dense orbit for instance. For finitely generated groups, the presence of many relations between the generators is obviously a strong constraint for both the existence and the deformation of an action. This elementary observation suggests that
Large group actions should be rigid (i. e. hard to deform).
An even more optimistic hope, which may work for special classes of groups (*e. g.* $SL(n,\mathbb{Z})$, for $n \geq 3$), is the following
Large group actions on a compact manifold should preserve a rigid geometric structure, at least on some open dense set.

By a geometric structure, one means a k-rigid A-geometric structure in the sense of M. Gromov. The essential feature of such a structure is that a germ of a diffeomorphism preserving it, is completely determined by its derivatives up to order k. A connection, a non degenerate metric or a

projective structure definesa rigid structure, while a symplectic or a complex structure do not. To preserve a flat connection, is much the same as to be linearisable, a classical theme of dynamical systems.
To preserve a rigid geometric structure on an open dense set is a strong property as the following theorem of M. Gromov [G] shows.
Theorem *If the pseudo-group of germs of isometry has a dense orbit, then it has an open dense orbit.*
An isometry of a structure is just a diffeomorphism preserving the structure. This theorem implies that a group action with a dense orbit and preserving a geometric structure on some open dense set is, up to a technical subtelty, locally modelled on an homogeneous situation. Therefore, finding an invariant geometric structure by a group action is an important step towards a description of this action. Actually, there is no such a thing as a definition

of a large group action. It is a rather vague and versatile notion. But, somehow, we have examples. Initiated by R. Zimmer's work [Z], a well studied class of large group actions is the case of volume preserving ergodic actions of Lie groups of real rank greater than 2, or lattices in these (*e. g* $SL(n,\mathbb{Z})$ for $n \geq 3$). Since then, however, other situations and groups have been studied, and we should not restrict ourselves to this class. E. Ghys and V. Sergiescu advocate for instance the case of the Thompson group [Gh-S].

Let's now give some examples of actions of lattices, where, to avoid technical definitions, we emphasize the case of $SL(n,\mathbb{Z})$.
1- The action of $SL(n,\mathbb{Z})$ on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. This action preserves a flat connection and is volume preserving and ergodic. From the dynamical point of view, an important feature is the presence of numerous Anosov diffeomorphims.
2- The action of $SL(n,\mathbb{Z})$ on the sphere S^n. This action preserves a projective structure but no measure.
3- Let G be a semisimple group containing a subgroup isomorphic to $SL(n,\mathbb{R})$. All the non compact subgroups of G, and in particular $SL(n,\mathbb{Z})$, will be acting on the right, ergodically and preserving a volume form on G/L, where L is a lattice in G. This action also preserves the Killing form of G. So far, all these actions preserve a geometric structure on the whole

manifold. A new class of example have been introduced by A. Katok and J. Lewis using the following simple but useful technique [K-L].
4- The action of $SL(n,\mathbb{Z})$ on T^n has a fixed point, namely 0. We can now blow up this point, that is replace it by the projective space of its tangent space. One easily checks that $SL(n,\mathbb{Z})$ still acts on this manifold although the geometric structure is just preserved on an open dense set. Other very interesting examples have been constructed by similar techniques.

In the case of lattices, R. Zimmer's version of G. Margulis's superrigidity is a good support for both the rigidity question and the existence of an invariant geometric structure. For other groups, we have a series of positive answers for the rigidity question. In my talk, I will try to survey these theorems (due to many mathematicians) concerning analytic actions, Anosov actions, rigidity properties, and the topological structure of manifolds admitting actions of large groups.

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1991 MS Classification:

Joachim Lohkamp, Universität Augsburg, Germany

to be announced

1991 MS Classification: 53 C 42

Pinkall, Ulrich, TU Berlin

Quaternionic line bundles and the differential geometry of surfaces

In this talk, which is based on joint work with Dirk Ferus and Franz Pedit, we discuss an extension of classical Riemann surface theory to what could be called "quaternionic Riemann surface theory". This extension provides a new approach to the differential geometry of surfaces immersed conformally in 3- and 4-space.

Meromorphic functions on a Riemann surface M are quotients of holomorphic sections of complex holomorphic line bundles. From the viewpoint of differential geometry they can be regarded as branched conformal immersions into the Riemann sphere. Thus, as surfaces in 3-space, they all are just the round 2-sphere. To obtain more general (branched) conformal immersions of M into 3-space we replace the complex line bundles over M by quaternionic line bundles and take quotients of "quaternionic holomorphic" sections.

The quaternionic line bundles L in this theory are all direct sums $L = E \oplus E$ where E is a complex line bundle over M. A "quaternionic holomorphic" structure on such a line bundle is given by a first order linear operator $D = d + Q$ on smooth sections of L. Here d is (the direct sum) of a $\bar{\partial}$-operator on the complex line bundle E, i. e., a holomorphic structure on E, and Q is a quaternionic linear endomorphism of L anti-commuting with the complex structure on L. The endomorphism Q is zero if and only if $L = E \oplus E$ holomorphically, in which case one simply deals with the usual theory of complex holomorphic line bundles. The length $|Q|^2$ is a 2-form and can be integrated over M to give a measure $||Q||^2$ of how far

away the quaternionic holomorphic structure is from a complex holomorphic structure.
Any conformal immersion f into 3-space induces a quaternionic holomorphic line bundle so that f itself becomes a quaternionic holomorphic section. The endomorphism Q is related to the Hopf differential and $||Q||^2$ is the Willmore energy of f. Conversely, the quotients of quaternionic holomorphic sections of a quaternionic holomorphic line bundle give rise to (branched) conformal immersions into 3 and 4-space.
Enough of the basic results survive to preserve the flavour of the classical algebraic geometry of Riemann surfaces:

1. Riemann-Roch holds for quaternionic holomorphic line bundles.

2. Quaternionic holomorphic sections have isolated zeros and vanish like $z^m + O(|z|^{m+1})$. Thus we can define the vanishing order.

3. If ϕ is a non-trivial quaternionic holomorphic section of L then $\pi \deg L + ||Q||^2 \geq \pi Z$ where Z is the sum of the vanishing orders of ϕ. Equality holds if and only if the quaternionic holomorphic structure comes from a complex holomorphic structure on L^{-1}.

4. There is a correspondence between quaternionic holomorphic line bundles over M and special maps f into quaternionic projective space. These maps, called quaternionic holomorphic, are such that projections to quaternionic lines (4-spheres) are conformal.

Using the above properties we discuss several geometric results. The first is an analogue of the fact that any rational curve in complex projective space of degree 1 is a projective line: given a quaternionic holomorphic map of a 2-sphere into quaternionic projective space of degree 1 for which $||Q||^2 < 3\pi$ then the image of such a map must lie in a quaternionic line.
The second result is a rigidity result for immersed 2-spheres in 3-space: two immersed 2-spheres whose mean curvature half densities $H|df|$ are the same and whose Willmore energy is less then 16π are congruent up to scale. The bound 16π is sharp.
Finally we point out how the various “Weierstrass representations” for conformally immersed surfaces in 3-space can be obtained from special quaternionic line bundles: on the quaternionic spin bundle $L = S \oplus S$, where S is a complex spin bundle over M, we consider quaternionic holomorphic structures D for which Q is real. Then any quaternionic holomorphic section ϕ of L squares to a closed R^3-valued 1-forms ϕ^2 on M which can be integrated (up to translational periods) to a conformal immersion f of M.
More applications include results concerning Willmore surfaces, isothermic surfaces and the Bonnet problem.

1991 MS Classification:
Leonid Polterovich, School of Mathematical Sciences, Tel Aviv University, Israel

Geometry on the Group of Hamiltonian Diffeomorphisms

The group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\Omega)$ of a symplectic manifold (M,Ω) plays a fundamental role both in geometry and classical mechanics. For a geometer, at least under some assumptions on the manifold M, this is just the connected component of the identity in the group of all isometries of the symplectic structure Ω. ¿From the mechanical point of view, $\mathrm{Ham}(M,\Omega)$ is the group of all admissible motions. It was discovered by H. Hofer (1990) that the group of Hamiltonian diffeomorphisms carries a natural Finsler metric with non-degenerate distance function. Intuitively speaking, the distance between a given Hamiltonian diffeomorphism f and the identity transformation is equal to the minimal amount of energy required in order to generate f. This geometry has been intensively studied for the past 8 years in the framework of modern symplectic topology. It serves as a source of refreshing problems, and gives rise to new methods and notions. Also, it opens up the intriguing prospect of being able to use an alternative geometric intuition in Hamiltonian dynamics. In the talk I am going to focus on the following aspects of these developments.

- A paradox: locally the space $\mathrm{Ham}(M,\Omega)$ looks like a flat one, while globally that is not the case.

- A conjecture: the diameter of $\mathrm{Ham}(M,\Omega)$ is infinite. I shall discuss some existing methods, including a recent proof of this conjecture for $M = S^2$.

- Length spectrum and a symplectic version of Yang-Mills theory.

- Asymptotic geometric invariants.

- Applications to dynamics and ergodic theory.

1991 MS Classification: 58xx
Ruan, Yongbin, University of Wisconsin-Madison, USA

Quantum cohomology and its applications

In a few years, quantum cohomology has grown from an infant in physics to young adult in mathematics with relations to different fields such as symplectic topology, algebraic geometry, quantum and string theory, integrable systems and gauge theory. The development in last few years has been explosive. Now, the foundations have been systematically studied and the ground is secure. As a young adult, quantum cohomology is looking forward to a steady and productive life ahead. Therefore, now is a particularly good time to assess the developments of last few years and make some proposals for the future. Quantum cohomology is such a diverse field that it is impossible to make a complete survey in 45 minutes. I will make no attempt to do so. I will refer B. Dubrovin's lecture in this congress for the algebraic side of the story.

The devoplement of quantum cohomology has roughly two distint periods: an early pioneer period and more recent period of technical sophistication. The main focus of early period was to establish a mathematical foundation for quantum cohomology. It was accomplished by author and G. Tian for semi-positive symplectic manifolds, a class which contains the most of interesting examples for applications such as Fano and Calabi-Yau manifolds. Another equally important activity of this period was the exploration of relation of quantum cohomology with other fields, like symplectic topology, string theory, gauge theory and birational geometry. These explorations lay down a foundation for future applications. I will give a rather detailed survey of this period.
The recent period focused on removing the semi-positivity condition: this was done recently by an array of authors. The main characteristic of this period is increasingly technical sophistication. In my view, the development of these new techniques is probably more important than the goal of removing semi-positivity condition itself. Therefore, I will mostly survey the main techniques in this period.
The most sucessful application of quantum cohomology right now is in enumerative geometry. Due to my limited expertise on this subject, I will leave it to the experts to give a proper survey. Instead, I will focus on other aspect of applications, for examples, functorial properties of quantum cohomology, application to symplectic topology and birational geometry. I believe that there is a great potential lying on these applications.

6. Topology

1991 MS Classification: 20, 53, 54, 55

Dranishnikov, Alexander, University of Florida/Steklov Math. Institute, USA

Dimension theory and large Riemannian manifolds

Dimension theory of compacta has a counterpart in coarse geometry. M. Gromov defined the notion of asymptotic dimension of a metric space and showed that hyperbolic groups have this dimension finite. Using Roe's version of the descent principle, G. Yu proved that the asymtotical finite dimensionality implies the Novikov conjecture for geometrically finite groups. This finite dimensionality problem in coarse geometry turns out to be quite parallel to the Alexandroff problem from the classical dimension theory. A connection between large scale dimension of metric spaces and covering dimension of Higson corona will be presented.

1991 MS Classification: 55

Dwyer, William G., University of Notre Dame, USA

Lie groups and p-compact groups

A *p-compact group* is the homotopical ghost of a compact Lie group; it is the residue that remains after the geometry and algebra have been stripped away. This talk will sketch the theory of p-compact groups, with the intention of illustrating the fact that many classical structural properties of compact Lie groups depend only on homotopy theoretic considerations.

1991 MS Classification: 57R55

Fintushel, Ronald, Michigan State University, USA;
Stern, Ronald J., University of California at Irvine, USA

Knot Surgery on 4-Manifolds

In this talk we shall discuss geometric operations on a given smooth 4-manifold which preserve the underlying topological structure and alter its smooth structure. In particular, we shall consider the following construction: Let X be a simply connected smooth 4-manifold which contains a smoothly embedded torus T of self-intersection 0. Given a knot K in S^3, we replace a tubular neighborhood of K with $S^1 \times (S^3 \setminus K)$ to obtain the *knot surgery manifold* X_K.

More formally, this procedure is accomplished by performing 0-framed surgery on K to obtain the 3-manifold M_K. The meridian m of K can be viewed as a circle in M_K; so in $S^1 \times M_K$ we have the smooth torus $T_m = S^1 \times m$ of self-intersection 0. Since a neighborhood of m has a canonical framing in M_K, a neighborhood of the torus T_m in $S^1 \times M_K$ has a canonical identification with $T_m \times D^2$. The knot surgery manifold X_K is given by the fiber sum

$$X_K = X \#_{T=T_m} S^1 \times M_K = (X \setminus T \times D^2) \cup (S^1 \times M_K \setminus T_m \times D^2)$$

where the two pieces are glued together so as to preserve the homology class $[\mathrm{pt} \times \partial D^2]$. This latter condition does not, in general, completely determine the diffeomorphism type of X_K, and X_K is taken to be any manifold constructed in this fashion.

We shall discuss consequences of this knot surgery construction. For example, under relatively mild hypotheses on (X, T), X_K will be homeomorphic to X, and its Seiberg-Witten invariant (viewed as a Laurent polynomial) will be $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t)$, where $\Delta_K(t)$ is the symmetrized Alexander polynomial of K in the variable $t = \exp(2[T_m])$. As a consequence, given (X, T) satisfying the hypotheses, this produces infinitely many nondiffeomorphic 4-manifolds, all homeomorphic to X. In particular, this works for $(K3, \text{elliptic fiber})$. If the knot K is fibered, and X is symplectic with T a symplectically embedded torus, then the knot manifold X_K is also symplectic. However, if the Alexander polynomial of K is not monic, the knot manifold X_K supports no symplectic structure.

We shall also use a similar technique to construct exotic embedded surfaces in 4-manifolds, and discuss what is known about geography of smooth 4-manifolds.

Reference: R. Fintushel and R. Stern, *Knots, links, and 4-manifolds*, to appear in Invent. Math.

1991 MS Classification: 57-XX Manifolds and cell complexes; 68-XX Computer Science; 81-XX Quantum Theory

Freedman, Michael H. ; Microsoft Corporation, Redmond, WA, U. S. A.

Topological views of computational complexity

Occasionally problems of fundamental importance arrive in mathematics from another field; usually from physics (differential equations, indefinite metrics, operator theory, generalized functions, attractors, instantons, strings); once from philosophy (self-reference); and now from computer science. I refer to the basic problem of computational complexity, most notably the conjecture that $P \neq NP$. This is essentially the assertion that many standard decision problems, such as the existence of a 3-coloring of a graph, can not be decided on by a computation whose time grows only polynomially in the size of the instance (the graph in this example). A proof of this conjecture would be truly remarkable, as it must foreclose myriad yet-unconceived approaches. Once appreciated, this thought has led many to wonder if the conjecture is undecidable.

The notion of computational "time" is at least slightly model dependent, and the incorporation of quantum mechanical principles appears to allow models of computation which are dramatically faster—at least on certain problems—so P/NP becomes a family of related questions depending on the computational model. Mathematicians should think of the P/NP questions, not only as *problems* within mathematics, but as a *threat* to our practice of mathematics, for the unexpected outcome would imply the existence of a rapid, as opposed to exhaustive, algorithm for finding short proofs—the only kind of proof people can find anyway.

The lecture will consist of observations which bring the perspective of topology to computational complexity. Possible topics include:

- A heuristic view of "decidability" in terms of "general position".
- Logical and coloring problems as examples of TQFTs (Topological Quantum Field Theories).
- TQFTs as a possible bridge between physics and computation. The "observables" of the Witten-Jones Chern-Simons theory are $\#P$-hard to compute, so if they could be instead measured (efficiently), $\#P$-hard problems could be solved (rapidly). ($\#P$ is a class of problems with natural number answers, e. g., the number of 3-colorings of a graph, analogous to, but presumably more difficult than, NP.)
- "Finite-type" invariants of graphs and their relation to "hard" problems. The notion of finite-type invariant can be borrowed from topology and used in many combinatirial settings.
- A partial generalization of the satisfaction problem in Boolean logic which is amenable to "homological", hence rapid, solution.

1991 MS Classification: 55Oxx
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Toward a global understanding of $\pi_*(S^n)$

Keywords and Phrases: Homotopy groups, spheres, periodic groups
Determining the homotopy groups of spheres is a problem which has occupied mathematicians for over 50 years. Early work used the fibration sequences of James at the prime 2 and Toda at odd primes. These sequences have been called EHP sequences after the names of the maps which occur. At two we have

$$\cdots \to \pi_k(S^n) \to \pi_{k+1}(S^{n+1}) \to \pi_{k+1}(S^{2n+1}) \to \pi_{k-1}(S^n) \to \cdots$$

The first map the the suspension homomorphism, the second is the Hopf invariant and the third is labeled P. The key feature of this collection of ideas is that knowing $\oplus_{j<k}\pi_{j+n}(S^n)$ for all n allowed one, in principle, to calculate $\pi_{k+n}(S^n)$. Since $\pi_*(S^1)$ is known, a group by group calculation seemed possible.
¿From this approach two things emerged. The calculation was very complicated and did not seem systematic. In particular, there was not a compelling pattern connecting adjacent groups. This is in sharp contrast to the dual problem of computing, $H^{n+k}(K(\mathbf{Z}, n))$, where $K(\mathbf{Z}, n)$ is the Eilenberg-Mac Lane space with one non-zero homotopy group. The latter computation was possible primarily because the answer was *free*. On the other hand $\pi_*(S^n)$ is not a very nice ring in any known structure.

Recent efforts have concentrated on finding subsets or quotients of homotopy groups with some global structure. One of the early results in this direction is the theorem of Serre which determines $\pi_*(S^n) \otimes \mathbf{Q}$. The next step in this direction determined all of $\pi_*(S^n)$ which is related to K-theory. This is the work of Adams, Mahowald, Davis, Thompson and Miller and was completed nearly 20 years ago. Just as $H^{n+k}(K(\mathbf{Z}, n); \mathbf{Z}/2)$ is easier to describe than these groups with integers for coefficients, so $\pi_*(S^n; \mathbf{Z}/2)$ has more structure. There is a graded ring $\mathbf{Z}[a]$ with the dimension of $a = 8$ and $\pi_*(S^n; \mathbf{Z}/2)$ is a module over this ring. The free part of this module structure is understood and this formed the key idea which allowed for the complete determination of the homotopy groups related to K-theory. In order to discuss the next case, we have to use a more complicated coefficient module. It is becoming clear what that should be. In this talk I will describe the new tools which are being used to understand the next component of homotopy and the new problems which have arisen.

1991 MS Classification: 57M
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A filtration of the set of integral homology 3-spheres

In 1989, Witten [3] proposed his famous formula of topological invariants of 3-manifolds, based on Chern-Simons gauge theory. The formula is given by using a path integral over all G connections on a 3-manifold, where G is a fixed Lie group. Following combinatorial properties of the invariants predicted by Witten's formula, the invariants, what we call the *quantum G invariant*, have been rigorously reconstructed by many researchers, say by using surgery presentations of 3-manifolds.
Since we have many Lie groups, we have obtained many quantum invariants of 3-manifolds in this decade. To control these many invariants we consider the following two approaches.

- Unify them into an invariant.
- Characterize them with a common property.

By the first approach, we expect the existence of the universal invariant among quantum invariants, though the universal quantum invariant of 3-manifolds is not found yet. Instead of universal quantum invariants, we have the universal invariant Ω among perturbative invariants of 3-manifolds, where the perturbative G invariant is obtained from the quantum G invariant by "asymptotic expansion". By the second approach, we obtain the notion of finite type such that the d-th coefficient of a perturbative invariant is of finite type of degree d. For detailed arguments see [2].
We prepare some notations. A *web diagram* on the empty set ϕ is a trivalent graph such that a cyclic order of three edges around each trivalent vertex of the graph is fixed. We denote by $\mathcal{A}(\phi; \mathbf{C})$ (resp. $\mathcal{A}(\phi; \mathbf{Z})$) the quotient vector space over $\mathbf{C}$ (resp. $\mathbf{Z}$) subject to the AS and IHX relations. We define the

degree of a web diagram to be half the number of trivalent vertices of the graph. We denote by $\mathcal{A}(\phi;\mathbf{C})^{(d)}$ and $\mathcal{A}(\phi;\mathbf{Z})^{(d)}$ their subspaces of degree d. We denote by $\hat{\mathcal{A}}(\phi;\mathbf{C})$ the completion of $\mathcal{A}(\phi;\mathbf{C})$ with respect to the degree; values of the above universal invariant Ω belong to this space $\hat{\mathcal{A}}(\phi;\mathbf{C})$. We have a map $\mathcal{A}(\phi;\mathbf{C}) \to \mathbf{C}$ obtained by substituting a Lie algebra to web diagrams; we call the map *weight system*. For precise definitions of these notations see for example [2].
For simplicity we restrict our attention to integral homology 3-spheres ($\mathbf{Z}$HS's) in the following.

Theorem 1 (see [2] for precise statement)
(1)(**Ohtsuki** for $G = SO(3)$) *The invariant Ω is universal among perturbative invariants, i. e. , the perturbative G invariant recovers from the invariant Ω by the weight system derived from substitution of the Lie algebra of G.*
(2)(**Le**) *The invariant Ω is universal among all finite type invariants i. e. , the map*

$$\{\mathbf{Z}HS\text{'}s\} \xrightarrow{\Omega} \hat{\mathcal{A}}(\phi;\mathbf{C}) \xrightarrow{proj} \mathcal{A}(\phi;\mathbf{C})^{(\leq d)} \xrightarrow{W} \mathbf{C}$$

is of finite type of degree d for any linear map W, and conversely any finite type invariant of degree d is obtained as the above composed map for some linear map W.

How fine do the invariants in Theorem 1 distinguish $\mathbf{Z}$HS's? (Since it is conjectured that the perturbative G invariant dominates the quantum G invariant, this question might be equivalent to the question: how fine do quantum invariants distinguish $\mathbf{Z}$HS's?)
We define the d-th equivalence relation $\sim_d$ among $\mathbf{Z}$HS's as follows. Two $\mathbf{Z}$HS's M and M' are *d-th equivalent*, denoted by $M \sim_d M'$, if $v(M) = v(M')$ for any finite type invariant v of degree $\leq d$. We have the ascending series of the sets of the equivalence classes;

$$\{\mathbf{Z}\text{HS's}\}/\sim_1 \longleftarrow \{\mathbf{Z}\text{HS's}\}/\sim_2 \longleftarrow \{\mathbf{Z}\text{HS's}\}/\sim_3 \longleftarrow \cdots .$$

Such equivalence relations have been originally studied by Habiro by using his "claspers"; to be precise his definition of the equivalence relations is slightly stronger than ours. Habiro showed that generators of $\{\mathbf{Z}\text{HS's}\}/\sim_{d+1}$ are given by web diagrams of degree d in some sense. As a corollary of results of Habiro [1] we have

Theorem 2 (a corollary of results in [1]) *The set $\{\mathbf{Z}HS\text{'}s\}/\sim_{d+1}$ becomes a commutative group which is isomorphic to $\mathcal{A}(\phi;\mathbf{Z})^{(\leq d)}_{conn}$, where $\mathcal{A}(\phi;\mathbf{Z})_{conn}$ denotes the subspace of $\mathcal{A}(\phi;\mathbf{Z})$ spanned by connected web diagrams. By taking the direct limit of these isomorphisms we have the following homomorphism*

$$\{\mathbf{Z}HS\text{'}s\} \longrightarrow \varprojlim_{d} \{\mathbf{Z}HS\text{'}s\}/\sim_d \cong \hat{\mathcal{A}}(\phi;\mathbf{Z})_{conn}. \tag{1}$$

It is known that there exists the invariant $\omega(M) \in \mathcal{A}(\phi; \mathbf{C})_{\mathrm{conn}}$ satisfying $\Omega(M) = \exp \omega(M)$. The map (1) gives a reconstruction of the invariant ω, which dominates all invariants in Theorem 1. If we could identify $\{\mathbf{Z}\text{HS's}\}$ with a subset of $\hat{\mathcal{A}}(\phi; \mathbf{Z})_{\mathrm{conn}}$ by the map (1), all invariants in Theorem 1 would be understood as functions of weight systems on the space $\hat{\mathcal{A}}(\phi; \mathbf{Z})_{\mathrm{conn}}$.

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1991 MS Classification:

Bob Oliver, Département de Mathématiques Université Paris-Nord, Villetaneuse, France

Vector bundles over classifying spaces

A vector bundle over a space X can be thought of as a collection of (complex) vector spaces, one for each point in X, which are combined in one topological space. For any X, let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of vector bundles over X, made into a commutative monoid by the operation of direct sum. Define $\mathbb{K}(X)$ to be the Grothendieck group of $\mathrm{Vect}(X)$; i. e., the abelian group of all formal differences $(x - x')$ for $x, x' \in \mathrm{Vect}(X)$ (with the obvious relations). When X is compact, then $K(X) \overset{\mathrm{def}}{=} \mathbb{K}(X)$, the K-theory of X, defines a cohomology theory; and in particular has nice properties such as forming exact sequences and Bott periodicity which make it (relatively) easy to calculate these groups.

When X is non-compact, and in particular when $\dim(X) = \infty$, then this is no longer the case, and the groups $\mathbb{K}(X)$ no longer define a cohomology theory. So instead, one must define $K(X)$ via classifying spaces. In general, for any topological group G, Milnor constructed a classifying space BG for G, with the property that for any space X, bundles over X with "structure group" G are in one-to-one correspondence with the set $[X, BG]$ of homotopy classes of maps $X \to BG$. In particular, if U denotes the infinite unitary group, then one defines $K(X) = [X, \mathbb{Z} \times BU]$ for arbitrary X, and this coincides with the above definition when X is compact. There is a natural homomorphism $\beta_X : \mathbb{K}(X) \to K(X)$ for each X, which is an isomorphism when X is compact, but not in general. In fact, the geometrically defined functor $\mathbb{K}(-)$ can behave very differently from $K(-)$: it is not a cohomology theory, and is in general much more difficult to compute than $K(-)$.

If G is a compact Lie group and BG is its classifying space, then $\mathbb{K}(BG)$ and $K(BG)$ can both be studied by comparison with the representation ring $R(G)$ of G. Let

$$\alpha_G : R(G) \longrightarrow \mathbb{K}(BG)$$

be the homomorphism defined by sending a complex G-representation V to its Borel construction $(EG\times_G V)$, regarded as a vector bundle over BG. Alternatively, if one thinks of a representation (with hermitian product) as a homomorphism $\rho : G \to U(n)$, then α_G sends ρ to the vector bundle classified by $B\rho : BG \to BU(n)$. The *completion theorem* of Atiyah and Segal says that

$$\beta_{BG} \circ \alpha_G : R(G) \longrightarrow K(BG)$$

extends to an isomorphism $\widehat{\alpha}_G : R(G)\widehat{\ } \xrightarrow{\cong} K(BG)$, where $R(G)\widehat{\ } = \varprojlim_n (R(G)/I^n)$ is completion with respect to the augmentation ideal $I = \operatorname{Ker}[R(G) \to \mathbb{Z}]$.

The new result which I want to describe, proven in joint work with Stefan Jackowski, is a description of the Grothendieck group $\mathbb{K}(BG)$ itself, also in terms of representations. The starting point for doing this was to consider the case where G is p-toral; i. e., an extension of a torus by a finite p-group. By theorems of Dwyer–Zabrodsky (when G is a finite p-group) and of Notbohm (when G is p-toral), $\alpha_G : R(G) \to \mathbb{K}(BG)$ is an isomorphism in this case. Hence, for an arbitrary compact Lie group G, there are homomorphisms

$$\mathbb{K}(BG) \xrightarrow{\text{restr}} \mathbb{K}(BP) \xrightarrow[\cong]{\alpha_P^{-1}} R(P),$$

for each p-toral subgroup $P \subseteq G$; and these combine to define a homomorphism

$$r_G : \operatorname{Vect}(BG) \longrightarrow R_{\mathcal{P}}(G) \stackrel{\text{def}}{=} \varprojlim_P R(P).$$

Here, the inverse limit is taken over all p-toral subgroups of G (for all primes p) with respect to inclusion and conjugation of subgroups.

Our main theorem is that r_G is an isomorphism; i. e., that $\mathbb{K}(BG) \cong R_{\mathcal{P}}(G)$, for any compact Lie group G. This is proven by first studying $\operatorname{Vect}_n(BG) \cong [BG, BU(n)]$ for each n, using techniques developed in joint work with Jim McClure, based on a decomposition of BG as a homotopy direct limit of spaces BP for p-toral $P \subseteq G$. These sets $\operatorname{Vect}_n(BG)$ are in themselves quite complicated in general (they contain non-vanishing higher derived functors of inverse limits, and can even be uncountable); but all of the "difficult" parts of $\operatorname{Vect}(BG)$ vanish when one takes the Grothendieck construction.

Among the consequences of this theorem are that β_{BG} is always injective, that α_G is surjective if G is finite or connected; and that $\operatorname{Coker}(\alpha_G)$ always

has finite exponent. In fact, $\mathbb{K}(BG) \cong R(G)$ (α_G is an isomorphism) whenever if $\pi_0(G)$ (the group of connected components of G) has the property that all of its elements have prime power order.
As one example, let G be a finite group, and let r_p (for any prime p) denote the number of conjugacy classes of elements of p-power order. Then $\mathbb{K}(BG) \cong \mathbb{Z} \times \prod_p (\mathbb{Z})^{r_p - 1}$, while in contrast $K(BG) \cong \mathbb{Z} \times \prod_p (\widehat{\mathbb{Z}}_p)^{r_p - 1}$.

1991 MS Classification: 57R55

Stern, Ronald J., University of California at Irvine, USA

Knot Surgery on 4-Manifolds

(joint talk with R. Fintushel)

1991 MS Classification: 53, 58

Taubes, Clifford H., Harvard University, USA

The Geometry of the Seiberg-Witten Invariants.

Fix an oriented, Riemannian 4-manifold X with a Spin$^{\mathbb{C}}$ structure. (The set of Spin$^{\mathbb{C}}$ structures is a canonical affine space for the second cohomology.)The Seiberg-Witten equations constitute a canonical system of differential equations on X. If X is compact and has non-negative definite intersection form, an algebraic count of the solutions gives a differential invariant of X. (Witten conjectured that these invariants reproduce the Donaldson invariants.) If X is symplectic, these invariants can be computed by an algebraic count of pseudo-holomorphic submanifolds. When X is not symplectic, the metric still provides a self-dual, harmonic 2-form. This form is symplectic away from its zero set, Z. And, the Seiberg-Witten invariants count certain pseudo-holomorphic submanifolds in $X - Z$ with boundary Z.
References: C. H. Taubes, The Seiberg-Witten and the Gromov invariants, *Math. Res. Letters* **2** (1995), 221–238; C. H. Taubes, Seiberg-Witten invariants and pseudo-holomorphic subvarieties for self-dual, harmonic 2-forms, preprint 1998.

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7. Lie Groups and Lie Algebras

1991 MS Classification: 11R39, 22E55
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Towards a stable trace formula

Suppose that G is a connected, reductive group over a number field F. For example, we could take F to be $\mathbb{Q}$, and G to be a representative from one of the four infinite families $GL(n)$, $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$. The trace formula for G is an identity that relates geometric objects with dual spectral objects. The geometric objects are attached to conjugacy classes in $G(F)$, and are closely related to harmonic analysis. The spectral objects are attached to automorphic representations, and are really the ultimate objects of study. They carry fundamental arithmetic information, that is believed to hold the answers to some of the deepest questions in number theory.

It is hard to extract arithmetic information from the trace formula by studying it in isolation. There are simply too many geometric terms. One should try instead to compare it with trace formulas for other groups. Such a comparison could lead to fundamental reciprocity laws between automorphic spectra on different groups. In the case of the classical groups $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$, it would provide the means for attacking a basic classification problem, that of reducing the representation theory of these groups to the much better understood theory of $GL(n)$.

The groups to be compared with a given G are the *endoscopic* groups for G. They form a family of quasisplit groups $\{G'\}$ that have many conjugacy classes in common with G. It is actually only the stable conjugacy classes –equivalence classes in $G(F)$ defined by conjugacy over an algebraic closure $\bar{F}$ — that can be transferred from G to other groups G'. On the other hand, the geometric terms in the trace formula are not stable distributions. They depend on more than just the averages of a test function over stable conjugacy classes. The basic problem can then be formulated as that of stabilizing the trace formula. It entails the construction of a stable trace formula –a refined identity whose individual terms are stable distributions. It also includes the further analysis required to establish identities between terms in the original trace formula and their stable counterparts on endoscopic groups.

The lecture will be a general discussion of questions relating to the construction and analysis of a stable trace formula. I can say nothing about the fundamental lemma, which is one of the key problems to be resolved. However, many other analytic and combinatorial problems have been solved. I shall describe the relationships of these problems to the local and global terms in the trace formula.

1991 MS Classification: 22E46 22E30 43A70 46E35 46E39
Bernstein, Joseph, Dept. Math. Tel Aviv University, Israel
Analytic structures on representation spaces of reductive groups

1. Let G be a real reductive group (for example, $G = GL(n, \mathbf{R})$). We would like to study analytic structures which are naturally defined on a representation (π, G, V) of the group G.

In order to explain what we are after let us consider first a model case. Namely, suppose we are given a C^∞ manifold M and a C^∞ vector bundle E on M and we would like to study possible analytic structures on the space V of the sections of E.
We may consider many different analytic structures on V: smooth sections , L^2 sections (or more generally, L^p sections), different kinds of Sobolev spaces of sections, of Besov spaces of sections and so on.

There is a convenient way to represent all these structures. Namely, we fix the space $V = V^\infty$ of smooth sections and study different topologies $\mathcal{T}$ on this space. For a given topology $\mathcal{T}$ we can then complete the space V with respect to $\mathcal{T}$.
For example, while the space V does not have canonical structure of a (pre)Hilbert space it clearly has a canonical structure of a Hilbertian space, i. e. a topological vector space with topology given by a norm N such that N^2 is a Hermitian form on V.
For every real number s we also can consider a topology $\mathcal{T}_s$ defined by a Sobolev norm S_s on the space V; the completion is the Sobolev space of sections V_s.

2. Now let us come back to the case of a representation (π, G, V). We assume that this representation is admissible, i. e. its restriction to the maximal compact subgroup $K \subset G$ has finite multiplicities. For simplicity of presentation let us also assume that this representation is irreducible.
By remarkable theorem of Casselman and Wallach there exists a canonical "smooth" version of this representation $(\pi, G, V = V^\infty)$. We would like to discuss some natural topologies which can be canonically constructed on V.

Consider as an example the simplest case $G = SL(2, \mathbf{R})$. In this case (for generic V) the space V can be realized as the space D_λ of smooth functions on the punctured plane $\mathbf{R}^2 \setminus 0$ homogeneous of degree $\lambda - 1$. Restricting to a unit circle $S^1 \subset \mathbf{R}^2 \setminus 0$ we get a morphism $\nu_\lambda : V \to \mathcal{F}$, where $\mathcal{F} = C^\infty(S^1)$ is the space of functions on the unit circle. Thus we can define s Sobolev structure on the space V using the norm $N_s(v) = S_s(\nu_\lambda(v))$.

The problem is that the space V has two natural realizations, as D_λ and as $D_{-\lambda}$, and the norms N_s obtained from these realizations are not equivalent. However, it is not difficult to check, that if we define the Sobolev norm N_s on the space V to be $N_s(v) = S_{s+r}(\nu_\lambda(v))$, where $r = \mathrm{Re}(\lambda)$, then the corresponding topology on the space V does not depend on the realization. This allows us to define a canonical s-Sobolev structure on the space V. In particular for $s = 0$ we get a canonical Hilbertian structure on V.

3. Let us discuss the case of a general reductive group G. For simplicity assume that G is split (e. g. $G = SL(n, \mathbf{R})$). Then again we can consider a series of representations $(\pi_\lambda, G, D_\lambda)$ parametrised by characters λ of the Cartan subgroup $A \subset G$; each of this representations can be realized in the space $\mathcal{F}$ of functions on the flag variety $X = G/B$, where B is a Borel subgroup.

We claim that the space $\mathcal{F}$ has canonical system of Sobolev norms S_s parametrized by points s of the $\mathbf{R}$ linear space $\mathfrak{a}^* = \mathrm{Mor}(A, \mathbf{R}^+)$; this space is dual to the Cartan subalgebra $\mathfrak{a} = \mathrm{Lie}(A)$.
Now, similarly to the $SL(2)$ case, we can define s Sobolev norm on the space V as $N_s(v) = S_{s+r}(\nu_\lambda(v))$, where $r = \mathrm{Re}(\lambda)$.

This construction defines a canonical system of Sobolev structures on a representation V of the group G which is parametrized by points $s \in \mathfrak{a}^*$ (at least for generic V). In particular, we define a canonical Hilbertian structure on V. In order to show that this definition extends to all admissible representations we use alternative definition of these structures based on the assymptotics of matrix coefficients.

Remarks. 1. Similar considerations suggest a conjecture that one can canonically define on V also L_p structures, L_p Sobolev structures and Besov structures.
2. If we consider representations of a p-adic reductive group G then we will find exactly the same analytic structures. They are parametrized by points s of the real vector space $\mathfrak{a}^* = \mathrm{Mor}(A, \mathbf{R}^+)$.

1991 MS Classification:
Ivan Cherednik, The University of North Carolina at Chapel Hill, USA
From double Hecke algebra to analysis

() The starting point of many mathematical and physical theories is the formula:

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2k} \mathrm{dx} \ = \ \Gamma(k+1/2), \ \Re k > -1/2. \qquad (1)$$

Let us give some examples.
(a) Its direct generalization to the Bessel functions, namely, the invariance of the Gaussian e^{-x^2} with respect to the Hankel transform, is a cornerstone of the Plancherel formula.
(b) The deformation of this formula

$$\int_{-\infty}^{\infty} (e^{x^2} - 1)^{-1} x^{2k} \mathrm{dx} \ = \ \Gamma(k+1/2)\zeta(k+1/2), \ \Re k > 1/2, \qquad (2)$$

is fundamental in the analytic number theory.

(c) Replacing x^{2k} by $\sinh(x)^{2k}$, we come to the theory of spherical and hypergeometric functions and to the spherical Fourier transform. The spherical transform of the Gaussian plays an important role in the harmonic analysis on symmetric spaces.
To employ modern mathematics at full potential, we do need to go from Bessel to hypergeometric functions. In contrast to the former, the latter can be studied, interpreted and generalized by a variety of methods in the range from representation theory and algebraic geometry to integrable models and string theory. However straightforward substituting $x^{2k} \to \sinh(x)^{2k}$ creates problems. The formula (1) has no sinh-counterpart, the Gaussian looses its Fourier-invariance, and the spherical transform is not selfdual anymore.
Difference setup. It was demonstrated recently that we can save these and related properties switching to the truncated theta-function:

$$\mu_k(x;q) \ = \ \prod_{j=0}^{\infty} \frac{(1-q^{j+2x})(1-q^{j+1-2x})}{(1-q^{j+k+2x})(1-q^{j+k+1-2x})}, \ 0<q<1, \ k \in \mathbf{C}. \tag{3}$$

Actually it is a unification of $\sinh(x)^{2k}$ and the Harish-Chandra function (A_1) serving the inverse spherical transform. So it is not surprising that the μ-transform is selfdual and has other remarkable properties. For instance, setting $q = \exp(-1/a), \ a > 0$,

$$(-i)\int_{1/4-\infty i}^{1/4+\infty i} q^{-x^2}\, \mu_k \mathrm{dx} = \sqrt{a\pi} \prod_{j=1}^{\infty} \frac{1-q^{j+k}}{1-q^{j+2k}} \tag{4}$$

provided that $\Re k > -1/2$. The limit of (4) multiplied by $(a/4)^{s-1}$ as $a \to \infty$ is precisely the imaginary variant of (1).
Once we managed Γ, it would be unexcusable not to try

$$\widetilde{\zeta}_q(s) = \ (-i)\int_{1/4-\infty i}^{1/4+\infty i} (q^{x^2}-1)^{-1}\mu_k \mathrm{dx} \text{ for } k = s-1/2, \ \Re s > 0. \tag{5}$$

The limit of $(a/4)^{s-1}\widetilde{\zeta}_q(s)$ as $a \to \infty$ is $\Gamma(s)\zeta(s)$ for $\Re s > 1$. In the critical strip, it is different:

$$\lim_{a\to\infty} a^{2s-2}\widetilde{\zeta}_q(s) \ = \ -\cot(\pi s)\Gamma^2(s-1/2) \ \text{ for } 0 < \Re s < 1. \tag{6}$$

However the zeros of ζ have q-deformations with interesting properties.
Double Hecke algebras provide justifications and generalizations. In the A_1-case, $\mathcal{H}$ is the quotient of the group algebra of

$$\Pi_q = \langle T, X, Y, q^{1/4}\rangle, \ TXTX = 1 = T^{-1}YT^{-1}Y, \ Y^{-1}X^{-1}YXT^2 = q^{1/2}, \tag{7}$$

by $(T-t^{1/2})(T+t^{-1/2})=0$. So the definition involves central q,t (and their roots). Renormalizing $T\to q^{1/4}T,\ X\to q^{-1/4}X,\ Y\to q^{1/4}Y^{-1}$,

$$\Pi_q\cong\Pi_1 \mod \langle q^{1/4}\rangle,\ \Pi_1\cong\pi_1(\{E\times E\setminus \mathrm{diag}\}/\mathbf{S}_2),\ E=\text{ elliptic curve,} \tag{8}$$

a special case of the calculation due to Birman.
Thanks to the topological interpretation, $PSL_2(\mathbf{Z})$ acts projectively on Π_1 and $\mathcal{H}$. For instance, the automorphism τ sending $Y\to q^{1/4}XY$ and fixing X,T,q,t corresponds to the standard parabolic generator.
The automorphism τ becomes the conjugation by the q-Gaussian q^{-x^2} in the following $\mathcal{H}$-representation on functions of x :

$$T\to t^{1/2}s+(q^{2x}-1)^{-1}(t^{1/2}-t^{-1/2})(s-1),\ X\to q^x,\ Y\to spT \tag{9}$$

for the reflection $sf(x)=f(-x)$ and the translation $pf(x)=f(x+1/2)$. The Fourier transform is associated with the anti-involution $\{\sigma: X\to Y^{-1}\to X\}$ preserving T,t,q.
Diagonalizing Y and its multi-dimensional counterparts we come to the Macdonald polynomials. Combining τ and σ, we prove the Fourier-invariance of these polynomials multiplied by the Gaussian and get (4). Switching to the roots of unity, we generalize the Verlinde algebras and the Gauss sums. Last but not the least, we obtain the classical p-adic theory of spherical functions (Iwahori, Matsumoto, Macdonald) as $q\to 0$.

1991 MS Classification: 22 11
Eskin, Alex, University of Chicago, USA
Counting problems and semisimple groups

Some natural counting problems admit extra symmetries related to actions of Lie groups. For these problems, one can sometimes use ergodic and geometric methods, and in particular the theory of unipotent flows, to obtain asymptotic formulas.
We will present counting problems related to diophantine equations, diophantine inequalities and quantum chaos, and also to the study of billiards on rational polygons.

1991 MS Classification: 22
Kottwitz, Robert, University of Chicago, USA
Harmonic analysis on semisimple p-adic Lie algebras

Certain topics in harmonic analysis on semisimple groups arise naturally when one uses the Arthur-Selberg trace formula to study automorphic representations of adele groups. These topics, which fall under the heading of “comparison of orbital integrals, ” have been surveyed in Waldspurger’s article in the proceedings of ICM94. As in Harish-Chandra's work, many questions in harmonic analysis on the group can be reduced to analogous questions on its Lie algebra, and in particular this is the case for "comparison of orbital integrals. " We will discuss some recent work of this type.

1991 MS Classification:

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Chtoucas de Drinfeld et applications

Etant donné X une courbe projective lisse géométriquement connexe sur un corps fini $\mathbb{F}_q$, F son corps des fonctions et $\mathbb{A}$ l'anneau des adèles de F, Drinfeld a défini les chtoucas de rang r sur X dans le but de réaliser dans la cohomologie de leurs schémas classifiants la correspondance de Langlands entre représentations automorphes cuspidales de $\mathrm{GL}_r(\mathbb{A})$ et représentations irréductibles de dimension r du groupe de Galois de F. Ce programme a été entièrement réalisé par lui dans le cas du rang $r = 2$.
Les schémas (ou plutôt les champs au sens de Deligne–Mumford) Cht^r de chtoucas de Drinfeld de rang r sont lisses sur $X \times X$ de dimension relative $2r - 2$ mais ils ne sont pas quasi–compacts ni même leurs composantes connexes. Pour tout sous–schéma fermé fini $I \hookrightarrow X$, il y a au–dessus de Cht^r un champ Cht_I^r représentable fini étale galoisien classifiant les chtoucas avec structures de niveau I. Les Cht_I^r sont munis de correspondances de Hecke associées aux éléments de l'algèbre de convolution de $\mathrm{GL}_r(\mathbb{A})$ et aussi de "morphismes de Frobenius partiels" au–dessus des endomorphismes $\mathrm{Frob} \times \mathrm{Id}$ et $\mathrm{Id} \times \mathrm{Frob}$ de $X \times X$.

Dans le cas $r = 2$, Drinfeld a défini dans un premier temps certains ouverts de type fini dans les Cht_I^2. Ces ouverts ne sont stables ni par les correspondances de Hecke ni par les morphismes de Frobenius partiels mais on peut malgré tout considérer leur cohomologie à supports compacts et l'action sur elle de l'endomorphisme de Frobenius. En combinant le théorème des points fixes de Grothendieck, le théorème de pureté de Deligne et la formule des traces de Selberg, Drinfeld a ainsi pu démontrer la conjecture de Ramanujan–Petersson pour les représentations automorphes cuspidales de $\mathrm{GL}_2(\mathbb{A})$.
Dans un second temps, Drinfeld a construit des compactifications de ces ouverts de type fini. Ces compactifications retrouvent une action des correspondances de Hecke et des morphismes de Frobenius partiels et elles sont reliées entre elles par des projections birationnelles. La correspondance de Langlands pour GL_2 a alors été établie par Drinfeld en faisant le lien avec la formule des traces de Selberg.

Considérant maintenant un rang r quelconque, la notion de polygone canonique de Harder–Narasimhan nous a permis de définir dans les Cht_I^r des ouverts de type fini qui généralisent ceux de Drinfeld en rang 2. Par combinaison avec une formule des traces d'Arthur–Selberg convenable, nous avons pu démontrer le résultat suivant :

THÉORÈME. — (i) *Soit* π *une représentation automorphe cuspidale irréductible unitaire de* $\mathrm{GL}_r(\mathbb{A})$. *Alors :*
- *Si* r *est impair,* π *vérifie la conjecture de Ramanujan–Petersson (c'est–à–dire que pour toute place* x *de* F *où le facteur* π_x *de* π *est non ramifiée, toutes les valeurs propres de Hecke* $z_i(\pi_x)$, $1 \leq i \leq r$, *sont de module* 1) *et on pose* $\varepsilon_\pi = 0$.
- *Si* r *est pair, ou bien* π *vérifie la conjecture de Ramanujan–Petersson et on pose* $\varepsilon_\pi = 0$, *ou bien pour toute place* x *où* π *est non ramifiée, la moitié parmi les* $z_i(\pi_x)$, $1 \leq i \leq r$, *sont de module* $q^{\frac{1}{4}\deg(x)}$ *et l'autre moitié de module* $q^{-\frac{1}{4}\deg(x)}$ *et on pose* $\varepsilon_\pi = \frac{1}{4}$.

(ii) *Pour deux telles représentations* π, π' *de* $\mathrm{GL}_r(\mathbb{A})$, $\mathrm{GL}_{r'}(\mathbb{A})$, *tous les zéros de la fonction de Rankin–Selberg* $L(s, \pi \otimes \check{\pi}')$ *sont sur la droite* $\mathrm{Re}\, s = \frac{1}{2}$ *si* $\varepsilon_\pi = \varepsilon_{\pi'}$ *et sur les droites* $\mathrm{Re}\, s = \frac{1}{4}$, $\mathrm{Re}\, s = \frac{3}{4}$ *si* $\varepsilon_\pi \neq \varepsilon_{\pi'}$.

Après cela, nous avons construit des compactifications des ouverts de type fini "naturels" dans les Cht^r généralisant celles de Drinfeld en rang 2 et plus récemment nous avons pu définir des structures de niveau arbitraire au–dessus de ces compactifications (en passant par la construction de compactifications équivariantes des quotients $\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$).

1991 MS Classification: 20E06, 20E08, 20E32, 20F32, 22D40, 22E40
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Products of trees, lattices and simple groups

The group of automorphisms of a locally finite tree, denoted $\mathrm{Aut}\, T$, is a locally compact group which exhibits behaviour analogous to that of a rank one simple Lie group. This analogy has motivated many recent works, in particular the study of lattices in $\mathrm{Aut}\, T$ by Bass, Kulkarni, Lubotzky and others. Recall that in the case of semisimple Lie groups, irreducible lattices in higher rank groups have a very rich structure theory and one encounters many deep and interesting phenomena such as (super)rigidity and arithmeticity. Motivated by this we study in several joint works with Marc Burger and with Robert J. Zimmer cocompact lattices in the group of automorphisms of a product of trees or rather in groups of the form $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ where each T_i, $i = 1, 2$ is a (bi-)regular tree. The results obtained concerning the structure of lattices in $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ enable us to show the existence and construct the first examples of finitely presented torsion free simple groups, see [1].

A lattice $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ is reducible if its projections on each factor $\mathrm{Aut}\, T_i$ are discrete. Denoting the closures of the projections of Γ by $H_i = \overline{\mathrm{pr}_i(\Gamma)}$, $i = 1, 2$, we observe that for cocompact lattices $H_i \neq \mathrm{Aut}\, T_i$. We show that imposing various "largeness" conditions on the groups H_i, $i = 1, 2$ allows one to obtain a rich structure theory for these lattices. A (closed) subgroup $H < \mathrm{Aut}\, T$ will be called *locally primitive* if for every vertex $x \in T$ its stabilizer in H acts as a primitive permutation group on the set of adjacent edges.

The following result may be viewed as an "arithmeticity theorem".

Theorem 1, [2]. *Let $\Gamma < Aut\, T_1 \times Aut\, T_2$ be an irreducible cocompact lattice. Let $H_i = \overline{pr_i(\Gamma)}$ be the closures of the projections, $i = 1, 2$. Assume that each $H_i < Aut\, T_i$ is locally primitive. Then one of the following possibilities holds:*
(1) Every linear image of Γ is finite.
(2) Γ has an infinite linear image over a field of characteristic 0. Then H_i is a p_i-adic analytic group for some prime p_i. The adjoint map yields a continuous surjection from $H_1 \times H_2$ onto a semisimple Lie group over some local fields and the image of Γ is an arithmetic lattice. Moreover the kernel of this homomorphism is a torsion free discrete subgroup of $H_1 \times H_2$.
(3) Γ has an infinite linear image over a field of positive characteristic p. Then there is a continuous map with unbounded image from $H_1 \times H_2$ into a simple Lie group over $F_p((t))$.

A related result is a geometric rigidity theorem asserting that if $\Gamma < \mathrm{Aut}\,(T_1) \times \mathrm{Aut}\,(T_2)$ is a cocompact lattice with locally primitive projections and Γ is isomorphic to a lattice $\Gamma' < \mathrm{Aut}\,(T_1') \times \mathrm{Aut}\,(T_2')$ then the isomorphism is induced by an isometry between $T_1 \times T_2$ and $T_1' \times T_2'$.
Further results, jointly with Marc Burger, study the normal subgroups structure of irreducible lattices in $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$. Again one needs to put conditions on the "size" of the projections of the lattice. For a locally compact group H let us denote by $H^{(\infty)}$ the intersection of all its cocompact closed normal subgroups. Assuming that the closures of the projections of an irreducible lattice act transitively on the boundary of the tree we have the following analog of Margulis' normal subgroup theorem:

Theorem 2, [1]. *Let $\Gamma < Aut\, T_1 \times Aut\, T_2$ be a cocompact lattice such that $H_i := \overline{pr_i(\Gamma)}$ acts transitively on ∂T_i and $H_i^{(\infty)}$ is of finite index in H_i, $i = 1, 2$. Then, any nontrivial normal subgroup of Γ is of finite index.*

Unlike finitely generated linear groups which are necessarily residually finite, lattices in $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ may be non residually finite. We establish a criteria for ensuring that certain lattices are not residually finite. Using this criteria together with theorem 2 we show

Theorem 3, [1]. *For every pair (n, m) of sufficiently large even integers there exists a finite square complex $\mathcal{Y}_{n,m}$ whose universal covering is the product $\mathcal{T}_n \times \mathcal{T}_m$ of regular trees of respective degrees n and m and whose fundamental group $\Gamma_{n,m}$ (which is a cocompact lattice in $Aut\, \mathcal{T}_n \times Aut\, \mathcal{T}_m$) satisfies:*
(1) The group $\Gamma_{n,m}$ is simple, finitely presented and torsion free.
(2) $\Gamma_{n,m} \equiv F *_G F$ where F, G are finitely generated free groups, $[F : G] \in \{n, m\}$.
(3) If $\Gamma_{n,m}$ is isomorphic to $\Gamma_{k,l}$, then the corresponding complexes $\mathcal{Y}_{n,m}$ and $\mathcal{Y}_{k,l}$ are isomorphic, and $\{n, m\} = \{k, l\}$.
(4) The cohomological dimension of $\Gamma_{n,m}$ is 2.
(5) $\Gamma_{n,m}$ is an automatic group.

We remark that the question whether amalgams $F *_G L$ of free groups can be simple was raised by P. M. Neumann [4], [3].

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1991 MS Classification:
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On representations of Lie superalgebras

We give a brief review and a summary of recent results concerning finite-dimensional representations of classical Lie superalgebras. The examples of these superalgebras are general linear Lie superalgebra $gl(m,n)$ and orhosymplectic Lie superalgebra $osp(m,n)$.
We present a solution to the problem of calculating characters of irreducible representations. In order to do this one needs to define Kazhdan-Lusztig polynomials for the category of finite-dimensional representations over classical Lie superalgebras and find an algorithm for their calculation. The last can be done using a geometric approach involing cohomology of G-sheaves on a compact homogeneous supermanifold G/P for different parabolic subgroups P. In particular we explain an analogue of Borel-Weil-Bott theorem for flag supemanifold.
Finally we discuss an unusual tensor product structure on the category of finite dimensional representations for the family of superalgebras $gl(m,n)$ which is closely related to all above topics.

1991 MS Classification: 22,55,14
Vilonen, Kari, Brandeis University, USA
Topological Methods in Representation Theory

In this talk we survey topological methods, all related to Morse theory, and the way they are applied in representation theory of reductive Lie groups. The point of view we take was first proposed by Kashiwara as a series of conjectures in 1987. It extends the theory of Beilinson-Bernstein which was explained by Beilinson in his lecture at the ICM in Warsaw in 1983. The Beilinson-Bernstein theory reduces questions in representation theory

to complex algebraic geometry, whereas our point of view leads us to geometry over the reals. Our results, which constitute joint work with Wilfried Schmid, include two geometric formulas for characters of representations and a proof of the Barbasch-Vogan conjecture.

1991 MS Classification: 17B65, 17B67, 81R10

Minoru Wakimoto, Kyushu University, Japan

Representation theory of affine superalgebras at the critical level

As is known well, the representation theory of affine Lie algebras have a number of important connection with various area of mathematics and physics, while the representation of superalgebras still remain quite mysterious, and only a few has been known about it because of a serious technical and intrinsic difficulties arising in its analysis.
The subject seems quite interesting and fascinating, since the associated Macdonald identities are of different kinds from those of usual affine Lie algebras, and also the Ramanujan's famous mock theta functions take place closely related to the denominator identities for certain affine superalgebras.
In this talk, I will explain an approach to the representation theory of superalgebras and, in particular, some of interesting features of representations of affine superalgebras at the critical level so far revealed and expected.

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8. Analysis

1991 MS Classification: 30,35
Astala, Kari, University of Jyväskylä, Finland

Analytic aspects of quasiconformality

A number of different topics in geometric analysis make use of quasiconformal phenomena, i. e. properies of mappings, spaces etc. that behave quantitatively like the corresponding conformal notions but are often much more flexible to use. The areas of applications include diverse fields ranging e. g. from complex dynamics and hyperbolic geometry to partial differential equations and singular integrals.
The present talk reviews some of the recent advances in the theory of quasiconformal mappings with emphasis on the analytic aspects of the developments.

1991 MS Classification: 32, 35, 42
Christ, Michael, University of California, Berkeley, USA

Singularity and Regularity — Local and Global

For some years a central question in complex analysis in several variables was whether the Bergman projection (the orthogonal projection of square integrable functions onto holomorphic functions) preserves the class of functions smooth up to the boundary, for arbitrary smoothly bounded pseudoconvex domains. This is closely linked with the regularity of solutions of the $\bar{\partial}$–Neumann problem, a boundary value problem for a system of Laplace type partial differential equations in which the boundary conditions depend on the complex geometry of the domain's boundary. The question was finally settled in the negative, but by indirect reasoning which applied only to special examples. Despite a later simplification by Y. -T. Siu, certain more refined questions remain open.
Global regularity of the Bergman projection is related to other unresolved problems concerning the hypoellipticity and global regularity of subelliptic and more degenerate second order partial differential operators, in the real analytic, Gevrey, and smooth categories. The positive results for these problems are roughly parallel, both in scope and in the techniques from which they can be obtained. The negative analyses are more *ad hoc* though common themes do exist. Large gaps between the positive and negative results persist, except in the lowest-dimensional case.
Proofs of the positive results motivate the construction of a metric on the cotangent bundle which is somewhat related to the usual Carnot-Caratheodory metric on the base space, but differs in fundamental respects. Asymptotic properties of this metric correspond well with sharp hypoellipticity results, in cases where such results are known.
This talk will survey some aspects of these problems, emphasizing the negative results.

References: D. Barrett, Behavior of the Bergman projection on the

Diederich-Fornæss worm, *Acta Math.* **168** (1992), 1-10.
M. Christ, Global C^∞ irregularity in the $\bar{\partial}$–Neumann problem for worm domains, *Journal Amer. Math. Soc.* **9** (1996), 1171-1185.
M. Christ, The Szegö projection need not preserve global analyticity, *Annals of Math.* **143** (1996), 301-330.
M. Christ, Analytic hypoellipticity in dimension two, preprint.
M. Christ, Hypoellipticity: Geometrization and speculation, preprint.
C. Kiselman, A study of the Bergman projection in certain Hartogs domains, *Proc. Symp. Pure Math.* **52** (1991), Part 3, 219-231.
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1991 MS Classification: 19K56, 46L80
Higson, Nigel, Pennsylvania State University, USA.
The Baum-Connes Conjecture

Let G be a second countable, locally compact group and denote by $C^*_{\text{red}}(G)$ the reduced C^*-algebra of G. It is the completion in the operator norm of the convolution algebra $L^1(G)$, acting as operators on $L^2(G)$. The Baum-Connes conjecture proposes a means of calculating the C^*-algebra K-theory groups $K_*(C^*_{\text{red}}(G))$ using group homology and the representation theory of compact subgroups of G. It originates in work of Kasparov and Mishchenko on the Novikov higher signature conjecture and ideas of Connes in foliation theory. The first cases of the conjecture were settled by Pimsner and Voiculescu, who investigated free and free abelian groups, and by Connes and Kasparov, who independently considered simply connected solvable groups. These results actually preceded the current formulation of the conjecture. The conjecture is true, and essentially trivial, for compact groups.

The formulation of the Baum-Connes conjecture requires Kasparov's equivariant KK-theory, using which equivariant K-homology theory for G-spaces may be developed. If $\underline{E}G$ denotes the universal proper G-space then the conjecture asserts that a certain natural assembly map

$$\mu\colon K^G_*(\underline{E}G) \to K_*(C^*_{\text{red}}(G))$$

is an isomorphism. If G is a discrete and torsion-free group (this is the most interesting and important case) then $K^G_*(\underline{E}G)$ is the K-homology of the classifying space BG. In this situation the resulting assembly map

$$\mu\colon K_*(BG) \to K_*(C^*_{\text{red}}(G))$$

is very closely related to its namesake in manifold theory. Indeed the Baum-Connes conjecture is a close cousin of the manifold-theoretic Borel conjecture and it actually implies the Novikov conjecture. These and other relations to topology are the cause a good deal of interest in the Baum-Connes conjecture among topologists and homotopy theorists.

In recent joint work with Kasparov the author has proved the following result. A *metrically proper* action of a locally compact group on Hilbert space is a continuous action by affine isometries such that $\|g \cdot v\| \to \infty$ as $g \to \infty$, for all vectors v.

Theorem The Baum-Connes conjecture holds for all groups which admit metrically proper actions on Hilbert space.

The theorem applies to amenable groups and also unifies a good deal of what is currently known about the Baum-Connes conjecture. On the other hand it applies to no non-compact group with Kazhdan's property T. Indeed the conjecture is currently known for no such discrete group, although by invoking a great deal of tempered representation theory the conjecture has been proved by Wassermann for linear reductive Lie groups. Future progress for discrete groups will probably require a penetrating analysis of the relationship between the harmonic analysis of discrete groups and K-theory for group C^*-algebras.

1991 MS Classification: 42A50

Lacey, Michael, Georgia Institute of Technology, USA

On the Bilinear Hilbert Transform

In 1964, Alberto Calderón defined a bilinear generalization of the Hilbert transform by

$$Hfg(x) := \text{p.v.} \int f(x+y)g(x-y)\,\frac{dy}{y}.$$

He raised the striking conjecture that this operator would map $L^2 \times L^2$ into L^1, in contrast to the well–known fact that the ordinary Hilbert transform does not preserve L^1. In joint work with Christoph Thiele, a range of inequalities for this transform have been established, including Calderón's conjecture. This talk will survey our argument.

Calderón's original motivation was the so–called commutator connected to the Cauchy integral on Lipschitz curves. The study of this integral lead to the development of many aspects of the modern Calderón–Zygmund Theory, but yielded no information about the bilinear Hilbert transform. Indeed, the fundamental obstacle was that in contrast to most operators of that theory, the bilinear Hilbert transform commutes with conjugation by exponentials in a natural way: Denoting $f_\lambda(x) = e^{i\lambda x} f(x)$, one sees immediately that $Hf_\lambda g_\lambda(x) = e^{2i\lambda x} Hfg(x)$. Thus, the transform displays a great deal of sensitivity to the frequency behaviors of the transformed functions.

There is one other operator which exhibits the same kind of behavior, and it appears in a celebrated theorem of Lennart Carleson on Fourier series. This Theorem, which ironically was established in 1965, asserts the pointwise convergence of the partial Fourier sums of a square integrable function on the unit circle. The operator that must be controlled to prove this result is the maximal partial Fourier sums of square–integrable function. It is again highly sensitive to the frequency behaviors of the function under consideration.

The proof which I developed with C. Thiele can be seen as an outgrowth and reëxamination of the deep and subtle techniques invented by L. Carleson in 1965. This talk will define the Hilbert transform, recall Carleson's Theorem, elaborate on it's relationship to the bilinear Hilbert transform, and describe elements of the proof of the Calderón conjecture, which is based upon combinatorial–geometric aspects of the time–frequency plane.

1991 MS Classification: 28A75, 30E20, 42B20
Mattila, Pertti, University of Jyväskylä, Finland
Rectifiability, analytic capacity, and singular integrals

This lecture is a survey on some interplay between geometric measure theory (rectifiability), complex analysis (analytic capacity) and harmonic analysis (singular integrals). The theory we shall discuss has been mainly developed during this decade. Vaguely, it deals with the following three principles:

1. Singular integrals of degree m behave nicely on an m-dimensional subset E of $\mathbb{R}^n$ if and only if E is in some sense rectifiable.
2. The analytic capacity of a 1-dimensional compact subset E of the complex plane is zero if and only if E is purely unrectifiable.
3. The analytic capacity of a 1-dimensional compact subset E of the complex plane is positive if and only if the Cauchy singular integral operator is L^2-bounded in a large part of E.

By the m-dimensionality of E we usually mean that the m-dimensional Hausdorff measure $H^m(E)$ is positive and finite. The rectifiability of a set means that H^m almost all of it can be covered with m-dimensional C^1-surfaces. The nice behaviour of singular integrals in 1 could refer to L^2-boundedness or the almost everywhere existence of principal values. In both cases there are theorems with some extra assumptions on the set E in question. A typical here relevant singular integral operator T is that related to the Riesz kernel:

$$Tg(x) = \int_E |x-y|^{-m-1}(x-y)g(y)dH^m y, x \in \mathbb{R}$$

A compact subset of the complex plane $\mathbb{C}$ has analytic capacity zero if and only if it is removable for bounded analytic functions, almost by definition. The highlight of this topic is that the principle 2 holds very generally. That is, we have the following recent result of G. David:
Theorem. *Let E be a compact subset of $\mathbb{C}$ with $H^1(E) < \infty$. Then the analytic capacity of E is zero if and only if E is purely unrectifiable, that is, $H^1(E \cap \Gamma) = 0$ for every rectifiable curve Γ.*
The proof of this result combines principles 1 and 3: if E has positive analytic capacity, the Cauchy operator is bounded on a large part of E, by a very general $T(b)$-theorem which David proves. By principle 1 this large part is rectifiable, and so E is not purely unrectifiable.
The main tool used in the proof is the following remarkable identity relating Menger curvature to the Cauchy kernel. It was found by M. S. Melnikov for a few years ago. The Menger curvature $c(z_1, z_2, z_3)$ of the triple $z_1, z_2, z_3 \in \mathbb{C}$

is the reciprocal of the radius of the circle passing through these points. The identity is

$$c(z_1, z_2, z_3)^2 = \sum_{\sigma} \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}$$

where σ runs through all six permutations of $\{1, 2, 3\}$. Melnikov and Verdera used this identity to give a simple geometric proof of the L^2-boundedness of the Cauchy singular integral on Lipschitz graphs.
There is a very similar removability problem as 2 which concerns Lipschitz harmonic functions. In the plane the exact analogue of David's theorem holds. The results in higher dimensions on that question as well as on principle 1 are much weaker, since we don't have anything like the above identity available.
Much of the standard Calderón-Zygmund-theory of singular integrals holds for general measures which satisfy the doubling condition, as has been known for a long time. Interesting recent work of F. Nazarov, S. Treil and A. Volberg, jointly, and of X. Tolsa shows that very large part of this theory is valid without any doubling condition at all. This work has been inspired by and related to the above three principles. In fact, the method developed by Nazarov, Treil and Volberg leads to a simpler proof of David's theorem.

Here are a few of the many relevant references.

G. David, *The unrectifiable 1-sets have vanishing analytic capacity*, to appear in Rev. Mat. Iberoamericana.

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X. Tolsa, *L^2-boundedness of the Cauchy integral operator for continuous measures*, to appear in Duke Math. J.

1991 MS Classification: 46 , 52 , 60
Milman, V. D., Tel Aviv University, Israel
Randomness and Pattern in Convex Geometric Analysis

We will analyze the reasoning behind certain counterintuitive geometric phenomena that occur in high-dimensional normed spaces as well as in high-dimensional convexity theory. A perceived random nature of high dimensional spaces is at the root of the reasons I will discuss in the talk, and the patterns it produces create the unusual phenomena we observe. In reverse, when we discover similar patterns in arbitrary, and apparently very diverse, convex bodies or normed spaces of high dimension, we interpret them as a manifestation of the randomness principle mentioned above. We demonstrate this interplay with a series of recent results, essentially obtained in the last two years. A significant part of these results was done in two series of works, one mainly in collaboration with G. Schechtman and the other in collaboration with A. A. Giannopoulos.

1991 MS Classification: 42B15, 22E30, 35L05

Müller, Detlev, Math. Seminar der C. A. -Universität Kiel, Ludewig-Meyn-Str.4, D-24098 Kiel, Germany

L^p-functional calculus on Lie groups and estimates for wave equations

Consider a "sum of squares" operator

$$L = -\sum_{j=1}^{k} X_j^2$$

on a manifold M of dimension $d \geq k$, where $X_1, \ldots, X_k$ are smooth, real vector fields on M satisfying the following bracket condition: $X_1, \ldots, X_k$, together with their iterated commutators $[X_{j_1}, [X_{j_2}, [\ldots [X_{j_{\ell-1}}, X_{j_\ell}] \ldots]]]$, span the tangent space of M at every point of M. If $k = d$, then L might for example be a Laplace-Beltrami operator. If $k < d$, then L will not be elliptic, but, according to a celebrated theorem of L. Hörmander, it will still be hypoelliptic. Operators of this type arize in various contexts, for instance in higher dimensional complex analysis. Assume in addition that L is essentially selfadjoint on $C_0^\infty(M)$ with respect to some volume element dx. Then the closure of L, again denoted by L, admits a spectral resolution $L = \int_0^\infty \lambda dE_\lambda$ on $L^2(M)$, and any function $m \in L^\infty(\mathbb{R}^+)$ leads to an L^2-bounded operator

$$m(L) = \int_0^\infty m(\lambda) dE_\lambda.$$

An important question is then under which additional conditions on the spectral multiplier m the operator $m(L)$ extends from $L^2 \cap L^p(M)$ to an L^p-bounded operator, for a given $p \neq 2$. If so, m is called an L^p-multiplier for L.

Since, without additional properties of M and L, there is little hope in finding answers to this questions, we shall assume that M is a Lie group G, with right-invariant Haar measure dx, and that $X_1, \ldots, X_k$ are left-invariant vector fields which generate the Lie algebra of G.

A main objective of the talk will be to survey some of the relevant developments concerning this question, and moreover to link it to questions concerning estimates for the associated *wave equation*, more precisely the following Cauchy-problem:

$$\begin{aligned} &\left(\tfrac{\partial^2}{\partial t^2} + L\right) u(x,t) = 0 \quad \text{on} \quad G \times \mathbb{R}, \\ &u(x,0) = f(x), \ \tfrac{\partial u}{\partial t}(x,0) = 0, \end{aligned} \tag{1}$$

whose solution is given by $u(\cdot,t) = \cos(t\sqrt{L})f$.
The classical model case is the Laplacian $L = -\Delta$ on $\mathbb{R}^d$. Here, $m(L)$ is a Fourier multiplier operator $\widehat{m(L)f}(\xi) = m(|\xi|^2)\hat{f}(\xi)$, and a sufficient condition for m to be an L^p-multiplier for $-\Delta$ follows from a well-known multiplier theorem going back to J. Marcinkiewicz, S. Mikhlin and L. Hörmander. This "MMH-theorem" states that $m(-\Delta)$ is L^p-bounded for all $1 < p < \infty$ (and of weak type (1,1)), provided that m is locally differentiable of degree α (in the sense that $m \in H^{\alpha}_{loc}(\mathbb{R}^+)$) for some $\alpha > d/2$, in a certain sense "uniformly on every scale". This result is sharp with respect to the critical degree $d/2$ only for $p = 1$; for $1 < p < \infty$, less restrictive conditions follow by suitable interpolation with the trivial L^2-estimate.
Theorems of "MMH-type" have meanwhile been proved by various authors for groups G of polynomial volume growth. For instance, for stratified nilpotent groups, M. Christ has proved such a theorem under the condition $\alpha > Q/2$, where Q is the so-called homogeneous dimension of the group G. If G is non-abelian, then $Q > d$, where d is the dimension of G as a manifold. These results make use of estimates for solutions of the *heat equation* $(\frac{\partial}{\partial t} + L)u = 0$ associated to L, sometimes for complex time parameters t. More recently, it was proved by E. M. Stein and myself, and independently by W. Hebisch, that there exist non-abelian groups, the so-called Heisenberg groups, on which the MMH-theorem does in fact hold under the weaker condition $\alpha > d/2$. It is not known whether this condition is sufficient on all groups of polynomial growth.
In another recent joint article, the following estimate for solutions to (1) has been established for the so-called sub-Laplacian L on a Heisenberg group:

$$||u(\cdot,t)||_{L^1} \leq C(1+|t|)^{\alpha}||(1+L)^{\alpha/2}f||_{L^1}, \tag{2}$$

for any $\alpha > (d-1)/2$.
This is the analogue of the corresponding estimate by A. Miyachi and J. Peral for the classical wave equation on Euclidian space. By some subordination principle it implies the sharp multiplier result for Heisenberg groups. Extensions of (2) to more general classes of groups would therefore be desirable.
In the last part of the talk, I shall concentrate on the Laplacian $L = -\Delta$ on $\mathbb{R}^d$. Then $m(L)$ corresponds to the radial Fourier multiplier $\xi \mapsto m(|\xi|^2)$. For such radial multipliers and $p \neq 2$, better L^p-estimates can be proved than those obtained from interpolating the MMH-estimate with the trivial L^2-estimate, by making use of the curvature of the sphere $|\xi| = 1$. The famous *Bochner-Riesz-conjecture*, which is concerned with Fourier multipliers of the form $m_\beta(\xi) = (1-|\xi|^2)^{\beta}_{+}$, describes the expected minimal smoothness conditions for these multipliers. This conjecture is again linked to the wave equation via the stronger *local smoothing conjecture*, due to C. Sogge, according to which the solution to (1) satisfies

$$||u||_{L^{p_d}(\mathbb{R}^d \times [1,2])} \leq C||(1-\Delta)^{\varepsilon}f||_{L^{p_d}(\mathbb{R}^d)}, \tag{3}$$

for all $\varepsilon > 0$, with $p_d = 2d/(d-1)$. The analogue of (3), in which the full L^p-norm on $\mathbb{R}^d$ is replaced by a mixed $L^p(L^2)$-norm in polar coordinates, has recently been proved in joint work with A. Seeger.
It is known that the validity of the local smoothing conjecture would imply that of other outstanding conjectures in Fourier analysis, like the "restriction conjecture" or the "Kakeya conjecture".

1991 MS Classification: 35 49
Müller, Stefan, MPI for Mathematics in the Sciences, Leipzig, Germany
Unexpected solutions of first and second order partial differential equations

The talk will present a general approach to construct Lipschitz solutions of $Du \in K$, where $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}^m$ and where K is a given set of $m \times n$ matrices, based on (an extension of) Gromov's method of convex integration. One application concerns variational problems that arise in models of microstructure in solid-solid phase transitions. Another application is the systematic construction of singular solutions of elliptic systems. Specifically one has the following result.

Theorem. *There exists a* 2×2 *second order strongly elliptic system*

$$\operatorname{div} a(Du) = 0$$

that admits a nontrivial Lipschitz solution $u : \mathbf{R}^2 \to \mathbf{R}^2$ *with compact support.*

One can also contruct such 2×2 systems that arise as the Euler-Lagrange equations of a variational integral. By scaling one easily checks that there exist solutions whose singular set has almost full measure.
The talk is based on joint work with Vladimir Šverák that was supported by the Humboldt foundation through the Max Planck research prize.

1991 MS Classification: 32B10,32B15,32B20,32D15,32H35,32H40
Pinchuk, Sergey, Indiana University, USA /Chelyabinsk State Technical University, Russia
Reflection Principle in Higher Dimensions

The reflection principle will be discussed with respect to the following
Conjecture. *Let* $D, D' \subset\subset \mathbb{C}^n\,(n > 1)$ *be bounded domains with real-analytic boundaries and let let* $f : D \to D'$ *be a biholomorphic (or, more generally, proper holomorphic) mapping. Then* f *extends holomorphically to an open neighborhood of* $\overline{D}$.
For $n = 1$ this is the classical Schwartz theorem and the reflection principle is the main tool for proving it. If the boundary ∂D of $D \subset \mathbb{C}$ is locally defined by an equation $\rho(z, \overline{z}) = 0$ and $\rho(z, \overline{w})$ is the complexification of $\rho(z, \overline{z})$, then the equation $\rho(z, \overline{w}) = 0$ can be locally solved as $z = h(w)$, where h is antiholomorphic, $h(w) = w$ for $w \in \partial D$ and $h \circ h(w) = w$

for all w. This defines an antiholomorphic reflection about ∂D. If h' is a reflection about $\partial D'$, then the analytic continuation of f across ∂D is given by $\tilde{f}(w) = h' \circ f \circ h(w)$.
For $n > 1$ an analogous reflection about hypersurface ∂D cannot be immediately defined because in order to solve $\rho(z, \overline{w}) = 0$ for $z = (z_1, ..., z_n)$ one needs at least $n - 1$ more equations.
There are two main types of the reflection principle for $n > 1$: analytic and geometric.
The idea of *the analytic reflection principle* is to differentiate f near ∂D in complex tangent directions in order to obtain remaining equations. This approach strongly depends on a preliminary study of boundary behaviour of f and was realized only in special situations (for example, for strictly pseudoconvex domains).
The geometric reflection principle is based on the technique of Segre varieties. For a given point w its Segre variety is a complex hypersurface $Q_w = \{z : \rho(z, \overline{w}) = 0\}$. This method was first introduced by S. Webster. Later significant contributions were done by K. Diederich and J. E. Fornaess. It was the main tool of the recent proof of the conjecture above for $n = 2$, obtained by K. Diederich and S. Pinchuk.
The talk will be focused on the geometric reflection principle.

1991 MS Classification: 30, 42, 46

Seip, Kristian, Norwegian University of Science and Technology, Norway

Developments from Nonharmonic Fourier Series

This lecture gives a survey of a subject which has been developed from classical work on nonharmonic Fourier series. In short, we present results on sampling and interpolation for a fairly wide class of spaces of holomorphic functions. It is shown how independent ideas originating with Paley and Wiener (1934), Duffin and Schaeffer (1952), and Beurling (1959) combine with modern techniques like L^2 estimates for $\overline{\partial}$ and Muckenhoupt weights into a quite general theory. We consider some important examples, through which we convey principle ideas of the subject.
The general setting is as follows. We are given a weighted L^p space of holomorphic functions defined on some domain Ω in the complex plane. We denote this space by $\mathcal{B}$ and assume that the functional of point evaluation $f \mapsto f(z)$ is bounded for each $z \in \Omega$. The norm of this functional is called the majorant of $\mathcal{B}$, and it is denoted by $M(z)$. If $\mathcal{B}$ is a Hilbert space, then $M(z) = \sqrt{K(z, z)}$, where $K(z, \zeta)$ is the reproducing kernel of the space. We say that a sequence of distinct points $\Lambda = \{\lambda_k\}$ in Ω is a set of sampling for $\mathcal{B}$ if $\|f\|_{\mathcal{B}} \simeq \|\{f(\lambda_k)/M(\lambda_k)\}\|_{\ell^p}$ for $f \in \mathcal{B}$, meaning that the ratio between the two norms is bounded from above and below by positive constants, independently of f. We say that Λ is an interpolating sequence for $\mathcal{B}$ if the interpolation problem $f(\lambda_k) = a_k$ has a solution $f \in \mathcal{B}$ whenever $\{a_k/M(\lambda_k)\} \in \ell^p$. Finally, we say that Λ is a complete interpolating sequence for $\mathcal{B}$ if it is both sampling and interpolating. Our aim is to give characterizations of such sequences.

Set $\Phi(z) = \log M(z)$, which is seen to be a subharmonic function. A crucial role is then played by its Laplacian $\Delta\Phi$, which induces density conditions for sampling and interpolating sequences. This statement will be made precise through a number of examples. Sharp density conditions of this kind were first stated by Beurling (1959), and we call such results Beurling-type density theorems.

We consider first situations in which complete interpolating sequences exist. The prime example is the Paley-Wiener space L^p_π $(1 < p < \infty)$, which consists of all entire functions of exponential type at most π, whose restrictions to the real line are in $L^p(\mathbb{R})$. The case $p = 2$ corresponds to the problem first studied by Paley and Wiener, namely that of describing unconditional bases of exponential systems for $L^2(-\pi, \pi)$. A characterization of complete interpolating sequences for L^2_π is given in work by B. S. Pavlov (1979), N. K. Nikolskii (1980), and A. M. Minkin (1992). The techniques of these authors are linked to the Hilbert space geometry of L^2; a simpler approach, yielding also a characterization in the general case $1 < p < \infty$ as well as for weighted spaces as described below, has later been given by Yu. I. Lyubarskii and the author (1997). In the latter work, the description of complete interpolating sequences is turned into a question about boundedness of the Hilbert transform in weighted L^p spaces of functions and sequences.

A remarkable extension of Fourier analysis and in particular the theory of Paley-Wiener spaces is given by L. de Branges in his book "Hilbert spaces of entire functions" (1968). For all de Branges-type spaces there exist complete interpolating sequences, in fact, there exist orthonormal bases of normalized reproducing kernels. In general, however, it is difficult to describe all complete interpolating sequences for such spaces.

We consider weighted spaces of de Branges-type, and put a certain regularity condition on the weight. To begin with, let $\mathcal{H}$ be a Hilbert space of entire functions whose norm is given by

$$\|f\|_w^2 = \int_{-\infty}^{\infty} |f(x)|^2 w(x)\,dx,$$

where w is a positive weight function. Assume further that $\mathcal{H}$ is closed under the operations $f(z) \mapsto \overline{f(\overline{z})}$ and $f(z) \mapsto f(z)(z - \overline{\zeta})/(z - \zeta)$, provided $f(\zeta) = 0$. It turns out that a natural regularity condition to be put on the weight is $w(x)M^2(x) \simeq 1$. If this holds, we say that w is a majorant weight. Then the following theorem holds: *The positive function w is a majorant weight if and only if w has the form*

$$\log w(x) = u(x) + g(x) - \int_{-\infty}^{\infty} (\log|1 - x/t| + (1 - \chi_{[-1,1]}(t))m(t)dt,$$

where $u \in L^\infty$, g is a real entire function, and $m(x) \simeq 1$. (The function g is an inessential part of the weight, because a replacement of w by we^{-g} corresponds to multiplying all functions in $\mathcal{H}$ by a factor $e^{g/2}$.) If $g(x) = ax$ for some real a, we say that $\mathcal{H}$ is a weighted Paley-Wiener space. It is seen that adding a positive constant b to m does not change the weight. The numbers a and b (the latter plays a role as "type") yield a natural parametrization of all weighted Paley-Wiener spaces associated with the weight w. Examples are provided, and we discuss some basic properties of weighted Paley-Wiener spaces. We show how characterizations of sampling, interpolating, and complete interpolating sequences extend to such spaces. Here the density function m plays the leading role, as it is in essence the Laplacian of $\log M$ along the real line.

Finally, we discuss the situation for spaces of entire functions which satisfy $\Delta\Phi(z) \simeq 1$ (weighted Fock spaces), and spaces of holomorphic functions in the unit disk which satisfy $\Delta\Phi(z) \simeq (1-|z|^2)^{-2}$ (weighted Bergman spaces). It is shown that for such spaces there are no complete interpolating sequences, a fact which permits complete descriptions of sampling and interpolation in terms of densities only.

The talk is based on material from the following references, as well as some unpublished joint work with Yurii Lyubarskii and Joaquim Ortega-Cerdà:

B. Berndtsson and J. Ortega-Cerdà, Interpolation and sampling in Hilbert spaces of analytic functions, *J. Reine Angew. Math.* **464** (1995), 109–120.

Yu. I. Lyubarskii and K. Seip, Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt's (A_p) condition, *Rev. Mat. Iberoamericana* **13** (1997), 361–376.

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1991 MS Classification: 35L05, 35R05, 35S30, 35S50

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Wave Equations with Low Regularity Coefficients

To a Riemannian manifold (M,g) is naturally associated the wave equation $\partial_t^2 u = \Delta_g u$, where Δ_g is the Laplace-Beltrami operator on M. For many applications it suffices to study this equation in local coordinates, in which it takes the form

$$\partial_t^2 u(t,x) \;=\; \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_{x_i}\big(\sqrt{g}\; g^{ij}(x)\;\partial_{x_j} u(t,x)\big)\,.$$

In case the metric coefficients $g^{ij}(x)$ are smooth functions of x, the local behavior of solutions is well understood through geometric optics and the Lax construction of the wave group for g. Such techniques, which lead to the use of Fourier integral operators, can be exploited to obtain space-time L^p norm estimates on solutions u which are important in the study of nonlinear versions of the wave equation. Examples include the Strichartz estimates, and the more recent null-form estimates of Klainerman-Machedon.

Much less is known in the case where the metric coefficients $g^{ij}(x)$ possess few derivatives with respect to x. For example, the following question is of interest: how many derivatives do the $g^{ij}(x)$ need in order to insure that, for example, the Strichartz estimates hold? We show that a sufficient condition is that the coefficients $R_{ijkl}(x)$ of the Riemann curvature tensor associated to g be bounded measurable functions; this is equivalent to uniform bounds on the sectional curvatures of (M, g). Counterexamples in joint work with C. Sogge show that L^q boundedness of the curvature is not sufficient if $q < \infty$.

The construction of the wave group which establishes this result is based upon a decomposition of phase space that has long been used to understand the behavior of oscillatory integral / Fourier integral operators on the L^p spaces. We show how this dyadic / parabolic decomposition, coupled with modified paraproduct techniques, leads to an appropriate geometric optics picture of solutions to wave equations for metrics with bounded curvature tensor. This construction is necessarily less precise than for smooth metrics, yet is sufficient to obtain basic estimates of interest for nonlinear wave equations.

1991 MS Classification: 46B07 46B20 46B09

Tomczak-Jaegermann, Nicole, Edmonton, Canada

From finite- to infinite-dimensional phenomena in Geometric Functional Analysis, on local and asymptotic levels

We shall present several directions in which close connections can be found between the geometry and linear-metric structure of infinite-dimensional Banach spaces and finite-dimensional phenomena. The geometry of convex bodies in $\mathbb{R}^n$ has a fundamental influence on structural phenomena in random quotients of the corresponding normed spaces; and this in turn can be used in infinite-dimensional constructions with various non-local properties. Other directions use, to various extents, a logical connection between the finite-dimensional (or local) structure of a Banach space and its infinite-dimensional structure.

Pairs of finite-dimensional spaces with unusually large norms of isomorphisms were discovered by Gluskin in the early 80's (presented at the ICM '86); these were random quotients of ℓ_1^n of dimension $[n/2]$, say; and extensive investigations of this class of spaces by a group of researchers followed. We shall discuss analogous phenomenona for general normed spaces. Let $X = (\mathbb{R}^n, K)$ be an n-dimensional normed space with unit ball K. A normed space $E = (E, P_E(K))$, where E is a "random" $[n/2]$-dimensional subspace of $\mathbb{R}^n$ and $P_E(K)$ is the image of K under the projection onto E, turns out to have very few well-bounded operators, depending on the volume of K. Precise quantitative formulations are asymptotic, as $n \to \infty$. On the other hand, the volume of K also controls "closeness" of $P_E(K)$ to an ellipsoid. This indicates a type of curious dichotomy for general random spaces.

We shall show how the above phenomena can be used in the constructions of infinite-dimensional Banach spaces that do not admit continuous operators, or sequences of operators, with natural (algebraic) properties. In fact, such constructions require just a few canonical operations, starting from an arbitrary Banach space, non-isomorphic to ℓ_2. This in particular leads to characterizations of a Hilbert space.
Another point of view that we shall discuss depends on stabilization of the structure of a given Banach space X over infinite-dimensional subspaces. For example, if X has a basis, the asymptotic structure of X comprises all finite-dimensional subspaces spanned by blocks of the basis living sufficiently far along the basis. So called asymptotic-ℓ_p spaces, for which the asymptotic structure is the simplest possible, generalize classical spaces ℓ_p, and they may still have an inherently different (and more interesting) geometry.
As discovered in a series of deep results presented at the ICM '94 in the talks by Gowers and by Odell and Schlumprecht, infinite-dimensional phenomena in general do not stabilize. Thus, a bridge between finite- and infinite-dimensional properties of Banach spaces is formed by localizing natural structural invariants to families of finite subsets of $I\!N$ with increasing complexity. As well, Gowers' Dichotomy Theorem may immediately reduce initially purely infinite-dimensional problems to local considerations.

1991 MS Classification: 42B 20, 42B 25, 11L 15
Wainger, Stephen, University of Wisconsin, Madison
Discrete Analogues of Singular and Maximal Radon Transforms

For each x in R^ℓ, let $\gamma(x,t)$ be a smooth k-dimensional surface through x in R^k. That is $\gamma(x,t)$ is a smooth mapping from $R^\ell \times R^k$ into R^ℓ with $\gamma(x,0) = x$. Let $f(x)$ be a function on R^ℓ. Then for suitably small η we define the singular and maximal Radon transforms respectively as

$$Sf(x) = \int_{|t|\leq\eta} f(\gamma(x,t))K(t)dt$$

and

$$Mf(x) = \sup_{0<\epsilon\leq\eta} \frac{1}{|B(\epsilon)|} \int_{|t|\leq\epsilon} |f(\gamma(x,t))|dt$$

where K is a Calderón-Zymund kernel on R^k, and $|B(\epsilon)|$ denotes the measure of the ball of radius ϵ in R^k. The following is a rough statement of the result we have in mind for singular and maximal Radon transforms.

Theorem 1: (Christ, Nagel, Stein and Wainger) S and M are locally bounded in L^p, $1 < p < \infty$, provided $\gamma(x,t)$ satisfies an appropriate curvature hypothesis.

Corollary: If $\gamma(x,t)$ is real analytic

$$\lim_{\epsilon\to 0} \frac{1}{|B(\epsilon)|} \int_{|t|<\epsilon} f(\gamma(x,t))dt \to f(x) \text{ a. e.}$$

for f in $L^p(R^\ell)$.

We are interested in discrete analogues of Theorem 1: We let P be a polynomial mapping from $R^\ell \times R^k \to R^\ell$ with integer coefficients. Denote by Z^ℓ and Z^k the lattice points in R^ℓ and R^k, that is points with integer coordinates. For $m \in Z^\ell$, set

$$\mathcal{S}f(m) = \sum_n{}' f(P(m,n))K(n)$$

where the prime denotes the term $n = 0$ is to be omitted, and

$$\mathcal{M}f(m) = \sup_N \frac{1}{N^k} \sum_{|n|\leq N} |f(P(m,n))|.$$

We are interested in estimates for $\mathcal{S}$ and $\mathcal{M}$ on $\ell^p(\mathbb{Z}^\ell)$. There are the following results:

Theorem 2: (Arkhipov and Oskolkov) (1987) If

$$P(m,n) = m + Q(n)$$

$$\|\mathcal{S}f\|_{\ell^2} \leq A\|f\|_{\ell^2}.$$

Theorem 3: (Bourgain) (1988-1989) If

$$P(m,n) = m + Q(n)$$

$$\|\mathcal{M}f\|_{\ell^p} \leq A\|f\|_{\ell^p}, \quad 1 < p \leq \infty.$$

Theorem 4: (Stein and Wainger) (1990) If

$$P(m,n) = m + Q(n)$$

$$\|\mathcal{S}f\|_{\ell^p} \leq A\|f\|_{\ell^p}, \frac{3}{2} < p < 3.$$

Theorem 5: (Stein and Wainger) (1997) Let

$$m = (m', m'') \quad \text{with} \quad m' \in z^k, m'' \in z^{\ell-k}.$$

If

$$P(m,n) = (m' + n, m'' + Q(m', n))$$

$$\|\mathcal{S}f\|_{\ell^2} \leq A\|f\|_{\ell^2}.$$

A related theorem is due to M. Wierdl.

Theorem 6: (1988) Let p_n denote the nth prime, and for m in Z set

$$Mf(m) = \sup_R \frac{1}{R} \sum_{n=1}^{R} |f(m - p_n)|.$$

Then

$$\|Mf\|_{\ell^p} \leq A\|f\|_{\ell^p}, 1 < p \leq \infty,$$

Theorems 3) and 6) have applications to ergodic theory.
In our lecture we shall discuss the difficulties that arise in attacking the discrete problems and how methods of analytic number theory are used in the attack.

9. Ordinary Differential Equations and Dynamical Systems

1991 MS Classification: 58F03, 58F23

de Melo, Welington, IMPA, Brazil

Renormalization and Rigidity in one dimensional smooth dynamical systems

If two smooth unimodal maps or real analytic critical circle maps have the same bounded combinatorial type then there exists a $C^{1+\alpha}$ diffeomorphism conjugating the two maps along the corresponding critical orbits for some $\alpha > 0$. The proof is based on the detailed understanding of the orbit structure of an infinite dimensional dynamical system, the renormalization operator.

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1991 MS Classification: 34, 58, 81

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Reducibility and point spectrum for linear quasi-periodic skew-products

A linear quasi-periodic skew-product is a dynamical system

$$(X, \theta) \to (A(\theta)X, \theta + \omega),$$

where $A(\theta)$ is a matrix valued function on the torus $\mathbb{T}^d$ and ω is a vector in $\mathbb{R}^d$. $A(\theta)$ may or may not be given as a "time-1-map" of a system of differential equations.

If $\{n\omega\}_{n\in\mathbb{Z}}$ is discrete in $\mathbb{T}^d$, the periodic case, then this system behaves basically as a constant system — it is *reducible* in some sense. In particular, there are no solutions decaying both in forward and backward time. If we think of the dynamical system as an operator on doubly infinite sequences in l^2 this means that 0 is not a *point eigenvalue* and, hence, there are no *point spectrum* at all.

If $\{n\omega\}_{n\in\mathbb{Z}}$ is dense in $\mathbb{T}^d$ the situation is very different and far from being completely understood. Best understood is when ω is Diophantine, i. e.

$$|< k, \omega >| \ \geq \ \frac{\kappa}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus 0,$$

for some constants $\kappa > 0$ and $\tau > d-1$. Then the system can be studied by perturbation theory of KAM-type and it has been known since the 60's that "most" systems sufficiently close to constants are reducible. These results

did not however apply to all, not even to "almost all", systems close to constants.
In the late 80's a perturbation theory was developed for the other extreme. Fröhlich-Spencer-Wittver and Sinai, independently, were able to prove that certain Schrödinger equations

$$-(u_{n+1}+u_{n-1})+V(\theta+n\cdot\omega)u_n \;=\; Eu_n$$

sufficiently far from constants had pure point spectrum. This implies that the corresponding skew-product

$$A(\theta) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & V(\theta)-E \end{pmatrix}$$

have point eigenvalues for all values of E in the spectrum.
In recent years these methods have been improved and, in particular, $SL(2,\mathbb{R})$ — related to the the Schrödinger equation — and $SO(3,\mathbb{R})$ have been well studied. We now have a fairly complete picture of the "close-to-constant" situation: the reducible systems are typical in measure theoretical sense — "almost all" systems are reducible; non-reducible systems are typical in a topological sense — generic systems are non-reducible. In $SO(3,\mathbb{R})$ the generic system is even uniquely ergodic as a dynamical system on $SO(3,\mathbb{R})\times\mathbb{T}^d$.
In the "far-from-constant" situation the picture is less complete but also here there has been some progress: pure point spectrum is typical for a very large class of Schrödinger equations when $d=1$.
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R. Krikorian Réductibilité presque partout des systèmes quasi-périodiques dans le cas $SO(3,R)$ 1039-1044C. R. Acad. de Paris, Série I **321** (1995)
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1991 MS Classification: 58F10
Hayashi, Shuhei, Waseda University, Japan
Hyperbolicity, stability and the creation of homoclinic points

A homoclinic point is a point in the intersection of stable and unstable manifolds of the same hyperbolic fixed point (or periodic point). There are two main results. The first is a perturbation result (the connecting lemma) to create homoclinic points, whose orbits (homoclinic orbits) make complex dynamic behavior and hyperbolicity. As an application of this perturbation technique, we get, for stable vector fields, the separation of singularities from periodic points, which is the crucial step in the proof of the Stability Conjecture (raised by J. Palis and S. Smale in the late 1960's) for flows. This is the second result and, as consequence, it turns out that two concepts, hyperbolicity and stability are essentially equivalent to each other for C^1 dynamical systems of a compact manifold. Moreover we consider an application toward the solution of a conjecture by J. Palis saying that it is a generic property of diffeomorphisms on a compact manifold to have either simple dynamics or homoclinic orbits. This is a step to understand nonhyperbolic dynamics in line with Palis' program.

1991 MS Classification:
Herman, Michael R., CNRS, Paris
Some open problems in dynamical systems

We will state some open problems in various domains of dynamical systems: Siegel singular disks—Invariant tori for monotone twist maps and monotone symplectic diffeomorphisms—Measure-preserving diffeomorphisms of 2–manifolds—Entropy and exponents—Existence of periodic orbits—Instabilities of Hamiltonian flows on $T^*\mathbf{T}^n$, $n \geq 3$, and the problem of topological stability—The oldest problem in dynamical systems (Newton, Lagange, Laplace, etc.): the topological stability of a classical system of N particles under universal gravitation.
We will quickly survey some of the known results.

1991 MS Classification: 58F11, 60J05, 58F15
Kifer, Yuri, Hebrew University of Jerusalem, Israel
Random Dynamics and its Applications

It is not always realistic to describe evolution of a physical system by repeated applications of exactly the same transformation. Thus the need to study actions of compositions of different maps (time dependent transformations) emerges in a natural way. The framework of random transformations is quite appropriate for this purpose if we have in mind results which hold true for a typical in some sense sequence of maps. Stochastic flows appearing as solutions of stochastic differential equations give another important motivation to deal with random transformations.
The set up consists of a probability space $(\Omega, \mathcal{A}, P)$ together with a $P-$preserving invertible map $\theta : \Omega \to \Omega$ and of a measurably depending on ω family of (in general, different) spaces X^ω together with a family of transformations $F_\omega : X^\omega \to X^{\theta\omega}$. The object of study is the action of compositions $F^n_\omega = F_{\theta^{n-1}\omega} \circ \cdots \circ F_{\theta\omega} \circ F_\omega$. The main notions of deterministic dynamics such as invariant measures and invariant sets are being replaced here by random "invariant" measures μ^ω on X^ω and random "invariant" sets $U^\omega \subset X^\omega$ satisfying $F_\omega \mu^\omega = \mu^{\theta\omega}$ and $F_\omega U^\omega = U^{\theta\omega}$. A suitable analogy of the Kolmogorov-Sinai entropy, called fiber or relativized entropy, appears in this set up, as well. In the case of smooth random transformations further developments lead to random "invariant" subbundles and submanifolds, the stochastic bifurcation theory, Lyapunov exponents and their relations with the fiber entropy, the dimension theory of random "invariant" sets and measures etc.

Among most interesting questions emerging in this framework is whether for a "typical" ω the compositions $F^n_\omega, n = 1, 2, ...$ act in a chaotic (stochastic) fashion. A similar question can be asked for compositions $F^n = F_{n-1} \circ \cdots \circ F_1$ from a given sequence $F_1, F_2, ...$ of deterministic transformations. Familiar signs of stochastic behavior are the central limit theorem, the law of iterated logarithm, large deviations type results etc. Namely, given a random measure μ^ω and a random function $g_\omega = g_\omega(x)$ with $\int g_\omega d\mu^\omega = 0$ consider $S_\omega(n, x) = \sum_{k=0}^{n-1} g_{\theta^k\omega}(F^k_\omega x)$ and veryfy whether for a typical ω the $\mu^\omega-$distribution (in $x \in X^\omega$) of $n^{-1/2}S_\omega(n, x)$ is close to normal, $S_\omega(n, x)$ has the order of $(2n \log\log n)^{1/2}$ for $\mu^\omega-$almost all x and the $\mu^\omega-$measure of points x for which $|n^{-1}S_\omega(n, x)| > \delta > 0$ is exponentially small in n.
This sort of stochastic behavior can be obtained for some classes of random transformations such as random expanding maps, random subshifts of finite type and certain random hyperbolic diffeomorphisms. For these transformations a version of random thermodynamic formalism can be developped yielding a rich collection of random "invariant" Gibbs measures μ^ω, in particular, a random analogy of the Sinai-Ruelle-Bowen measure, which describe the stochastic behavior indicated above.
The theory has nice applications to randomly evolving graphs $G(\omega)$ which are determined by a random incidence matrix $\Pi(\omega) = (\pi_{ij}(\omega), i = 1, ..., m(\omega); j = 1, ..., m(\theta\omega))$ so that $i_0, i_1, ..., i_n$ is a path in the graph $G(\omega)$ if $\pi_{i_k i_{k+1}}(\theta^k\omega) = 1$ for all $k = 0, ..., n-1$. Then the spaces X^ω above coincide with the sets of all infinite (one or two sided) paths in $G(\omega)$ with F_ω acting as the left shift. Employing the thermodynamic formalism for random subshifts of finite type it is possible to study statistical characteristics of paths in $G(\omega)$ with specific properties.
Random versions of Perron-Frobenius theorem for random positive matrices and Markov chains in random dynamical environments are also closely related to this topic. The latter is connected also with random walks on groups with stationary changing distributions which leads to random "harmonic" functions and random boundaries. Existence of nonconstant bounded random "harmonic" functions depends on positivity of certain fiber entropy type quantity which emerges also in computations of Hausdorff dimensions of random "harmonic" measures.
Another application of this theory yield computations of fractal dimensions of random sets U^ω. Consider, for instance, a random positive integer valued function $m = m(\omega)$ and the corresponding "random base" expansion of numbers $x \in [0, 1]$ given by $x = \sum_{k=0}^\infty (m(\omega)m(\theta\omega)\cdots m(\theta^k\omega))^{-1}x_k$, $x_k \in \{0, 1, ..., m(\theta^k\omega) - 1\}$. Specifying digits which can appear in such expansions or prescribing frequencies of digits we obtain random Cantor and other sets U^ω which are "invariant" under F_ω given by $F_\omega x = m(\omega)x$ (mod 1). The Hausdorff and other dimensions of such random sets can be obtained via random ergodic theory characteristics of F_ω.
Other applications of the above set up include some statistical mechanics models such as spin glasses type systems etc.

1991 MS Classification: 34, 35, 76
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Towards a Qualitative Theory of Hamiltonian PDEs

In our lecture we review some recent results on qualitative theory of Hamiltonian PDEs and related equations, assuming that the space-variable x belongs to a bounded domain (so-called "finite volume case"). We treat the equations as dynamical systems in a functions space formed by functions of x.

We start with equations which are Hamiltonian perturbations of an integrable PDE (such as the Korteweg-de Vries equation). Due to the finite-volume assumption we have imposed, these PDEs can be written as infinite chains of Hamiltonian equations. The most famous result on qualitative behaviour of a *finite* Hamiltonian chain, close to integrable, is the KAM-theorem. The theorem deals with solutions of a perturbed system which behave in the same way as solutions of the unperturbed integrable one and proves that these solutions occupy most part of the phase space when the perturbation tends to zero. This theorem admits a non-trivial and non-straightforward version, a KAM theorem for PDEs, which states that: *solutions of a perturbed integrable PDE, behaving in an integrable fashion, are asymptotically dense in a function space when the perturbation tends to zero.*

Closures in a function phase-space of solutions, constructed by the KAM theorem for PDEs, form finite-dimensional tori, invariant for the perturbed equation. They manifest rigidity of this equation's flow. In the finite-dimensional case *all* Hamiltonian equations possess some nontrivial rigidity, first observed by M. Gromov: the flow-maps of the equations preserve a non-trivial characteristic of subsets of the phase space, known as the symplectic capacity. In particular, the flow-maps cannot send a big ball into a narrow cylinder since symplectic capacity of the former is bigger than of the latter; this is Gromov's non-squeezing property. To discuss whether this property remains true for Hamiltonian PDEs, we first have to understand which norm among infinitely many non-equivalent norms of the space of functions should be used to measure the ball's radius. The right norm, a "Darboux norm of the function space", turns out to be a norm of low smoothness (in all examples it is weaker than the C^1-norm). Hence, *Gromov's property holds true for the flow, formed by generalised solutions of a Hamiltonian PDE, provided that the flow is well defined.* A flow formed by *classical* solutions acts in a space, formed by sufficiently smooth functions. For this flow Gromov's property is wrong. This happens due to an important phenomenon, discussed below.

Let us consider the following class of PDEs:

$$
\begin{aligned}
&\langle\text{non-linear homogeneous Hamiltonian equation}\rangle \\
&\qquad + \langle \delta_1\text{-small linear damping}\rangle \\
&\qquad + \langle \delta_2\text{-small linear dispersion}\rangle = \langle\text{order-one forcing}\rangle. \qquad (1)
\end{aligned}
$$

If $\delta_1 = 0$, then this equation is Hamiltonian. Still, the most important are equations (1) with small non-zero δ_1 since exactly equations of this form describe turbulence in different physical media. Let us denote $\delta = (\delta_1^2 + \delta_2^2)^{1/2}$. The following property holds for a large class of equations (1): *solutions of* (1) *develop a short space-scale of size* $\delta^\gamma, \gamma > 0$. The exponent γ may be estimated from below and from above. This property admits an elegant reformulation in spectral terms, related to some classical heuristic results from the theory of turbulence. Proof of this short-scale behaviour of solutions is based on a "non-Gromov" property of the flow, defined by equation (1) with zero r. h. s. in the space of smooth functions: *in finite time, the flow pulls the whole phase-space through its very narrow subset, formed by fast oscillating functions.*

The three groups of results sketched above are surrounded by an ocean of open problems. Some of them will be discussed in the lecture.

1991 MS Classification: 58, 57

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Counterexamples to the Seifert Conjecture

In 1950, H. Seifert proved the existence of a closed orbit for a flow which in some sense approximates the Hopf fibration on the 3-dimensional sphere. This result gave rise to the question as to whether every vector field on S^3 must have a periodic orbit. The assertion that such an orbit always exists became later known as the Seifert conjecture and was believed to be true until a surprising counterexample was given by P. A. Schweitzer. The idea was to locally perturb a non-singular vector field having a discrete collection of circular orbits to eliminate these circles without forming new ones. Schweitzer achieved that by constructing an aperiodic C^1 plug that allows control over the global configuration of orbits.

A sequence of fundamental results came about as a consequence of the work of A. Weinstein, P. Rabinowitz, C. Viterbo, H. Hofer, E. Zehnder, and others. Of particular importance is the 1987 paper by C. Viterbo with a proof of the Weinstein conjecture in $\mathbb{R}^{2n}$: a hypersurface of contact type carries a closed characteristic. In relation to both the Seifert and the Weinstein conjectures, a momentous advancement was made in 1993 by H. Hofer who proved the existence of a closed orbit for a smooth Reeb vector field on S^3. Subsequently, extensions of Hofer's result for S^3 (for instance, concerning unknotted orbits) were contributed by H. Hofer, K. Wysocki, and E. Zehnder. J. Etnyre and R. Ghrist proved a version of the Seifert conjecture in hydrodynamics.

In this abstract, attention is paid mainly to counterexamples to the Seifert conjecture, i. e., constructions of non-singular vector fields without circular orbits on S^3, or, almost equivalently, on any closed 3-manifold, and on some higher-dimensional manifolds. All of these examples are based on a construction of an appropriate plug–a flow exhibiting a special kind of symmetry, defined on a chart that is inserted in a manifold. This construction was first used by F. W. Wilson in the mid 1960's, and then by P. A. Schweitzer in 1972, who for the first time used an aperiodic plug to break circular orbits.

There are two types of examples of aperiodic flows on S^3 directly strengthening Schweitzer's construction. One, a $C^{3-\epsilon}$ flow, was given by J. Harrison in the mid 1980's. The other, a volume preserving C^1 plug, was constructed by G. Kuperberg in 1995, who also proved that every 3-manifold possesses a C^∞ volume preserving flow with no fixed points and with a discrete collection of closed orbits. Thus, by inserting the plug, he shows that every 3-manifold admits a C^1 volume preserving aperiodic flow.

The examples given by Schweitzer, Harrison, and G. Kuperberg have only non-compact orbits, but they have at least two minimal sets of covering dimension 1. Earlier F. W. Wilson showed a way of modifying smooth non-singular vector fields so that there are only minimal sets of codimension 2. This gives the existence of aperiodic flows on S^{2n+1}, $n \geq 2$. A modified Seifert conjecture asserted that on certain manifolds flows must have minimal sets of codimension more than one, in particular, that every vector field on S^3 has either singular points or 1-dimensional minimal sets.

A C^∞ (actually real analytic, as noted by W. Thurston) counterexample to the Seifert conjecture was given by K. Kuperberg in 1993. Extending the construction, G. Kuperberg and K. Kuperberg solved the modified Seifert conjecture showing that S^3 admits a C^ω flow with one minimal set only, of dimension 2. In general, a foliation of any codimension of any manifold can be modified in an analytic fashion so that all minimal sets have codimension 1. It is not known whether S^3 admits a C^3 aperiodic vector field with 1-dimensional minimal sets only. It is worth mentioning that these constructions do not provide volume-preserving flows even if the differentiability requirement is relaxed to C^1.

An interesting category is that of piece-wise linear flows. G. Kuperberg and K. Kuperberg proved for any $1 \leq k \leq n-1$, every 1-foliation of an n-manifold, $n \geq 3$, can be modified in a piece-wise linear fashion so there are no closed leaves but all minimal sets are k-dimensional. G. Kuperberg proved that every 3-manifold possesses a transversely measured 1-dimensional piece-wise linear foliation with discrete closed leaves.

The Hamiltonian version of the Seifert conjecture in dimension 3 has not been solved, but there are interesting examples in higher dimensions. In 1994, V. Ginzburg and M. Herman independently constructed examples of smooth compact hypersurfaces without closed characteristics in $\mathbb{R}^{2n}$, $n \geq 4$, resolving the case of Hamiltonian flows on spheres of dimension 7 or higher. At the same time, M. Herman found a $C^{3-\epsilon}$ counterexample to the Hamiltonian Seifert conjecture in dimension 5 (i. e. , for a compact hypersurface in $\mathbb{R}^6$). In 1997, V. Ginzburg improved the previous results so as to obtain a smooth proper function $H : \mathbb{R}^{2n} \to \mathbb{R}$, for $2n \geq 6$, with a regular level set on which the Hamiltonian flow has no closed orbits.

1991 MS Classification: 30D05, 30F40, 58F11, 58F23

McMullen, Curtis T., Harvard University, USA

Rigidity and inflexibility in conformal dynamics

This talk will present advances in the theory of conformal dynamical systems, especially rational maps on the Riemann sphere and Kleinian groups. We will discuss a connection between the rigidity of hyperbolic 3-manifolds and universal scaling phenomena in dynamics, and its role in the theory of 3-manifolds that fiber over the circle, renormalization of unimodal maps, critical circle maps and Siegel disks.

1991 MS Classification:

Grzegorz Świątek, The Pennsylvania State University, USA

Induced Hyperbolicity for One-Dimensional Maps

In the early 1980s M. Jakobson proved a theorem which asserted that in the logistic family $x \to ax(1-x)$, the mapping has a probabilistic absolutely continuous invariant measure for a set of parameters a with positive measure. The crucial step of the proof was the construction of an expanding map Φ, defined on the union of countably many intervals, so that on each connected component of its domain Φ was an iterate of the original quadratic map, and the range of this restriction was a fixed interval. Hence, from the original clearly non-expanding transformation, a uniformly expanding one was constructed by taking iterates in a piecewise fashion.

In the early 1990s J. -Ch. Yoccoz proved a theorem about local connectivity of Julia sets of some quadratic polynomials, including many Julia sets which contained the critical point. He also showed that for this class of polynomials certain combinatorial data, which for real maps reduces to the kneading sequence, determines that polynomial uniquely. The proof is based on the construction of partitions of the phase space which are not far from being Markov: most pieces are mapped in a univalent way onto other pieces, except for the pieces which contain the critical point. The partitions were subjected to a process of infinite inductive refinement under which the critical pieces shrank to the point.

It was then observed that a powerful tool for studying both unimodal maps of the interval and complex polynomials in the plane is obtained when these ideas are applied jointly. When appropriate iterations of the original polynomial are applied on pieces of Yoccoz partitions, the result is a map which is expanding on all pieces except for the one which contains the critical point. Such transformations are called *induced mappings*. When a sequence of increasingly refined partitions is considered, the critical branches disappear in the limit and an expanding map of Jakobson's type is obtained.

The original construction of Jakobson required exclusion of unsuitable parameters without regard for their topological dynamics, although a set of positive measure was left at the end. The new approach based on Yoccoz' discovery works for all polynomials without neutral or attracting periodic orbits, with the only exception of a well defined class of infinitely tunable ones.
This, or a closely related approach, played the key role in the solution of several important problems. For real quadratic polynomials, and by extension for unimodal maps with non-degenerate critical point, the infinitely tunable case turned out to be quite manageable, due to the phenomenon of the so-called a priori bounds, discovered by D. Sullivan. The combination of inducing and a priori bounds was the basis of the result of G. Levin and S. Van Strien about local connectivity of Julia sets for all real unimodal polynomials. J. Graczyk and the author used the same ingredients, and another phenomenon realted to inducing and known as the *decay of geometry*, to prove that periodic windows are dense in the logistic family.
Finally, there are new results we wish to present which use inducing applied to complex polynomials. These results were obtained together with J. Graczyk and S. Smirnov and concern the behavior of the polynomials on the boundary of the Mandelbrot set, typical with respect to the harmonic measure. Among other properties, it turns out that such polynomials have Julia sets which are conformally removable and of Hausdorff dimension strictly less than 2. The first result extends a theorem by P. Jones and the second is complementary to a result by M. Shishikura which asserts that a complex polynomial in the residual subset of the boundary of the Mandelbrot set has Julia set with Hausdorff dimension equal to 2.

1991 MS Classification: 34 58 70
Xia, Zhihong, Northwestern University, USA
Arnold diffusion and variational construction of connecting orbits

We use variational method to study Arnold diffusion and instabilities in high dimensional Hamiltonian systems. Our method is based on a generalization of Mather's theory on twist maps and their connecting orbits to a higher dimensional setting. Under some general nondegeneracy conditions, which can be verified in *a priori* unstable systems, for example, by the so-called Melnikov functions, we can construct transition chains of arbitrary fixed length, crossing gaps of any size between invariant KAM (lower dimensional) tori. Other results, such as shadowing properties, symbolic dynamics and nonintegrability, etc. , can also be obtained by our method.
In Arnold's original example, the perturbation is carefully chosen so that it does not touch any invariant tori. As an application of our results, we may choose the perturbation of any form in Arnold's example and we conclude that the same results (in fact stronger) hold, as long as the Melnikov integrals have the similar forms.

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10. Partial Differential Equations

1991 MS Classification: 35Q55, 58E05
Bethuel, Fabrice, Université de Paris-Sud à Orsay, France

Recent results for Ginzburg-Landau models

These models were first introduced by V. Ginzburg and L. Landau around 1950 in order to describe superconductivity. Similar models appeared soon after for various phenomena: Bose condensation, superfluidity, non linear optics. A common property of these models is the major role of topological defects, termed in our context vortices.
In a joint book with H. Brezis and F. Helein, we considered a simple model situation, involving a bounded domain Ω in R^2, and maps v from Ω to R^2. The Ginzburg-Landau functional, then writes

$$E_\epsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{4\epsilon^2}\int_\Omega (1-|v|^2)^2$$

Here ϵ is a parameter describing some characteristic lenght. We are interested in the study of stationary maps for that energy, when ϵ is small (and in the limit ϵ goes to zero). For such map the potential forces $|v|$ to be close to 1 and v will be almost S^1-valued. However at some point $|v|$ may have to vanish, introducing defects of topological nature, the vortices. An important issue is then to determine the nature and location of these vortices.
We will also discuss recent advances in more physical models like superconductivity, superfluidity, as well as for the dynamics: as previously the emphasis is on the behavior of the vortices. These results involve a very large number of contributions by many authors: Almeida, Colliander, Jerrard, Hardt, Lin, Mironescu, Rivière, Saut, Serfaty, Shafrir, Sigal and Struwe among others.

1991 MS Classification: 35, 43, 49, 53
Hélein, Frédéric, Ecole Normale Supérieure de Cachan, France

Phenomena of compensation and estimates for partial differential equations

Quantities like the Jacobian determinant of a mapping play an important role in several partial differential equations in Physics and Geometry. The algebraic structure of such nonlinearities allow to improve slightly the integrability or the regularity of these quantities, sometimes in a crucial way. Focused on the instance of $\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$, where a and $b \in H^1(\mathbb{R}^2)$, we review some results obtained on that quantity for 30 years and applications to partial differential equations arising in Geometry, in particular concerning the conformal parametrisations of constant mean curvature surfaces and the harmonic mappingss between Riemannian manifolds.
Reference: Frédéric Hélein, *Harmonic maps, conservation laws and moving frames*, Diderot éditeur, Paris 1997.

1991 MS Classification: 35, 49, 60

Robert R. Jensen, Loyola University Chicago, USA

Viscosity solutions of elliptic partial differential equations

In my talk and its associated paper I shall discuss some recent results connected with the uniqueness of viscosity solutions of nonlinear elliptic and parabolic partial differential equations. By now, most researchers in partial differential equations are familiar with the definition of viscosity solution, introduced by M. G. Crandall and P. L. Lions in their seminal paper, "Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier order, " *C. R. Acad. Sci. Paris* 292 (1981), 183–186. Initially, the application of this definition was restricted to nonlinear first order partial differential equations—i. e., Hamilton-Jacobi-Bellman equations—and it was shown that viscosity solutions satisfy a maximum principle, implying uniqueness. In 1988 an extended definition of viscosity solution was applied to second order partial differential equations, establishing a maximum principle for these solutions and a corresponding uniqueness result. In the following years numerous researchers obtained maximum principles for viscosity solutions under weaker and weaker hypotheses. However, in all of these papers it was necessary to assume some minimal modulus of spatial continuity in the nonlinear operator, depending on the regularity of the solution, and to assume either uniform ellipticity or strong monotonicity in the case of elliptic operators. The results I shall discuss are related to attempts to weaken these assumptions on the partial differential operators—e. g., operators with only measurable spatial regularity, and operators with degenerate ellipticity.

1991 MS Classification:

Lindblad, Hans, University of California at San Diego, La Jolla, CA, USA

Minimal regularity solutions of nonlinear wave equations

Recently there has been much interest in proving existence and uniqueness of solutions of nonlinear wave equations with low regularity initial data. One reason is that many equations from physics can be written as a system of nonlinear wave equations with a conserved energy norm. If one can prove local existence and uniqueness assuming only that the energy norm of initial data is bounded then global existence and uniqueness follow from the global energy bound. Therefore it is interesting to find the minimal amount of regularity of initial data needed to ensure local existence for the typical nonlinear wave equations.

In the lecture I will present counterexamples to local existence with low regularity data for the typical nonlinear wave equations. In the semi-linear case the counterexamples are sharp, in the sense that with slightly more regularity one can prove local existence. I will present in some detail new counterexamples for the quasi-linear case. It is natural to look for existence in Sobolev spaces, since the Sobolev norms are more or less the only norms that are preserved for a linear wave equation. The counterexamples involve constructing a solution that develops a singularity along a characteristic for all positive times. In the quasi-linear case it also involves controling the geometry of the characteristic set. The norm is initially bounded but becomes infinite for all positive times, contradicting the existence of a solution in the Sobolev space associated with the norm. The counterexamples are half a derivative more regular than what is predicted by a scaling argument. This argument use the fact that the equations are invariant under a scaling to obtain a sequence of solutions for which initial data is bounded in an appropriate Sobolev norm. The counterexamples were not widely expected since for several nonlinear wave equations, one does, obtain local existence down to the regularity predicted by scaling.

On the other hand, the classical local existence theorems for nonlinear wave equations are not sharp in the semi-linear case. These are proved using just the energy inequality and Sobolev's embedding theorem. Recently these results were improved using space-time estimates for Fourier integral operators, know as Strichartz' estimates, and generalizations of these. There are many recent results in this field by for example Klainerman-Machedon, Grillakis, Ponce-Sideris and Tataru. In particular, Klainerman-Machedon proved that for equations satisfying the 'null condition', one can go down to the regularity predicted by the scaling argument mentioned above. I will present joint work with Sogge where we prove local existence with minimal regularity for a much studied simple class of model semi-linear wave equations. I should mention that there is a somewhat paralell development for KdV and nonlinear Schroedinger equations, with work of for example Bourgain and Kenig-Ponce-Vega.

Whereas the techniques of harmonic analysis were essential in improving the local existence results, the Strichartz estimates are not the best possible global estimates since they don't catch the right decay as time tends to infinity if the initial data has compact support. The classical method, introduced by Klainerman, to prove global existence for small initial data is to use the energy method with the vector fields coming from the invariances of the equation. However, this method requires quite a lot of regularity of initial data and also the energy method alone doesn't give optimal estimates for the solution since it is an estimate for derivatives. I will present joint work with Georgiev and Sogge where we obtain better global estimates using techniques from harmonic analysis taking into account the invariances or symmetries of the wave equation. We obtaine estimates with mixed norms in the angular and spherical variables, with Sogge, and weighted Strichartz' estimates with Georgiev and Sogge. Using these new estimates we prove that a certain class of semi-linear wave equations have global existence in all space dimensions. This was a conjecture by Strauss, following an initial result by John.

1991 MS Classification: 35, 42

Machedon, Matei, University of Maryland, USA

Fourier Analysis of Null Forms and Non-linear Wave Equations.

The non-linear terms of many equations, including Wave Maps and Yang-Mills have a special, "null" , structure. In joint work with Sergiu Klainerman, I use techniques of Fourier Analysis, such as generalizations and refinements of the restriction theorem applied to null forms to study the optimal Sobolev space in which such non-linear wave equations are well posed. This talk covers a survey and recent developments of these ideas.

1991 MS Classification: 35Bxx, 35Qxx

Merle, Frank, Université de Cergy-Pontoise, France

Blow-up Phenomenon For Critical Nonlinear Schrödinger and Zakharov Equations.

We are interested in time singularity formation in dispersive Hamiltonian systems with infinite speed of propagation such as the nonlinear Schrödinger equation (NLS). We will consider the critical nonlinear Schrödinger equation and critical Zakharov equation (NLS coupled with a wave equation) in whole space. These equations appear in particular in plasma physics, laser beam propagation, etc. .. The Cauchy problem of these equations is locally well-posed in energy space and the question is to describe the singularity formation in this space.

For NLS, a virial identity yields existence of singular solutions. The notion of criticallity is related to a conformal invariance of the equation. We first see that the space L^2 (invariant by the flow) plays a crucial role. There is an L^2 concentration of the solution at the blow-up time of an amount at least greater than a universal number M_0. From the conformal invariance and the existence of a periodic solution Q, we have an explicit blow-up solution S such that $|S|_{L^2} = M_0$. Note that other blow-up solutions can be generated from S.

We first show that up to the invariance of the equation, S is the unique blow-up solution of minimal mass M_0. In particular, solutions at the critical mass level M_0 or below are, up to the invariance of the equation, either Q, S, or a solution defined for all time and behaving like a linear solution as the time goes to infinity.

We then consider the question of continuation of the solution after the blow-up time in the case of the minimal blow-up solution S. We show that the time singularity is in some sense unstable in time and there exists regular continuations of the solution after the blow-up time. However, the deterministic character(through one parameter in S^1) is lost through the blow-up and we describe how this occurs.

We are now interested in the structural stability of such qualitative properties of blow-up solutions, considering the Zakharov equations which can be seen as a more realistic physical approximation near the singularity. Note that there are no conformal invariance nor simple arguments for obstruction to global existence.
First, from a virial type identity, existence of a large class of blow-up solutions can be exhibited (in finite or infinite time). L^2 concentration properties of the blow-up solutions can be proved.
From a bifurcation argument at infinity (related to the solution S for critical NLS), we prove the existence of explicit blow-up solutions for Zakharov equations with the same qualitative structure as S for NLS equation. Existence of such solutions will have various conscequences (for example, instability of periodic solutions).
To conclude, the blow-up rate for any blow-up solution of the Zakharov equation (even when the approximation parameter related to NLS is small) is greater or equal to the one given by S. In particular, the blow-up rate numerically observed for blow-up solutions of NLS (different from the one given by S) is unstable under perturbations related to the Zakharov equation.

1991 MS Classification:

Gustavo Ponce, University of California, Santa Barbara, CA 93106, USA

On nonlinear dispersive equations

I shall present results obtained jointly with Carlos E. Kenig and Luis Vega concerning properties of solutions to nonlinear dispersive models. We consider equations of the form

$$\partial_t u = iP(\nabla_x)u + F(u, \partial_x u, \dots), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{1}$$

where the real symbol $P(i\vec{\xi})$ represents the dispersive relation and $F(\cdot)$ denotes the nonlinearity. These models which include the generalized Korteweg-de Vries equation ($n = 1,\ P(i\xi) = \xi^3,\ F = u^k \partial_x u,\ k = 1, 2...$), the generalized Benjamin-Ono equations, and systems related to the nonlinear Schrödinger equation, arise in several physical situations and have been shown to be relevant in different mathematical contexts. We are interested in the initial value problem (IVP) and the periodic boundary value problem (PBVP) associated to (1) and mainly study the following two related problems.
First, we deal with the problem of the optimal regularity of the datum $u(x, 0) = u_0(x)$ which guarantees that the corresponding IVP and PBVP are well-posed. The notion of well-posedness includes existence, uniqueness, continuous dependence of the solution upon the data u_0 in some function space $\mathcal{X}$, persistence property (i. e. the time evolution of the solution $u(t)$ describes a continuous curve in $\mathcal{X}$), and so the solutions generate a dynamical system in the function space $\mathcal{X}$. The regularity of the data is

measured in the classical Sobolev spaces $H^s(\mathbb{R}^n) = (1-\Delta)^{s/2}L^2(\mathbb{R}^n)$, and $\dot{H}^s(\mathbb{R}^n) = (-\Delta)^{s/2}L^2(\mathbb{R}^n)$ with $s \in \mathbb{R}$, in which solutions of the associated linear problem are given by the unitary group $\Gamma = \{e^{itP(\nabla_x)} : t \in \mathbb{R}\}$. We have developed new techniques, mainly coming from harmonic analysis, which provide a better understanding of the relationship between the dispersive (linear) part and the degree and the structure of the nonlinearity F. These include optimal estimates for smoothing effects and the maximal function associated to the group Γ. The methods allow us to establish local and global well-posedness results for low values of the index s and prove that in several cases these are the best possible.
The second problem addresses the existence of solutions to some nonlinear dispersive models of Schrödinger type. These include the Davey-Stewartson, Ichimori and Zakharov-Schulman systems. In these systems the dispersion is represented by a non-degenerate second order operator

$$P = \partial^2_{x_1} + .. + \partial^2_{x_k} - \partial^2_{x_{k+1}} - ..\partial^2_{x_n},$$

(thus for $1 \geq k < n-1$ the operator P is not elliptic) and the nonlinearity F involved non-local operators of "orden one". In many cases the existence of solutions for the IVP associated to these systems cannot be obtained by standard methods such as : classical method, energy estimates, L^qL^p-estimates, etc., even assuming high regularity on the datum u_0. In this case we refine and extend arguments used in the previous case to answer the local well-posedness question for several forms of these systems.

1991 MS Classification: 35K

Safonov, Mikhail, University of Minnesota, USA

Estimates near the boundary for solutions of second order parabolic equations

We consider linear uniformly parabolic equations of second order, with *measurable* coefficients, in a *Lipschitz* cylinder $Q = \Omega \times (0,T)$, in both the *divergence* form

$$Lu = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) - u_t = 0$$

and the *non-divergence* form

$$Lu = \sum_{i,j=1}^{n} a_{ij}D_iD_ju - u_t = 0$$

where $D_i = \partial/\partial x_i$. We will discuss different (boundary, backward, etc.) forms of the Harnack principle, the Hölder continuity of the quotient of two positive solutions vanishing on a portion of the parabolic boundary, the doubling property of the L-caloric measure on the boundary, the classical Fatou theorem. A general approach to these problems will be demonstrated, which is based on the "standard" Harnack inequality and an elementary estimate for oscillation of solutions near the boundary.

1991 MS Classification: 35-R30
Uhlmann, Gunther, University of Washington, Seattle, USA

Inverse Boundary Value Problems for Partial Differential Equations

Inverse boundary problems are a class of problems in which one seeks to determine the internal properties of a medium by performing measurements along the boundary of the medium. These inverse problems arise in many important physical situations, ranging from geophysics to medical imaging to the non-destructive evaluation of materials.
The appropriate mathematical model of the physical situation is usually given by a partial differential equation (or a system of such equations) inside the medium. The boundary measurements are then encoded in a certain boundary map. The inverse boundary problem is to determine the coefficients of the partial differential equation inside the medium from knowledge of the boundary map.
In this talk we will survey part of the significant progress which has been made in the last twenty years in this area. Many of the advances have been a consequence of the construction of *complex geometrical optics* solutions for the class of partial differential equations under consideration.
The prototypical example of an inverse boundary problem is the inverse conductivity problem, also called electrical impedance tomography, first proposed by A. P. Calderón. Let Ω be a bounded domain with Lipschitz boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sources or sinks of current the equation for a voltage potential u on Ω is given by

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega\,. \tag{1}$$

In this case the boundary map is the voltage to current map; that is, the map assigns to a voltage potential on the boundary the corresponding induced current flux at the boundary. This boundary map is also called the Dirichlet to Neumann map, since the voltage potential is Dirichlet data associated to equation (1), and the induced current flux is Neumann data. The inverse problem in this case is to recover in Ω the function γ from the Dirichlet to Neumann map.
Many results on this particular inverse problem have been obtained using complex geometrical optics solutions to equation (1). We will also discuss other examples of inverse boundary problems, including examples associated to the Schrödinger equation in the presence of a magnetic field, Maxwell's equations, the Dirac equation, and the Lamé system of elasticity.

1991 MS Classification:
D. R. Yafaev, IRMAR, Université Rennes-1, France.

Scattering theory: some old and new problems

Scattering theory is, roughly speaking, perturbation theory of self-adjoint operators on the (absolutely) continuous spectrum. It has its origin in mathematical problems of quantum mechanics and is intimately related to the theory of partial differential equations. This talk is devoted to a survey of different tools of this theory ranging from the abstract functional analysis to an advanced (and sometimes non-standard) calculus of pseudo-differential operators. Some recently solved problems, such as asymptotic completeness for the Schrödinger operator with long-range and multiparticle potentials, as well as open problems, are discussed. Special attention is paid to the properties of the scattering matrix which is the main observable of the theory.

Here are the detailed contents of the talk:

1. Two problems of scattering theory: eigenfunction expansions and large time asymptotics of unitary groups

2. Hamiltonians of quantum mechanics

3. Methods of scattering theory: “trace class“ and “smooth”. The absence of a bridge between these two methods. An open concrete problem

4. The multiparticle Schrödinger operator with short-range pair potentials
a) Setting up the problem
b) The Mourre estimate and the limiting absorption principle
c) The radiation conditions a priori estimate on the resolvent
d) The asymptotic completeness
e) An unsolved problem: the discrete version (Heisenberg model)

5. The multiparticle Schrödinger operator with long-range pair potentials
a) The asymptotic completeness
b) Some cases of the break-down of asymptotic completeness:
new channels of scattering

6. Scattering of waves on unbounded obstacles

7. The scattering matrix for the two-particle Schrödinger operator with a short-range potential. Its eigenvalues and eigenfunctions

8. The scattering matrix for the two-particle Schrödinger operator with a long-range potential
a) The scattering matrix as a pseudo-differential operator with an oscillating symbol
b) The spectrum of the scattering matrix
c) The diagonal singularity of the scattering amplitude

9. The scattering matrix for the N-particle Schrödinger operator: some results and open questions

11. Mathematical Physics

1991 MS Classification: 81T30
Paul S. Aspinwall, Duke University, Durham, USA
Geometry and String Duality

(See section 4)

1991 MS Classification:
Bogomolny, E. B., CNRS-URA 62, Institut de Physique Nucléaire, Orsay, France
Spectral Statistics

The aim of the lecture is to present an overview of statistical properties of energy eigenvalues of deterministic systems and related questions. The following subjects are shortly discussed: Distribution of energy eigenvalues for integrable and chaotic systems. Universality within standard random matrix ensembles. Distribution of zeros of number-theoretical zeta functions. Arithmetical models. Intermediate statistics. Approach to universal limit. Random polynomials.

1991 MS Classification: 46, 47, 81, 83
Buchholz, Detlev, Universität Göttingen, Germany
Scaling algebras in local relativistic quantum physics

Relativistic quantum physics in a pseudo-Riemannian spacetime manifold $(\mathcal{M}, g)$ can conveniently be described by a C^*-dynamical system $(\mathcal{A}, \alpha)$, where $\mathcal{A}$ is a C^*-algebra, describing the physical observables in $\mathcal{M}$, and α a representation of the isometry group of $(\mathcal{M}, g)$ by automorphisms of $\mathcal{A}$. The Maxwell-Einstein principle of locality is implemented in this setting by exhibiting a distinguished pre-cosheaf (net) of subalgebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ which is labelled by the spacetime regions $\mathcal{O} \subset \mathcal{M}$.
In high energy physics the structure of the observables in very small spacetime regions $\mathcal{O}$ (at "small spacetime scales") is of great interest. It can be explored with the help of the new mathematical concept of *scaling algebra*. Given a dynamical system $(\mathcal{A}, \alpha)$, the elements of the associated scaling algebra $\underline{\mathcal{A}}$ consist of certain specific functions $\underline{A} : \mathbf{R}_+ \longmapsto \mathcal{A}$ which establish a connection between the observables at different spacetime scales $\lambda \in \mathbf{R}_+$. The action of the isometry group of $(\mathcal{M}, g)$ as well as the net structure of $\mathcal{A}$ can be lifted to $\underline{\mathcal{A}}$.
The subset of physical states in the dual space $\mathcal{A}^*$ of $\mathcal{A}$ fixes a weak topology on $\underline{\mathcal{A}}$ which allows one to proceed to arbitrarily small spacetime scales, $\lambda \to 0$. The resulting scaling limit is described by C^*-dynamical systems $(\mathcal{A}^{(0)}, \alpha^{(0)})$ in the fibers of the Minkowskian tangent bundle of $(\mathcal{M}, g)$. For the physically important case of Minkowski spacetime the general structure of $(\mathcal{A}^{(0)}, \alpha^{(0)})$ has been completely determined and classified.
Of particular interest are the outer local endomorphisms of $(\mathcal{A}^{(0)}, \alpha^{(0)})$ since they contain information on the symmetry properties of the original dynam-

ical system at small spacetime scales. It has been shown that these morphisms fit into the setting of the Doplicher–Roberts duality theorem, and thus give rise to a compact "gauge group" $G^{(0)}$ and "field net" $(\mathcal{F}^{(0)}, \alpha^{(0)})$ which extends $(\mathcal{A}^{(0)}, \alpha^{(0)})$. These results can be used to give a precise mathematical meaning to intuitive physical notions such as "color symmetry" and "quark confinement". Part of these results is joint work with R. Verch.

1991 MS Classification: 82B43, 82B26, 60K35, 05C80

Chayes, Jennifer T., Microsoft Research, Redmond, USA

Birth of the Infinite Cluster: Finite-Size Scaling in Percolation

Percolation is the simplest and most widely studied model of a random medium. Among the numerous applications of percolation are calculations of the distribution of oil in a porous medium, determination of the behavior of the diffusion coefficient in a turbulent plasma, and the question of intractability in the satisfiability problem in theoretical computer science. A fundamental feature of the percolation model is that it undergoes a critical phase transition from a disordered phase to a phase with long-range order and hence transport. Technically, the transition occurs only in an infinite system. Beyond the transition point, the ordered phase is characterized by the presence of an infinite cluster. The critical regime has been studied extensively by both analytical and numerical methods. This work is a detailed study of the phase transition in percolation, in particular of the question of finite-size scaling: Namely, how does the critical transition behavior emerge from the behavior of large, finite systems? Our results rigorously locate the proper window in which to do critical computation and establish features of the phase transition. This work is a finite-dimensional analogue of classic work on the critical regime of the random graph model of Erdös and Rényi. No prior knowledge of percolation or phase transitions is assumed in this talk. (This is joint work with C. Borgs, H. Kesten and J. Spencer.)

1991 MS Classification: 35, 58, 76

Collet, Pierre, CNRS, Ecole Polytechnique, Palaiseau, France

Extended Dynamical Systems

The theory of extended dynamical systems deal with the time evolution of systems where the spatial extension is important. This is a natural generalization of dynamical systems in finite dimensional phase space which were successfully used to model physical situations with only few degrees of freedom excited. A system can be thought as extended when the size of the spatial domain where it is defined is large compared to the size of the structures exhibited by the solutions. A simple example is given by the waves on a sea whose wavelength is much smaller than the sea size. In this situation it is natural to try to approximate the spatially finite system by an infinite one (this is analogous to the thermodynamic limit of statistical mechanics). The most challenging problem is to explain the origin of the structures (with their well defined scale), and to understand their time evolution. This is an area of high activity among experimental and theoretical Physicists.

We will discuss three questions which have been investigated recently from a rigorous point of view, mostly for dissipative systems inspired by hydrodynamic problems (namely non-linear partial differential equations of parabolic type).

The first problem is to prove that the evolution semi-flow is well defined for all times. The novelty of the problem is that one wants to work with solutions which are only bounded (and regular) but which do not necessarily tend to zero at infinity (for example waves). Therefore the problem of existence and regularity of solutions of the partial differential equations governing the time evolution has to be re-investigated with new methods.

The second problem deals with the occurrence of instabilities when the time evolution depends on a parameter. There is of course an analogy with standard bifurcation theory, however in the present context, when one looks at the linearized evolution around a basic solution (often homogeneous) one finds a continuous spectrum. When a relevant parameter varies, this spectrum may penetrate into the right half plane and the simple solution becomes unstable. Instabilities provide a mechanism for the appearance of solutions with well defined structures. Amplitude equations have been determined which are analogous to (truncated) normal forms. They describe the large space-time modulations of the structures near the threshold.

The third problem deals with the understanding of large time evolution. Some results have been derived for particular classes of solutions (weakly non-linear stability, interaction of structures). We will also give some results about a possible quantitative approach to the description of the so-called space-time chaos which may occur when the structures disappear or become disordered.

1991 MS Classification: 81T30

Dijkgraaf, Robbert, University of Amsterdam, The Netherlands

The Mathematics of Fivebranes

Fivebranes are one of the richest but puzzling objects in string theory. They are quantum systems associated to real six-dimensional manifolds that involve a two-form with a three-form field strength that satisfies a self-duality condition. Through recent developments such as the eleven-dimenional origin of string theory (also known as M-theory) and the existence of extended objects and their relation to gauge theory, much has become known about the quantization of fivebranes. Fivebranes relate such diverse subjects as the geometry of Calabi-Yau threefolds, moduli spaces of vector bundles on complex surfaces, generalized Kac-Moody algebras, automorphic forms for the group $O(3, 2, \mathbf{Z})$ and the theory of elliptic genera. There is also a close analogy between fivebranes and the study of two-dimensional conformal field theory and its application to three-manifold invariants. Finally there are interesting relations with the formalism of non-commutative geometry.

1991 MS Classification: 58F10 - 58F30 - 58F36 - 70F15 - 70K20 - 34D10 - 34E15

Giorgilli, Antonio, Università di Milano, Italy

On the problem of stability for near to integrable Hamiltonian systems

The problems addressed are concerned with the "stability" of near to integrable Hamiltonian systems in the light of the strong development of the quantitative methods of perturbation theory in the second half of this century. The word "stability" is used here in a quite wide sense, which includes a considerable weakening of the traditional concept of stability, as widely studied, e. g., by Lyapounov. The basis of the results that I'm going to illustrate is on the one hand the celebrated KAM theory on persistence of conditionally periodic motions, and, on the other hand, the Nekhoroshev's theorem on stability over exponential times.
Let me first illustrate the framework. I will consider the problem that Poincaré has called "the general problem of dynamics", namely a canonical system of differential equations with Hamiltonian $H(p,q) = h(p) + \epsilon f(p,q,\epsilon)$; here $p \in \mathbf{R}^n$ and $q \in \mathbf{T}^n$ are action–angle variables, n is the number of degrees of freedom and ϵ is a small parameter. It is well known that for $\epsilon = 0$ the orbits are dense on invariant tori $\mathbf{T}^k$, with some positive $k \leq n$, which are subsets of a torus $p =$ const. The motion is either quasiperiodic or periodic according to the existence of resonance relations among the unperturbed frequencies $\omega(p) = \frac{\partial h}{\partial q}$.
The main achievement of KAM theorem is the proof that for most initial data, characterized by strongly non–resonant frequencies, the motion is still quasi periodic, the orbit lying on an invariant torus $\mathbf{T}^n$ which is close to the unperturbed one. This relaxes the condition for stability that requires the result to hold for an open set of initial data: the complement of the surviving invariant tori is open and dense, but has small measure.
The theory of Nekhoroshev aims at proving that the action variables satisfy a bound $|p(t) - p(0)| < B\epsilon^b$ for all times $|t| < A\exp(1/\epsilon^a)$, with positive constants A, C, $a < 1$ and $b < 1$. This relaxes the condition that the stability result should hold for an infinite time interval; however, the exponential dependence of the estimated stability time on the inverse of the perturbation makes the result interesting for physical systems. Following Littlewood, I will call this phenomenon *exponential stability.*

My purpose here is to report on some progress made on this subject during the last decade. Most of the attention will be concentrated on the exponential stability. I will address in particular the following points: (a) the actual applicability of the concept of exponential stability to physical systems; (b) the extension of the concept of exponential stability to system with a very large number of degrees of freedom, and possibly to infinite systems; (c) the relations between the existence of KAM invariant tori and the exponential stability of Nekhoroshev's theory.

The relevance of the exponential stability for physical systems depends on the size of the perturbations that is actually allowed. However, the analytical estimates that are available are usually very pessimistic, unable to give realistic results. For a practical application it is more convenient to perform explicitly the perturbation expansions up to some finite order in the perturbation parameter, and to exploit the asymptotic character of the resulting series. This requires a lot of algebraic manipulation that the available computers do allow, so that the method may be effectively applied at least to simple cases. In this spirit several attempts have been made to investigate the stability of the Lagrangian triangular equilibria of the restricted problem of three bodies in the Sun–Jupiter case. The best result currently available is that there is a rather big neighbourhood of the equilibrium which is stable for a time interval as large as the estimated age of the universe; this neighbourhood is big enough to include a few of the known asteroids.

The actual applicability of the exponential estimates to systems with a large number of degrees of freedom is questionable because the available values of the constants entering the expression above depend in a bad manner on the number of degrees of freedom. The worse dependence is that of the constant a in $\exp(1/\epsilon^a)$: one has typically $a = O(1/n)$. The latter estimate has been shown to be optimal, in general. However, a further relaxation of the concept of stability shows that some version of the theory may still be applied. Precisely, instead of looking for an almost constant value of *all* the actions of the system one attempts to prove only the existence of at least one almost invariant quantity. A significant case concerns systems that satisfy the following requirements: (i) there a natural splitting into two distinct subsystems characterized by well separated frequencies or characteristic time intervals, and (ii) the two subsystems are weakly coupled. A first example is a FPU type chain with alternating heavy and light mass points: the spectrum splits naturally into two different parts, called the acoustic and the optical part, with a quite large gap between the frequencies. In this case the perturbation parameter is the ratio ω_-/ω_+ between the maximal acoustic frequency and the minimal optical frequency. A second example is, for instance, a system of identical diatomic molecules moving on a line and interacting with a short range analytic potential. The perturbation parameter is the ratio τ/ω between the typical interaction time during a collision and the internal frequency of vibration of the molecules. In both these examples the high frequencies modes turn out to be (at least approximately) equal, so that the high frequency subsystem is completely resonant. Precisely the complete resonance allows us to prove that the exchange of energy between the two subsystems is practically frozen for a time interval $\exp(1/\epsilon)$, i. e., that the exponent a in the exponential estimate is

1, not $1/n$. This fact may be very relevant for applications to Statistical Mechanics.

The relations between the existence of KAM tori and the phenomena of exponential stability may be investigated by considering the neighbourhood of an invariant KAM torus. This leads to the concept of superexponential stability. Here is the argument, short but essentially complete. According to Kolmogorov's approach, the neighbourhood of the torus is described by an Hamiltonian which is very similar to the Hamiltonian describing the neighbourhood of an elliptic equilibrium, with nonresonant frequencies. The perturbation parameter is the distance ρ from the invariant torus. On the other hand, a local application of Nekhoroshev's theorem shows that the Hamiltonian may be given the form of the general problem of dynamics with a perturbation that is $O\big(\exp(-1/\rho^a)\big)$. Therefore, using the global formulation of Nekhoroshev's theorem one proves that the neighbourhood of a torus is exponentially stable in $O\big(\exp(-1/\rho^a)\big)$, that is, it is stable up to a time $O\big[\exp\big(\exp(1/\rho^a)\big)\big]$. The name "superexponential stability" refers to the double exponential appearing in the estimate.

1991 MS Classification: 81-02

Graf, Gian Michele, ETH Zürich, Switzerland

Stability of matter in classical and quantized fields

Stability of matter is the statement — rooted in quantum mechanics and electrostatics — that the energy per particle has a lower bound which is independent of the state of matter and, in particular, of its amount. This fact is responsible for preventing matter from shrinking indefinitely in size, as well as for basic thermodynamic properties such as extensivity.

In recent years considerable activity was directed to the issue of stability in the case of matter interacting with an *electromagnetic field*. We shall review the results which have been established by various groups, in different settings: relativistic or non-relativistic matter, classical or quantized electromagnetic fields. Common to all of them is the fact that electrons interact with the field both through their charges *and* the magnetic moments associated to their spin. It is in fact the inclusion of spin which makes the problem of stability distinctively different from its counterpart with no electromagnetic field present. More precisely, spin allows for electron states which have no kinetic energy (zero modes) w. r. t. suitable inhomogeneous magnetic fields. This amounts to invalidating the uncertainty principle, which is however restored, in a sense, by taking into account both the kinetic energy of the electron *and* the field energy.

Stability of matter in presence of magnetic fields requires that $Z\alpha^2$ (where Z is the largest nuclear charge in the system) as well as the fine structure constant α itself, do not exceed some critical value. This is related to the fact that in a zero mode the remaining non-zero energies of the Coulomb interaction and of the field exhibit the same scaling under dilatations. If one imposes an ultraviolet cutoff to the field, as it occurs in unrenormal-

ized quantum electrodynamics, then stability no longer implies a bound on α, $Z\alpha^2$. We conjecture that it should do so again in a properly renormalized theory.
An important technical tool, which is however of independent interest, is given by Lieb–Thirring type inequalities for the sum (or some other moment) of the eigenvalues of a one–particle Schrödinger, or related, operator. We shall present such estimates for the case of a Pauli operator with an arbitrary inhomogeneous magnetic field. The bounds involve the field strength and/or of that of the field gradient. As a corollary, we obtain a bound on the density of zero modes in a given magnetic field.

1991 MS Classification: 82B23
McCoy, Barry, State University of New York, Stony Brook, NY, USA
Rogers-Ramanujan identities: A Century of progress from mathematics to physics

In this talk I will summarize the discoveries of the last five years in the theory of Rogers-Ramanujan identities; discoveries which have been made because of the interchange of ideas between mathematics and physics. This interchange of ideas is quite remarkable because when L. J. Rogers discovered his celebrated identities in 1894 there was not one shred of reason to suppose that they had anything to do with physics. Furthermore the brilliant developments by Ramanujan, Schur, Watson, Bailey, Slater and Andrews and many connections with number theory and combinatorics which were made in the first three quarters of the century contained not one sentence concerning quantum mechanics, statistical mechanics, or physics experiments.
The relation between Rogers-Ramanujan identities and physics was first seen by Baxter in 1981 when he found that the original identities of Rogers appeared in order parameter computations for the hard hexagon model. A second glimpse of the relation is seen by combining the relation between conformal field theory and Virasoro algebra established by Belavin, Polyakov and Zamolodchikov in 1984 with the 1988 suggestion of Kac that every representation of the Virasoro algebra should produce a Rogers-Ramanujan identity.
Conformal field theory and, equivalently, representations of Virasoro Algebra are commonly studied by the techniques of bosonisation and this leads to compact expressions for the characters. However, there is an equivalent description of the Hilbert space in terms of fermions which leads to q-series representations of the characters and the identity of these two representations leads to Rogers-Ramanujan identities. This fermionic representation has a direct physical interpretation in terms of particles which the bosonic representations does not possess. The fermionic representation is a generalization of fermi statistics in quantum mechanics.
In the past five years we have found methods of proof which exploit these ideas to prove Rogers-Ramanujan identities for large classes of conformal field theory models including the minimal models $M(p,p')$, many coset models, and models with $N=1$ and $N=2$ supersymmetry. We have found that

for cases such as the cosets $(A_1^{(1)})_1 \times (A_1^{(1)})_1/(A_1^{(1)})_2$ and $(E_8)_1 \times (E_8)_1/(E_8)_2$ which both describe the same CFT have different but equal q-series representations of the same characters. These different coset constructions lead to different integrable perturbations and thus it may be conjectured that each fermionic representation of a character leads to a different integrable perturbation. I will present the current status Rogers-Ramanujan identities for the integrable $\phi_{1,3},\ \phi_{2,1},\ \phi_{1,5}$ and $\phi_{1,2}$ perturbations and will present examples where we have more identities than known integrable perturbations of conformal field theory. I will conclude with a discussion of the relation of the quasi-particle representation of the conformal field theory with the massive integrable perturbations and lattice models and with some remarks on extensions to the much larger space of nonintegrable perturbations.

1991 MS Classification: 60K35, 82B27
Schonmann, Roberto, University of California at Los Angeles, USA
Metastability and the Ising model

This address will survey recent progress on a classical non-equilibrium statistical mechanics problem, in the context of a well-studied idealized interacting particle system, called kinetic Ising model. The problem concerns the speed and the patterns of relaxation of statistical mechanical systems in the proximity of the phase-transition region, and is related to the problem of understanding the metastable behavior of systems in such regions.
For instance, it is well known that a ferromagnetic material which is in equilibrium under a negative external magnetic field relaxes to equilibrium very slowly after the magnetic field is switched to a small positive value. A detailed mathematical analysis of such a phenomenon can only be performed on simplified models. It is widely accepted that an appropriate model for this and many other purposes is a kinetic Ising model: a Markov process which endows the traditional Ising model with a reversible stochastic dynamics. On each vertex of an infinite lattice Z^d, we have variables (called spins) which take the values -1 or $+1$. The system evolves in continuous time as a Markov process which is time-reversible and has as invariant measures the canonical Gibbs measures of statistical mechanics. When the "temperature" parameter appearing in the definition of the model is small enough, there is a phase transition which takes place when the "external field" parameter, h, changes sign (this corresponds to the change from a negative to a positive orientation of most spins). The question then arises of how the system relaxes to equilibrium when h is small (positive say), and the system is initially in the configuration with all spins -1.
In equilibrium the spins have to be mostly $+1$, with the -1 spins forming finite clusters in a background of $+1$ spins. Simulations have shown that the relaxation mechanism is driven by the behavior of the clusters (droplets) of $+1$ spins which are formed in the Sea of -1 spins. While small droplets tend to shrink and disappear, large ones tend to grow and are responsible for the relaxation. This phenomenon has long been understood on non-rigorous heuristic grounds, and can be used to predict for instance the order of mag-

nitude, as $h \searrow 0$, of the relaxation time for the process. The prediction is that the relaxation time grows as $\exp(\lambda/h^{d-1})$, where λ is a constant which can be computed. The value of λ is, in particular, related to the equilibrium surface tension of the Ising model through the Wulff construction, which solves a variational problem. In this talk I will describe an approach which led to rigorous proofs of results of this type and other related ones.

1991 MS Classification: 81T40
Smirnov, Fedor A., CNRS, LPTHE, Université Pierre et Marie Curie, Paris, France

Space of local fields in integrable field theory and deformed Abelian integrals

this is LaTex file
In this talk we consider the restricted sine-Gordon model (RSG) which coincides with a massive perturbation of minimal model of conformal field theory (CFT). The space of states of the model is the Fock space of particles (H_p). The space of local fields (H_f) is the same as in conformal case: it is a direct sum of tensor products of Verma modules of two Virasoro algebras corresponding to right and left chiralities (for simplicity we consider one chiral sector). The highest vectors correspond to primary fields. The Verma modules in question are reducible, the space of independent local fields ($\widehat{H}_f$) is obtained by factorizing over submodules of "null-vectors". The chiral sector of minimal models of CFT provides a quantization of Korteweg-de Vries hierarchy. The procedure of factorizing over the null-vectors corresponds to imposing the equations of motion.
Every operator in the RSG model is determined by complete set of its matrix elements in the space of states H_p. For a local field the matrix elements taken between vacuum and n-particle states are called form factors. Due to the crossing-symmetry the form factors actually define all other matrix elements, i. e. they define the local field. Putting it differently, we can say that the form factors provide a pairing between H_p and H_f. Analyzing the explicit formulae for the form factors we find the following unexpected and mathematically attractive feature of this pairing. The form factors are described in terms of deformed hyper-elliptic integrals. Different states of particles correspond to different "differentials" while different local operators correspond to different "cycles" in these integrals. Moreover, one finds the same number of possible "cycles" as the number of linearly independent vectors in the space H_f. The problem arises of explaining the origin of null-vectors in this context.
The deformed hyper-elliptic integrals satisfy identities similar to those satisfied by usual hyper-elliptic integrals. Most remarkably, there is a deformation of the Riemann bilinear identity. As a result of these identities certain local operators vanish which are shown to correspond to null-vectors in H_f. Calculating the characters we find that there is correct number of null-vectors.
Part of these results is joint work with D. Bernard and O. Babelon.

References:
F. A. Smirnov, On the deformation of abelian integrals, *Lett. Math. Phys.* **36** (1996) 267
O. Babelon, D. Bernard, F. A. Smirnov, Null vectors in integrable field theory, *Commun. Math. Phys.* **186** (1997) 601

1991 MS Classification:
Yau, Horng-Tzer, Courant Institute of Mathematical Sciences, New York University, New York, NY, 10012, USA

Scaling Limit of Particle Systems, Incompressible Navier-Stokes Equation and Boltzmann Equation

Although the basic microscopic physics laws of the classical or quantum mechanics are well established by the Newton or the Schroedinger equations, the macroscopic physics are described by phenomenological equations such as the Euler equation, the incompressible Navier-Stokes equation or the Boltzmann equation. Our goal is to understand mathematically the connection between these two descriptions by taking scaling limits of the microscopic equations with suitably chosen scalings of space, time, and other macroscopic parameters of the systems. Furthermore, we would like to study the corrections or fluctuations to these phenomenological equations and other problems intrinsic to the microscopic descriptions such as metastability and behavior of tagged particles. Our understanding of the Newton or the Schroedinger equation with many degrees of freedoms, however, is extremely limited and rigorous treatments of the scaling limits of many-body dynamics from the first principles are far beyond the current technique. There are three possible directions:
(A) Classical dynamics of one or a few particles in a given environment such as a classical Lorentz gas or some billiard problems.
(B) Quantum dynamics of one or a few particles in a given environment such as a quantum Lorentz gas.
(C) Stochastic dynamics of many-body systems, i. e. , the classical dynamics are modified by adding certain noise (elliptic operators).
The approach (C) retains the many-body feature while (A) and (B) treat the first principle dynamics. In this talk, we shall review some results related to (B) and (C).
We shall first describe a stochastic particle system based on random walks with collision rules on the standard cubic lattice $\mathbb{Z}^d$, $d = 3$ and state precisely a law of large numbers stating that, for initial distributions corresponding to arbitrary macroscopic L^2 initial data, the distributions of the evolving empirical momentum densities are supported entirely on global weak solutions of the incompressible Navier-Stokes equations. Furthermore the viscosity will be characterized by the Green-Kubo formula and variational principles. The key analytic inputs are a method solving the fluctuation-dissipation equation of the lattice gases and multiscale estimates based on the logarithmic Sobolev inequality. We shall also review some applications to these methods including an estimate on the mixing rates of

random rotations on S^N (Kac model of Boltzmann equation) and $SO(N)$ as $N \to \infty$.

Our second result concerns the quantum Lorentz gases which describe one electron in an environment given by random impurities. There are two relevant limits: the Boltzmann-Grad limit (or the low density limit) and the weak coupling limit. Our result states that the phase space density of the quantum evolution defined through the Wigner transform converges weakly to a linear Boltzmann equation globally in time. The collision kernel is given by the quantum differential scattering cross section in the Boltzmann-Grad limit and by its semi-classical approximation in the weak coupling limit. Notice that in the Boltzmann-Grad limit it is the true quantum differential scattering cross section that appears in the Boltzmann equation, instead of its semi-classical approximation. Thus the Boltzmann-Grad limit is not a semi-classical limit. Our method is based on a renormalized perturbation expansion with cutoff and works for a general class of statistical laws of the random impurities including Gaussian and Poisson distributions.

12. Probability and Statistics

1991 MS Classification: 60J25, 60K35
Aldous, David, University of California at Berkeley, USA
Stochastic Coalescence

Consider N particles, which merge into clusters according to the rule: a cluster of size x and a cluster of size y merge at (stochastic) rate $K(x, y)/N$, where K is a specified rate kernel. This Marcus-Lushnikov model of coalescence, and the underlying deterministic approximation provided by the Smoluchowski coagulation equations, have an extensive scientific literature. A recent reformulation is the *general stochastic coalescent*, whose state space is the infinite-dimensional simplex (the state $\mathbf{x} = (x_i, i \geq 1)$ represents unit mass split into clusters of masses x_i), and which evolves by clusters of masses x_i and x_j coalescing at rate $K(x_i, x_j)$. Existing mathematical literature (Kingman's coalescent, component sizes in random graphs, fragmentation of random trees) implicitly studies certain special cases. Recent work has uncovered deeper constructions of special cases of the stochastic coalescent in terms of Brownian-type processes. Rigorous study of general kernels has only just begun, and many challenging open problems remain.

1991 MS Classification: 49L99 60H30 62B10 90A09
Avellaneda, Marco, Courant Institute, New York University, USA
Inverse Problems in Quantitative Financial Modeling

A key issue in modern quantitative finance is the *calibration* of asset pricing models. In order to correctly reflect market conditions, asset pricing models are required to price correctly a given set of "reference" securities. Accordingly, the parameters of the model must be adjusted as prices change. Mathematically, the calibration problem can be formulated as searching

for a (multi-dimensional) diffusion measure for which the expected values of certain functions are known. This is a *moment problem* in path-space, which may be formulated as an inverseproblem for a parabolic PDE. I shall

describe a solution to this type of problem which uses the technique of *minimization of relative entropy.* This method has the advantage of bieng numerically tractable and stable with respect to perturbations in the inputs. Furthermore, the output of the model has strong financial significance. The mathematical tools used to solve this problem are nonlinear

Hamilton-Jacobi-Bellman-PDEs(quasi-linear as well as fully non-linear) and numerical optimization algorithms, such as L-BFGS. I will present some of applications for hedging the volatility risk of options and hedging the yield-curve risk of fixed-income portfolios. I will also like to point out some open problems in this field that I find mathematically interesting.

1991 MS Classification: 60K25

Bramson, Maury, University of Minnesota, USA

State Space Collapse for Queueing Networks

The diffusive limits of queueing networks, known as heavy traffic limits, are a topic of continuing interest. Until recently, the class of networks for which these limits had been derived was restricted. An important ingredient in such work is the demonstration of state space collapse, which says that, in the limit, the process must live on an appropriate invariant subspace. In an accompanying talk, Williams provides general conditions under which state space collapse suffices for heavy traffic limits. Here, we discuss how state space collapse can be reduced to the problem of showing the stability of the associated fluid model equations, which are the deterministic analog of the queueing network. We also discuss how these equations can be solved for various queueing networks, such as first-in first-out (FIFO) networks of Kelly type and certain static priority networks.

1991 MS Classification:

Mark Freidlin, University of Maryland, College Park, USA

Random and Deterministic Perturbations of Nonlinear Oscillators

We consider perturbations of an oscillator with one degree of freedom:

$$\ddot{q}_t^\epsilon + f(q_t^\epsilon) = \epsilon b(\dot{q}_t^\epsilon, q_t^\epsilon) + \sqrt{\epsilon}\sigma(\dot{q}_t^\epsilon, q_t^\epsilon) \circ \dot{W}_t, \quad q_0^\epsilon = q \in R^1.$$

Here W_t is the Wiener process in R^1, $0 < \epsilon << 1$. A typical example of the deterministic part of the perturbation is a small friction: $b(q, \dot{q}) = -\beta\dot{q}$. Let $F(q) = \int_0^q f(s)ds, \quad \dot{q} = p, \quad H(p,q) = p^2/2 + F(q)$. We assume that the potential $F(q)$ is generic (this means that $F(q)$ has a finite number of non-degenerate critical points) and $\lim_{|q|\to\infty} F(q) = \infty$. Let Γ be the graph homeomorphic to the set of connected components of the level sets of the Hamiltonian $H(p,q)$ provided with the natural topology: $\Gamma = \{I_1, ..., I_n; O_1, ...O_m\}$. Each vertex O_i belongs either to one edge (exterior vertex) or to three edges (interior vertex). A point $y \in \Gamma$ can be characterized by two coordinates (H, k), where k is the number of the edge containing y, and H is the value of the Hamiltonian on the level-set component corresponding to y.
Define a mapping $Y : \mathrm{R}^2 \to \Gamma$ so that $Y(x) = (H(x), k(x)), \quad x \in R^2$, where $k(x)$ is the number of the edge containing the point corresponding to the trajectory of the non-perturbed system ($\epsilon = 0$) starting at $x \in R^2$. The function $k(x)$, as well as $H(x)$, is a first integral for the non-perturbed system. The pair $X_t^\epsilon = (p_t^\epsilon, \quad q_t^\epsilon), p_t^\epsilon = \dot{q}_t^\epsilon$, forms a diffusion process. The long-time behavior of this process as $\epsilon \downarrow 0$ is determined by the averaging principle. The slow component of X_t^ϵ is the projection $Y(X_t^\epsilon)$ of X_t^ϵ on Γ. We prove that the processes $Y_t^\epsilon = Y(X_{t/\epsilon}^\epsilon)$ converge weakly in the space of continuous functions on Γ to a diffusion process Y_t on the graph $\Gamma = \{I_1, ..., I_n; O_1, ...O_m\}$.

Diffusion processes on a graph Γ are defined by a family of second order differential operators $L_1,..L_n$ and by gluing conditions at the vertices. The operator L_k defines Y_t inside $I_k \subset \Gamma$. The gluing conditions at a vertex $O_i \in \Gamma$ define, roughly speaking, what the trajectory is doing after is touches O_i. To describe the limiting process Y_t, denote by $G_k(z)$ the domain in R^2 bounded by $C_k(z) = Y^{-1}(z,k)$ if (z,k) is not a vertex. Let $S_k(z)$ be the area of $G_k(z)$. One can consider $S_k(z)$ as a function on Γ. This function has, in general, discontinuities at the interior vertices. If O_i is an interior vertex and $I_{i_1}, I_{i_2}, I_{i_3}$ are the edges attached to O_i, denote $S_j(O_i)$ the limit of $S_{i_j}(z)$ as the point $(z,i_j) \in I_{i_j}$ approaches O_i. Then in the case $b(p,q)=0$, $\sigma =$ unit matrix, the operators L_k defining the process on I_k have the form

$$L_k = \frac{1}{2S_k'(z)}\frac{d}{dz}(S_k(z)\frac{d}{dz}).$$

A smooth inside the edges and continuous on Γ function $u(y), y \in \Gamma$, belongs to the domain of the generator A of the limiting process Y_t, if and only if $Au(y)$ is continuous on Γ and at any interior vertex $O_i \in \Gamma$ the following gluing condition is satisfied

$$\sum_{j=1}^{3} S_j(O_i)D_j u(O_i) = 0.$$

Here D_j is the operator of differentiation in the coordinate H along I_{i_j} attached to $O_i, j = 1,2,3$. The operators L_k and the gluing conditions at the vertices define the limiting process in a unique way. The characteristics of the limiting process on Γ are calculated also for any $\sigma(p,q)$ and $b(p,q)$. Let now $\sigma \equiv 0$, so that just deterministic perturbations are considered. it turns out that the slow component $Y(X^\epsilon_{t/\epsilon})$ of the deterministically perturbed system in any reasonable sense converges as $\epsilon \downarrow 0$ to a stochastic process on Γ. In this case the limiting process $Y_t = (z_t, k_t)$ is deterministic inside each edge I_k:

$$\begin{aligned}\dot{Y}_t =& b_k(Y_t), Y_t \in I_k,\\ b_k(z) =& \frac{1}{S_k'(z)}\int_{G_k(z)} \frac{\partial b(p,q)}{\partial p} dpdq.\end{aligned}$$

At an interior vertex O_i the trajectory branches with certain probabilities between the edges attached to the vertex. The branching probabilities are expressed through the function $b(p,q)$ and the curve $Y^{-1}(O_i)$. To give rigorous sense to this statement, we consider random perturbations of the deterministic system $\dot{X}^\epsilon_{t/\epsilon} = (p^\epsilon_{t/\epsilon}, q^\epsilon_{t/\epsilon})$ and then calculate the limit when first $\epsilon \downarrow 0$ and then the random perturbations tend to zero. It is important to underline that the limit is the same for a wide class of random perturbations of the equation or of the initial conditions. So that the structure of the limiting stochastic process is an intrinsic property of the oscillator and its deterministic perturbations, but not of the random perturbations which we use for regularization of the problem. We also will discuss some similar problems for systems with many degrees of freedom.

1991 MS Classification:

Ghosh, J. K., Indian Statistical Institute, India, and Purdue University, USA

Bayesian Density Estimation

Consider a sequence of independent real valued random variables $\{X_i\}$ with unknown probability density f_o. A Bayesian makes inference about f_o by putting a prior probability measure on the space of all densities $\mathcal{P}$ equipped with a topology and a Borel structure. The posterior probability measure of the random f given the observed $X_1, \cdots, X_n$ is the Bayesian's summary. In particular, the posterior expectation of f is the Bayes estimate of f_o. Recently, various methods depending on different priors, namely, Dirichlet mixtures of normals and Gaussian processes for $\log f$, have become computationally feasible and are beginning to be widely used even though the posteriors cannot be written down explicitly. Unlike non-Bayesian methods they do not have a problem of choosing an optimal bandwidth and can adapt to the location of f_o.
A natural question of validation arises. Does the Bayes estimate converge to f_o or the posterior to δf_o in various natural senses of convergence ? Answer to such questions, for finite dimensional $\mathcal{P}$, which go back to Laplace, is usually yes. In infinite dimensional problems, like density estimation, the answer is often no. These problems are hard because explicit formulas for posterior are rare. We will explore recent positive results in the general area of Bayesian non-parametrics with main focus on the problem of density estimation. In particular, we will provide validation for the most popular method of Bayesian density estimation. Kolmogorov's metric entropy for a function space plays an important role in he development.
The talk will be technically self-contained.

1991 MS Classification: 62E20, 11P21, 60F05

Götze, Friedrich, Universität Bielefeld, Germany

Lattice Point Problems and the Central Limit Theorem in Euclidean Spaces

The distribution of sums of n independent identically distributed random vectors taking values in an d-dimensional Euclidean space is approximated by a multivariate Gaussian distribution on regions defined by quadratic forms like ellipsoids and hyperboloids. Provided that the random vectors have finite fourth moments and non degenerate covariance operators error bounds of order $\mathcal{O}(n^{-1})$ are derived for regions and distributions centered at zero and dimensions $d \geq 9$. The rate is sharp improving previous bounds of order $\mathcal{O}(n^{-d/(d+1)})$ by Esseen (1945). For non centered problems the approximation is given by an Edgeworth expansion. For $d \geq 12$ upper and lower bounds of the error are of order $\mathcal{O}(n^{-1})$ and depend explicitely on the 12 largest eigenvalues only (joint work with V. Ulyanov). Comparable upper bounds are proved for the error in asymptotic expansions for the distribution of degenerated U-statistics.

Fourier inversion together with discretization and conditioning techniques reduces the problem of proving error bounds to the investigation of weighted multivariate trigonometric sums of quadratic forms on bounded sets in $\mathbb{Z}^d$. One of the basic tools is an inequality which extends Weyl estimates for such trigonometric sums and allows to control their products at different frequencies.

These techniques are applied to the estimation of the the lattice point remainder, i. e. the difference between the number of lattice points in $\mathbb{Z}^d$ and the corresponding Lebesgue volume of an open ellipsoid centered at zero and scaled by a factor R. It is of order $\mathcal{O}(R^{d-2})$ resp. $o(R^{d-2})$ for general resp. *irrational* ellipsoids and $d \geq 9$. The rate is sharp and improves previous bounds of order $\mathcal{O}(R^{d-2d/(d+1)})$ for general ellipsoids by Landau (1915).
As a consequence the maximal gap between successive values of a positive definite *irrational* quadratic form tend to zero for values growing to infinity. This has been conjectured by Davenport and Lewis (1972) for all dimensions $d \geq 5$.

The results extend to hyperbolic cases like the asymptotic approximation of the number of lattice points in a cube of side length $R \to \infty$ which result in values of an irrational *indefinite* form in a fixed interval by the Lebesgue volume. For $d \geq 9$ this yields explicit bounds in terms of trigonometric sums for the convergence in this quantitative Oppenheim problem. This convergence has been proved by Eskin, Margulis and Mozes (1997) for signatures (p, q), $p \geq 3$, $q \geq 1$ using ergodic theory on groups.
These results are joint work with V. Bentkus.

1991 MS Classification: Primary 62G09, Secondary 62G05
Hall, Peter, Australian National University, Canberra, Australia
Applications of Intentionally Biased Bootstrap Methods

For centuries the sample mean has been recognised as an estimator of the population mean — or in contemporary notation, $\bar{X} = \int x\, d\widehat{F}(x)$ is an estimator of $\mu = \int x\, dF(x)$, where $\widehat{F}$ denotes the empirical distribution function computed using a sample drawn from a distribution F. The idea that the sample median is an estimator of the population median is implicit in work of Galton about 120 years ago. Thus, the notion that a parameter may be regarded as a functional of a distribution function, and estimated by the same functional of the standard empirical distribution, is a rather old one, even though it was perhaps only recognised as a general principle relatively recently.
Efron's (1979) classic paper on the bootstrap vaulted statistical science forward from these simple ideas. Efron saw that when substituting the true F by an estimator $\widehat{F}$, the notion of a 'parameter' could be interpreted much more widely than ever before. It could include endpoints of confidence

intervals or critical points of hypothesis tests, as well as error rates of discrimination rules. It could encompass tuning parameters in a wide variety of estimation procedures (even the nominal levels of intervals or tests can be regarded as tuning parameters), and much more.

Another key ingredient of the methods discussed by Efron (1979) was recognition that in cases where the functional of $\widehat{F}$ could not be computed directly, it could be approximated to arbitrary accuracy by Monte Carlo methods. The combination of these two ideas — the substitution or 'plug in $\widehat{F}$' rule, and the notion that Monte Carlo methods can be used to surmount computational obstacles — has been little short of revolutionary.

In 'standard' settings, where the appropriate way of applying the bootstrap is relatively clear, the uniform bootstrap offers an unambiguous approach to inference. Therein lies part of its attraction — there are no tuning parameters to be selected, for example. However, the lack of ambiguity can also be a drawback. In particular, the rigidity of the conventional bootstrap algorithm makes it relatively difficult to modify uniform-bootstrap methods so as to include constraints on the parameter space.

The weighted bootstrap, where resampling weights are not required to be identical, offers a way around this difficulty. It provides an opportunity for 'biasing' bootstrap estimators so as to fulfill constraints. Moreover, we may interpret the notion of a 'constraint' in a very broad sense, like that of a 'parameter'. Nevertheless, an unambiguous approach to choosing the weights is required. Biased-bootstrap methods provide a solution to that problem.

The biased bootstrap requires two inputs from the experimenter: a measure of the distance of the weighted-bootstrap distribution from the uniform-bootstrap distribution, and the constraints. The first is generic to a wide range of problems, and will be discussed from that viewpoint in the lecture. The second is problem-specific, and will be introduced through a range of examples. A general form of the biased bootstrap consists of choosing the weights so as to minimise distance subject to the constraints being satisfied. The method encompasses Owen's (1988) technique of empirical likelihood, as well as related techniques such as Chen's (1997) approach to constrained density estimation, and a novel approach to bootstrapping the generalised method of moments model suggested by Imbens, Johnson and Spady (1998).

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1991 MS Classification: 62G07,62G20,62C20
Johnstone, Iain, Stanford University, USA
Oracle Inequalities and Nonparametric Function Estimation

Statistical theory aims in part to articulate when and why certain applied methods of data analysis succeed. With emergence of large, often instrumentally acquired datasets, recent decades have seen a focus on "nonparametric" models in which the number of model parameters grows with the size of available data.
A common caricature is to posit observed data $y \in \mathcal{R}^n$ with structure $y = K\theta + \epsilon z$, Here θ is an unknown function which one desires to "estimate" or "recover", K is a linear operator and $z \in \mathcal{R}^n$ is a vector of standard Gaussian noise of known scale ϵ. An estimator $\hat{\theta}(y)$ is evaluated in terms of, for example, its mean squared error, or "risk" $R(\hat{\theta}, \theta) = E_\theta|\hat{\theta}(y) - \theta|^2$, where E_θ denotes expectation with respect to the probability distribution of y given θ and $|\cdot|$ denotes Euclidean distance.
A class of models $\{\mathcal{M}_J, J \in \mathcal{J}\}$ is given. Typically $\mathcal{M}_J$ are subsets, often linear, of $\mathcal{R}^n$ to which θ might belong. For each model J, there is a "natural" estimator $\hat{\theta}_J$ with risk $R(\hat{\theta}_J, \theta)$. For a fixed, unknown θ, the "ideal risk" would be $\inf_J R(\hat{\theta}_J, \theta)$. This risk would be attainable with the aid of an oracle who divulges the optimal J. Lacking access to oracles, we seek estimators $\hat{\theta}(y)$ with risk properties that mimick the ideal risk as closely as possible.
Here is one example of such an inequality. Suppose that K is the identity on $\mathcal{R}^n$ and that the data model is expressed in terms of coefficients in an orthonormal basis $\{\psi_i\}$:

$$y_i = \theta_i + \epsilon z_i \qquad\qquad i = 1, \dots n.$$

Here z_i are independent Gaussian noises of mean zero and variance one. Let an estimator $\hat{\theta}_\lambda$ be built componentwise via *soft thresholding* of the observed data: $\hat{\theta}_{\lambda,i}(y)$ equals 0 if $|y_i| \le \lambda$ and equals $y_i - \lambda \,\mathrm{sgn} y_i$ otherwise. Set $\lambda = \epsilon\sqrt{2\log n}$. Then, whatever be $\theta \in \mathcal{R}^n$ and $\epsilon > 0$,

$$E_\theta \sum_1^n (\hat{\theta}_{\lambda,i}(y) - \theta_i)^2 \le (2\log n + 1)[\epsilon^2 + \sum_1^n \min(\theta_i^2, \epsilon^2)].$$

From the inequality, one learns, for example (i) the ideal risk $\sum_1^n \min(\theta_i^2, \epsilon^2)$ connects compressibility of the signal θ in basis $\{\psi_i\}$ with the quality of estimation/recovery. It also establishes a link with approximation theory, (ii) near minimax optimality properties simultaneously over a class of parameter spaces Θ, and (iii) the factor $2\log n$ cannot be improved: a heuristic limiting the detection of sparse signals in Gaussian noise shows why.
Time permitting, we will discuss two other such oracle inequalities: (i) a linear shrinkage inequality involving the classical James-Stein estimator, and
(ii) an oracle inequality for "empirical atomic decomposition" in which the orthobasis $\{\psi_i, i = 1, \dots n\}$ is replaced by a possibly highly redundant *dic-*

tionary, such as might be approriate when the class of models for the signal θ is not well represented in terms of unconditional bases.

1991 MS Classification: 60
Le Gall, Jean-François, Ecole Normale Supérieure, France

Branching processes, random trees and superprocesses

We will survey a number of recent developments concerning the genealogy of branching processes, together with their applications to measure-valued processes and interacting particle systems, as well as some connections with semilinear partial differential equations.

A Galton-Watson branching process gives at time k the number of individuals in the k-th generation of a population where each individual gives rise, independently of the others, to a random number of children distributed according to a given offspring distribution. The genealogy of such a branching process is obviously described by a finite collection of discrete genealogical trees. One may ask about a similar description of the genealogy of continuous-state branching processes, which are the continuous analogues of Galton-Watson processes. Roughly speaking, continuous-state branching processes are obtained as scaling limits of Galton-Watson processes, starting from a "large" initial population and considering the evolution over a "long" time interval. Their distribution is characterized by a branching mechanism function ψ playing the same role as the offspring distribution of the discrete case. In the special case of Feller's diffusion ($\psi(u) = u^2$), it has been known for some time that the genealogical structure is coded by linear Brownian motion, in the sense that the contour process of the tree (describing the motion of a particle that "runs up and down along the branches of the tree") is a Brownian excursion. This fact was made explicit by Aldous, who has studied the corresponding continuous genealogical tree known as the Continuum Random Tree (CRT). In a joint work with Le Jan, we show that, for a more general continuous-state branching process, the continuous genealogical tree can be coded by a functional of a Lévy process (i. e. a real-valued process with stationary independent increments) with no negative jumps, whose Laplace exponent is precisely the branching mechanism ψ. This fact provides a unified perspective on parts of the theories of Lévy processes and of branching processes.

Measure-valued branching processes, also called superprocesses, combine a branching phenomenon (described by a continuous-state branching process) with a spatial motion, which can be for instance Brownian motion in $\mathbf{R}^d$. Roughly speaking, this means that the (infinitesimal) individuals of the continuous population move independently in space according to the law of Brownian motion. Superprocesses have been studied extensively in the last ten years, in particular by Dawson, Dynkin and Perkins. The genealogical structure of continuous-state branching processes provides a powerful tool for investigating path properties of superprocesses as well as their connections with partial differential equations. The case $\psi(u) = u^2$ in particular leads to the Brownian snake construction of quadratic superprocesses. A typical analytic application (joint work with Dhersin) is a nonlinear Wiener test characterizing those domains in $\mathbf{R}^d$ where there exists a positive solution of $\Delta u = u^2$ with infinite boundary conditions. Other relations between superprocesses and partial differential equations, including the classification of positive solutions of $Lu = \psi(u)$ in a general domain, have been studied recently by Dynkin and Kuznetsov.

If time permits, we will briefly describe connections of the previous developments with other areas of probability theory and mathematical physics. This applies in particular to the asymptotics of certain problems of statistical mechanics. Derbez and Slade have shown that the Continuum Random Tree arises in scaling limits of lattice trees in high dimensions, and very recently Hara and Slade proved a similar result for the infinite incipient cluster in critical percolation on the lattice. Finally, a recent joint work with Bramson and Cox has established that the asymptotic behavior of the classical voter model can be analysed in terms of the genealogical structure of Feller's diffusion.

1991 MS Classification: 60J27
Siegmund, David, Stanford University, USA
Genetic linkage analysis: an irregular statistical problem

The goal of gene mapping, or linkage analysis, is to locate genes that affect particular traits, e. g., genes that affect human susceptibility to particular diseases or genes that affect productivity of agriculturally important species. Recent changes in technology that provide large numbers of informative genetic markers at known locations throughout a genome suggest new statistical problems concerned with the design and analysis of gene mapping experiments. An artificially simplified, but illuminating genetic model leads to the following class of statistical problems. Observations are available on a doubly indexed set of random variables $Z(c, i\Delta)$, where $c = 1, \cdots, 23$ indexes the set of human chromosomes of genetic lengths ℓ_c and $i\Delta, 0 \leq i\Delta \leq \ell_c$ are the locations of markers spaced at intermarker distance Δ along each chromosome. For different values of c the random variables are independent. For each fixed c, $Z(c,t)$ is a Gaussian process in t, which satisfies

$$\mathrm{Cov}[Z(c,s), Z(c,t)] = \exp(-\beta|t-s|). \tag{1}$$

For most or perhaps all values of c

$$E[Z(c,t)] = 0 \text{ for all } t, \tag{2}$$

while for some c', $0 < \tau < \ell_{c'}$ and $\xi > 0$

$$E[Z(c',t)] = \xi \exp(-\beta|t-\tau|). \tag{3}$$

The values of c', τ, and ξ are all unknown. Thus the data consist of a large number of zero mean Ornstein-Uhlenbeck processes observed at equally spaced "time" points. A small number of these processes are superimposed on a double exponential drift defining a "peak" of an unknown height ξ at an unknown location τ. The problem is to decide which chromosomes, if any, harbor such a location τ and estimate the location by a confidence region.

The purpose of this paper is (i) to explain briefly the genetic background of the preceding problem, (ii) propose a framework for its solution that is useful as a point of departure for discussing more realistic versions of the problem, and (iii) describe some alternative models designed to capture the complicating features arising in practice. Special consideration is given to (a) the problem of multiple comparisons that arises through examining the large number of variables $Z(c, i\Delta)$ in searching for the relatively few values of c', t where the expected value is substantially different from 0, (b) the power to identify correctly the values c' as a function of the principal parameters τ, Δ, ξ as well as other parameters that enter into more realistic models, and (c) estimation of τ by confidence regions. Some of these problems can be understood in terms of recent literature on change-point problems, to which they are closely related. Particular attention will be given to complex and quantitative traits, which one expects to involve multiple, possibly interacting, genes, and to the relative efficiency of different strategies.

1991 MS Classification: 60K40,82D30
Sznitman, Alain-Sol, ETH Zurich, Switzerland

Brownian motion and random obstacles

The investigation of Brownian motion and random obstacles exhibits a rich phenomenology and displays paradigms which appear in several other areas of random media. The lecture will try to give an overview of recent developments such as the pinning effect, or the confinement property. It will discuss some of the techniques which have emerged such as the method of enlargement of obstacles, or the Lyapunov exponents. It will also point out connections with other areas of random media.

1991 MS Classification: primary 60G07; secondary 60H10, 60J65.
Tsirelson Boris, Tel Aviv University, Israel

Within and beyond the reach of Brownian innovation

A discrete-time random process is a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables; the corresponding distribution is a probability measure μ on the space $\mathbb{R}^{\mathbb{N}}$ of all sequences $(x_1, x_2, \dots)$ of real numbers. In particular, independent random variables U_n uniform on $(0,1)$ correspond to the product-measure $\mathrm{mes}^{\mathbb{N}} = \mathrm{mes} \otimes \mathrm{mes} \otimes \dots$ where mes is Lebesgue measure on $(0,1)$. An arbitrary μ is the image of $\mathrm{mes}^{\mathbb{N}}$ under some (non-unique) transformation $f : (0,1)^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ of the form

$$\begin{aligned}&f(u_1,u_2,\dots)=(x_1,x_2,\dots)\quad\text{when}\\&x_1=f_1(u_1),\ x_2=f_2(u_2,x_1),\ x_3=f_3(u_3,x_2,x_1),\ \dots\end{aligned}\tag{1}$$

Thus, arbitrary random variables X_n can be produced from "standard" random variables U_n by taking $X_n = f_n(U_n, X_{n-1}, \dots, X_1)$, which is an elementary case of *innovation.* A non-elementary case of innovation is a stochastic differential equation (SDE)

$$dX(t) = \sigma\Big(t,\ \big(X(s)\big)_{s\in[0,t]}\Big)\,dB(t) + v\Big(t,\ \big(X(s)\big)_{s\in[0,t]}\Big)\,dt\,. \tag{2}$$

Here the idea is to produce a continuous-time random process $X(\cdot)$ from the Brownian motion $B(\cdot)$, the latter being basically a vast collection of small independent increments $B(t+\Delta t) - B(t)$. The common feature of the two forms of innovation is as follows. Equation (1) implies

$$\mathbb{E}\,\big(\,\phi(X_1,X_2,\dots)\,\big|\,U_1,\dots,U_n\,\big) = \mathbb{E}\,\big(\,\phi(X_1,X_2,\dots)\,\big|\,X_1,\dots X_n\,\big) \tag{3}$$

for all $n \in \mathbb{N}$ and all bounded Borel functions $\phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, while for (2),

$$\mathbb{E}\Big(\phi\big(\big(X(s)\big)_{s\in[0,\infty)}\big)\,\Big|\,\big(B(s)\big)_{s\in[0,t]}\Big) = \mathbb{E}\Big(\phi\big(\big(X(s)\big)_{s\in[0,\infty)}\big)\,\Big|\,\big(X(s)\big)_{s\in[0,t]}\Big) \tag{4}$$

for all $t \in (0,\infty)$ and all bounded Borel functions ϕ on the relevant space of functions. In the modern jargon of *filtrations* (increasing families of σ-fields) we say that the filtration of (X_n) is *immersed* into the filtration of (U_n), and similarly, the filtration of $X(\cdot)$ is immersed into the filtration of $B(\cdot)$. Unexpected phenomena emerge, however, when passing from discrete time (1) to continuous time (2). Indeed, something strange already happens for discrete time if we extend the past back infinitely far; $\big(X_n\big)_{n\in-\mathbb{N}} = (\dots, X_{-2}, X_{-1})$. For example, consider the following very simple equation of the type (1) but for $n \in -\mathbb{N}$:

$$X_n = \lambda U_n + X_{n-1} \qquad (\mathrm{mod}\ 1); \tag{5}$$

that is, $X_n \in (0,1)$ and $X_n - (\lambda U_n + X_{n-1}) \in \mathbb{Z}$. (Here $\lambda \in (0,\infty)$ is a parameter, and once again, the U_n are independent and uniform on $(0,1)$.) Equation (5) determines uniquely the joint distribution of all X_n, but fails to determine a function $f : (0,1)^{-\mathbb{N}} \to \mathbb{R}^{-\mathbb{N}}$ such that $(X_n)_{n\in-\mathbb{N}} = f\big((U_n)_{n\in-\mathbb{N}}\big)$. In fact, (5) implies that X_{-1} is independent of $(U_n)_{n\in-\mathbb{N}}$, and the filtration of (X_n) is not a part of the filtration of (U_n). We call (X_n) a *weak solution* of (5). It is not an innovation! However, an innovation for (X_n) exists. For $\lambda = 1$ it is simple: just take $X_n = U_n$ instead of (5).
There exists $(X_n)_{n\in-\mathbb{N}}$ admitting no innovation (Vershik 1970); its filtration is *non-standard.* The phenomenon of (5) has its continuous-time counterpart, an SDE that fails to innovate (Tsirelson 1975, investigated further by Beneš, Stroock, Yor, Le Gall), though its (weak) solution admits another innovation (Émery and Schachermayer 1998). There is an

SDE whose solution admits no innovation (Dubins, Feldman, Smorodinsky, Tsirelson 1994); its filtration is *non-Brownian.* Brownian motions in graphs (treated as one-dimensional topological spaces), known as Walsh's Brownian motions or spider-martingales, generate non-Brownian filtrations (Tsirelson 1996), differing from Brownian filtrations by the *splitting multiplicity* (Barlow, Émery, Knight, Song, Yor 1997). There is a pure martingale generating a non-Brownian filtration (Émery and Schachermayer 1998). The SDE for the sticky Brownian motion generates a non-Brownian filtration (Tsirelson and Warren 1998).

We know of three filtration invariants: dimension (understood long ago, integer-valued or ∞), splitting multiplicity (integer-valued or ∞), and *cosiness* (two-valued). The n-dimensional Brownian motion ($n = 1, 2, \ldots ; \infty$) generates a cosy filtration of dimension n and splitting multiplicity 2. The latter, combined with well-known facts about submartingales, subharmonic functions and Green functions, proves (Tsirelson 1997) a conjecture of Bishop (1991): in any dimension, the natural projection of the Martin boundary to the topological boundary is at most 2 to 1 almost everywhere. Splitting multiplicity is robust under measure changes and invertible time changes, while cosiness is not. Splitting multiplicity > 2 implies non-cosiness. All known non-Brownian filtrations of instant dimension 1 are non-cosy.

A Brownian filtration, by definition, is generated by a finite or countable collection of independent Brownian motions. On the other hand, a Brownian motion in a topological group G is defined as a continuous G-valued random process with stationary independent increments on the left, starting from the unit of G. Thus, a filtration is Brownian if and only if it is generated by *some* Brownian motion in the Hilbert space (separable, infinite-dimensional). Every Brownian motion in a Lie group generates a Brownian filtration (Yosida 1952). The same holds for some infinite-dimensional groups: the group of diffeomorphisms (Baxendale 1984) and the group of unitary operators (Tsirelson 1998). The proof of the latter involves continuous tensor products (Feldman 1971, Arveson 1989, Tsirelson and Vershik 1995) and continuous quantum measurements (Davies 1976).

Problem 1. Is there a Brownian motion in *some* Polish group, that generates a non-Brownian filtration?

Here, "non-Brownian" may be replaced with "non-cosy", since a cosy filtration generated by a Brownian motion in a group is necessarily Brownian (Tsirelson 1998). If the answer is positive, we face another problem:

Problem 2. Does every random process admit an innovation by some Brownian motion in some Polish group?

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1991 MS Classification: 60

Williams, Ruth, University of California at San Diego, USA

Reflecting Diffusions and Queueing Networks

Certain diffusion processes, known as reflecting Brownian motions, have been proposed as approximations for heavily loaded queueing networks. Although there is now a substantial theory for these diffusions, limit theorems justifying their role as approximations for queueing networks have only been proved in some cases. Indeed, since a surprising example of Dai and Wang (1993) it has been known that these approximations are not always valid, and a challenging open problem has been that of establishing general conditions under which they do hold.

In this talk, some foundational theory for the diffusions will be described, including an invariance principle. The history of heavy traffic limit theorems for queueing networks will be briefly reviewed. Then an indication will be given of how the invariance principle can be used to obtain general sufficient conditions under which a heavy traffic limit theorem holds for open queueing networks. One of the key sufficient conditions will be that a property called state space collapse holds. In a companion talk, Bramson will relate state space collapse to the behavior of a balanced fluid limit (or law of large numbers approximation) for a queueing network. The combination of these results yields new heavy traffic limit theorems for some classes of networks. Queueing networks with a first-in-first-out service discipline will be used as examples throughout the talk.

13. Combinatorics

1991 MS Classification: 05C
Bollobás, Béla, University of Memphis, USA, and University of Cambridge, England

Hereditary properties of graphs: asymptotic enumeration, global structure, and colouring

We shall study the asymptotic behaviour of rather general properties of graphs. The basic questions we address are simply how many graphs there are with a given property, and what the structure of a typical graph is with that property. Under the mild assumption that the property is hereditary, significant progress has been made in recent years towards answering these questions.

A *property* of graphs is an infinite class of graphs closed under isomorphism. A natural measure of the size of a property $\mathcal{P}$ is the sequence $(c_n) = (c_n(\mathcal{P}))$ defined by $|\mathcal{P}^n| = 2^{c_n \binom{n}{2}}$, where $\mathcal{P}^n$ is the set of graphs in $\mathcal{P}$ with vertex set $[n] = \{1, 2, \ldots, n\}$. What can one say about the sequence $(c_n(\mathcal{P}))$ for various classes of properties? More generally, what can one say about the probability $\mathbb{P}_{n,p}(\mathcal{P})$ that a random graph $G_{n,p} \in \mathcal{G}(n,p)$ belongs to $\mathcal{P}$? In order to study this probability, define $(d_{n,p})$ by $\mathbb{P}_{n,p}(\mathcal{P}) = 2^{-d_{n,p}\binom{n}{2}}$. If $p = 1/2$, so that $\mathcal{G}(n,p)$ is the set of all $2^{\binom{n}{2}}$ graphs on $[n]$ with the uniform distribution, then $\mathbb{P}_{n,1/2}(\mathcal{P}) = 2^{(c_n - 1)\binom{n}{2}}$, that is, $d_{n,1/2} = 1 - c_n$.

Ideally, one would like to find precise approximations of c_n and $d_{n,p}$ in terms of simple characteristics of $\mathcal{P}$; better still, one would like to achieve this by finding a property $\mathcal{P}_0$ that is easily described and approximates $\mathcal{P}$ in the sense that $\mathcal{P}_0 \subset \mathcal{P}$ and the probability of $\mathcal{P}_0^n$ is almost as large as that of $\mathcal{P}^n$. For example, as shown by Erdős, Kleitman and Rothschild, in the uniform case $\mathcal{G}(n, 1/2)$, the class $\mathcal{T}$ of triangle-free graphs can be approximated by the class $\mathcal{B}$ of bipartite graphs: $\mathcal{B} \subset \mathcal{T}$ and $|\mathcal{B}^n|/|\mathcal{T}^n| \to 1$, so that $c_n(\mathcal{T}) = \frac{1}{2} + \frac{5}{2n} + O(\frac{\log n}{n^2})$. Kolaitis, Prömel and Rothschild extended this result to the properties of being K_{r+1}-free and r-partite.

For a general property $\mathcal{P}$, nothing of interest can be said about the sequences (c_n) and $(d_{n,p})$. However, for the rather large and important class of *hereditary* properties, it was shown by Alekseev, and by Thomason and the author (continuing earlier work of Prömel, Steger, and others) that (c_n) is monotonically decreasing and so converges. A property $\mathcal{P}$ is said to be *hereditary* if it is invariant under the deletion of vertices: if a graph G is in $\mathcal{P}$ then every *induced subgraph* of G is in $\mathcal{P}$. For a family $\mathcal{F} = \{F_1, F_2, \ldots\}$ of graphs, the class $\mathrm{Her}(\mathcal{F})$ consisting of all graphs that contain no F_i as an *induced* subgraph is a hereditary property, and every hereditary property is of this form.

In order to find an approximation of a hereditary property $\mathcal{P}$ by a simple property and thereby describe $c(\mathcal{P}) = \lim_{n\to\infty} c_n(\mathcal{P})$ in terms of easily determined characteristics of $\mathcal{P}$, define an (r,s)-colouring of a graph H as a colouring of the vertices of H with colours $1, 2, \ldots, r$, such that two vertices of colour i are adjacent if and only if $i \leq s$. Denote by $\mathcal{P}_{r,s}$ the hereditary property of all (r,s)-colourable graphs, and define the *colouring number* $r(\mathcal{P})$ of a hereditary property P as

$$r(\mathcal{P}) = \max\{r : \text{ there exists } \ s \in \{0, 1, \ldots, r\} \ \text{ such that } \ \mathcal{P}_{r,s} \subset \mathcal{P}\}.$$

The colouring number $r(\mathcal{P})$ is finite unless $\mathcal{P}$ is the trivial property of all graphs, and it is immediate that $c(\mathcal{P}_{r,s}) = r(\mathcal{P}_{r,s}) = 1 - 1/r$. If $\mathcal{P}$ is a non-trivial property with colouring number $r = r(\mathcal{P})$, and $\mathcal{P}_{r,s} \subset \mathcal{P}$ then $\mathcal{P}_{r,s}$ is precisely the desired approximation of $\mathcal{P}$:

$$c(\mathcal{P}) = c(\mathcal{P}_{r,s}) = 1 - 1/r.$$

Given a hereditary property $\mathcal{P}$, the *$\mathcal{P}$-chromatic number* $\chi_{\mathcal{P}}(G)$ of a graph G is the minimal integer k for which the vertex set of G has a partition into k parts such that each part spans a graph which belongs to $\mathcal{P}$. Thus if $\mathcal{P}$ consists of all graphs with no edges then $\chi_{\mathcal{P}}(G)$ is just the usual chromatic number $\chi(G)$. It has been known for ten years that, for every $0 < p < 1$, almost every random graph $G_{n,p}$ has chromatic number $(k_p + o(1))n/\log n$, where $k_p = -\log(1-p)/2$, and in 1995 this was extended to the $\mathcal{P}$-chromatic number of random graphs for an arbitrary hereditary property $\mathcal{P}$, but only for the case $p = 1/2$.

For $p \neq 1/2$ the picture is much more intricate. For a hereditary property $\mathcal{P}$ and a fixed probability $0 < p < 1$, the sequence $(d_{n,p}(\mathcal{P}))$ is also monotonic, so tends to a limit $d_p(\mathcal{P})$. Although for $p \neq 1/2$ we cannot find an easily described property approximating $\mathcal{P}$, nevertheless we can find a sequence of properties $\mathcal{P}_j \subset \mathcal{P}$ with $d_p(\mathcal{P}_j) \to d_p(\mathcal{P})$. This approximation can be used to prove rather sharp results about the $\mathcal{P}$-chromatic number of a random graph $G_{n,p}$, for all constant values of p.

The new results are joint work with Andrew Thomason; for related results, see his poster entitled *Hereditary hypergraph properties.*

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1991 MS Classification: 90C27, 05C40
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Applications of Relaxed Submodularity

Let V be a finite set and $b : 2^V \to \mathbf{Z}$ a set-function. The submodular inequality for two subsets $X, Y \subseteq V$ is, as follows:

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y). \tag{1}$$

Function b is called fully submodular if (1) holds for every two subsets $X, Y \subseteq V$. (When the reverse inequality holds in (1) we speak of the supermodular inequality. When equality holds everywhere, b is called modular.) Submodular functions proved to be extremely powerful in combinatorial optimization. One intuitive explanation for this is that submodular functions, as recognized in the early eighties, may be considered as discrete counterparts of convex functions. For example, Lovász observed that a (natural) linear extension of an arbitrary set-function h to a real function on $\mathbf{R}^V$ is convex if and only if h is submodular. Another occurence of this relationship is the discrete separation theorem asserting that if an integer-valued supermodular function p is dominated by an integer-valued submodular functions b, then there is an integer-valued (!) modular function m for which $p \leq m \leq b$. Recently, this kind of analogy has been developed systematically by K. Murota into a theory relating convex analysis and discrete optimization.
In applications, however, very often the submodular inequality is not fulfilled for every pair of sets. Accordingly, several types of submodular functions have been introduced, analyzed and applied. For example, b is intersecting submodular if (1) is required only when $X \cap Y$ is nonempty. Some further relaxations are: T-intersecting, crossing, ST-crossing, skew, and near submodular functions. The main goal of the present paper is to outline recent developments on these functions, along with applications.

1. T-intersecting supermodular functions. For a directed graph $G = (V, E)$ with a specified node s and terminal set $T \subseteq V$ one may want to find a minimum cost subgraph in which there are k edge-disjoint paths from s to every element of T. This has been known to be NP-complete in general (even for $k = 1$) and solvable in polynomial time if $T = V$. With the help of T-intersecting supermodular functions we solve the problem for general T provided the cost of every edge entering a node in $V - T$ is zero. This extension enables one to solve a variation of the problem when k node-disjoint paths are required to exist from s to the elements of T.

2. Crossing sub- and supermodular functions. A set-function b is called crossing submodular if (1) holds for crossing pairs X, Y (that is, $X - Y \neq \emptyset, Y - X \neq \emptyset$, $X \cap Y \neq \emptyset, X \cup Y \neq V$).
(a) We exhibit a new, simplified necessary and sufficient condition for the feasibility of a submodular flow problem constrained by a crossing submodular function. As a consequence, several orientation theorems will be deduced, in particular, we describe a characterization of mixed graphs having a k-edge-connected orientation.
(b) A recent direction of investigations is motivated by connectivity augmentation problems. The general question is to find a graph or digraph that covers a supermodular function. For example, the following theorem, due to A. Benczúr and A. Frank [1996], holds: *given a non-negative, integer-valued, symmetric, crossing supermodular function p, there is a graph (V, E) of at most γ edges so that there are at least $p(X)$ edges in the cut $[X, V - X]$ for every subset $X \subseteq V$ if and only if* (i) $\sum_i p(V_i) \leq 2\gamma$ *holds for every family* $\{V_1, \ldots, V_t\}$ *of disjoint non-empty subsets of V and* (ii) *there is a member $V_j \in \mathcal{F}$ so that $p(V_1 \cup V_j) = 0$ for every partition $\mathcal{F} := \{V_1, \ldots, V_h\}$ of V with $h \geq \gamma + 2$ and $p(V_1) = 1$.* This result may be used to characterize graphs or hypergraphs which can be made k edge-connected by adding at most γ new edges.

3. ST-crossing supermodular functions. Let $\mathcal{F}$ be a family of subsets of V and $p : \mathcal{F} \to \mathbf{Z}_+$ a function. For two non-empty subsets S, T of V family $\mathcal{F}$ is called ST-crossing and p ST-crossing supermodular if $X, Y \in \mathcal{F}$, $X \cap Y \cap S \neq \emptyset$, $T - (X \cup Y) \neq \emptyset$ imply that $X \cap Y, X \cup Y \in \mathcal{F}$ and the supermodular inequality holds for these pairs. (When $S = T = V$, we are back at ordinary crossing functions.) We will report on the following theorem (due to A. Frank and T. Jordán, 1995): *given an ST-crossing supermodular function p on $\mathcal{F}$, there is a digraph (V, E) of at most γ edges so that every set $X \in \mathcal{F}$ admits at least $p(X)$ entering edges if and only if $\sum_i p(V_i) \leq \gamma$ for every family $\{V_1, \ldots, V_t\}$ of independent members of $\mathcal{F}$ where independence means that $V_i \cap V_j \cap S = \emptyset$ or $T \subseteq V_i \cup V_j$ $(1 \leq i < j \leq t)$.*
This result implies a solution to the node connectivity augmentation problem of directed graphs. As a recent development, we describe a (purely combinatorial) strongly polynomial algorithm for computing the minimum digraph in question, as well as the maximum independent family in the special case $p \equiv 1$. This way an algorithm is available for finding a minimum number of new edges whose addition to a given k-connected digraph results in a $(k+1)$-connected digraph. The algorithm also finds a surprising application in combinatorial geometry: with its help, a vertically convex region

R of the plane, bounded by horizontal and vertical segments, can be covered by a minimum number of rectangles belonging to R.

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Ordering the symmetric group: Why use Iwahori-Hecke algebras?

As a Coxeter group, the symmetric group is generated by simple transpositions $\sigma_i := (i, i+1)$. Taking subwords of *reduced* products of σ_i induces an order on permutations, the *Bruhat order*.
Instead of enumerating all subwords, one can replace decompositions $\sigma_i\sigma_j \cdots \sigma_h$ by products $(1+\pi_i)(1+\pi_j)\cdots(1+\pi_h)$ and expand them in the *0-Hecke algebra* (whose generators π_i satisfy the braid relations, and $\pi_i^2 = -\pi_i$). One then obtains the sum of all permutations below a given one.
Another method consists in taking left factors of permutations (as words in $\{1, 2, \dots\}$), sorting them and comparing them componentwise. This amounts to associating to a permutation, a Young tableau encoding all the cosets μP, where $\mu \in S_n$ and where the P's are all the maximal parabolic subgroups. One can instead use double cosets $P\,\mu\,P'$ and associate to a permutation the minimum representatives β of all the double cosets, such that β is below μ in the Bruhat order: $\mathcal{S}_n \ni \mu \mapsto B(\mu) := \{\beta : \beta \leq \mu\}$. The order becomes inclusion of sets :

$$\mu \leq \zeta \Leftrightarrow B(\mu) \subseteq B(\zeta)\ .$$

This furnishes an embedding of the symmetric group, as an ordered set, into the (distributive) lattice of *alternating sign matrices*.
However, being able to compare permutations does not provide a way to describe intervals for the Bruhat order. For example, since the symmetric group is a rank-poset, combinatorists would like to describe the *Poincaré polynomial* of an initial interval $[1, \mu]$ (for geometers this polynomial is equal to the number of points of a Schubert variety over the finite field F_q). For some permutations (which correspond to non-singular Schubert varieties) this polynomial is known to be symmetric and to factor into q-integers $1 + q + \cdots + q^k$. One can lift such a factorization, not to the group algebra of $\mathcal{S}_n$, but to the *Iwahori-Hecke algebra* (the generators T_i satisfy the braid relations, and $(T_i + 1)(T_i - q) = 0$). For example the element

$$c_\mu := \sum\nolimits_{\mu \leq [4312]} (-q)^{\ell(\mu)} T_\mu = (T_3 - q)(T_2 - \frac{q^2}{1+q})(T_1 - \frac{q^2}{1+q})(T_3 - q)(T_2 - q)$$

becomes, under the specialization $T_1 = T_2 = T_3 = -1$, equal to $-(1+q)(1+q+q^2)^2$.
This special element c_μ belongs in fact to the *Kazhdan-Lusztig basis* which is a distinguished linear basis of the Hecke algebra of fundamental importance in representation theory and in geometry. The coefficients of the matrix

expressing the Kazhdan-Lusztig basis into the standard basis T_μ are called *Kazhdan-Lusztig polynomials.* These are polynomials in q with (small) integral positive coefficients, that are, unfortunately, hard to obtain, despite their elementary recursions. It is still a challenge to provide a combinatorial description of these polynomials. Such a description would have applications to properties of the Bruhat order, to desingularizations of Schubert varieties, multiplicities of Verma modules, &c.
In the case of the minimum representatives β which code the order (and correspond to *determinantal varieties*, i. e. varieties defined by the vanishing of minors of a given order of a generic matrix) the Kazhdan-Lusztig polynomials are just Gauss polynomials, also called *q-binomial coefficients.* We shall show that the above stated embedding of the symmetric group into a lattice furnishes an explicit combinatorial description of Kazhdan-Lusztig polynomials in the case of "3412-avoiding" permutations.

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1991 MS Classification: 52C99, 52C10, 68U05, 68R99
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Mathematical snapshots from the computational geometry landscape

The aim of the talk is to introduce several mathematically interesting notions, techniques, and results that emerged in the field of computational geometry in recent years.
Computational geometry is a branch of theoretical computer science considering the design of efficient algorithms for computing with configurations of simple geometric objects in Euclidean spaces. The space dimension is usually considered a constant, and many problems are studied mainly in the plane or in the 3-dimensional space. Computational geometry constituted sometimes around the year 1980 and it has been developing very rapidly since then.
Although the main emphasis in this field is on algorithms, there are purely mathematical aspects which deserve to be known to a wider mathematical public. Quite subjectively, the speaker has selected several such topics for the talk (necessarily omitting many other, maybe even nicer things). Since the results to be mentioned include the ideas of a large number of researchers (speaker's own results being a tiny part only), it is not possible to give explicit credits to all of the contributors, and names will be mentioned only occasionally.
The first topic planned for the talk is the complexity of arrangements of geometric objects. For example, suppose that we have n arbitrary segments in the plane, and we consider one connected component of the complement of

their union (a 2-dimensional cell of the arrangement). What is the maximum possible number of vertices (segments' endpoints and intersections) on its boundary? This brings us to the long-studied combinatorial problem of Davenport-Schinzel sequences and their generalizations. Recently, there has been a considerable progress concerning higher-dimensional analogues and other variants of the problem.

The second topic starts with the following problem. Given n lines in the plane and an number $r > 1$, cover the plane with as few (possibly unbounded) triangles as possible, in such a way that the interior of each triangle is intersected by no more than $\frac{n}{r}$ of the lines. It turns out that one can always use $O(r^2)$ triangles, and such a collection of triangles (called a $(1/r)$-cutting for the given collection of lines) can serve for subdividing a geometric or algorithmic problem concerning the given lines into smaller subproblems. For example, a simple proof of a result of Szemerédi and Trotter concerning the maximum possible number of incidences of n lines and n points can be given using $(1/r)$-cuttings (recently, a still simpler proof has been found using a lemma on the number of crossings in a graph drawing, but the technique with cuttings seems more generally applicable).

Time permitting, some more topics will be mentioned, such as the so-called range searching problem and its surprising relations to Heilbronn's problem in combinatorial geometry, or the attempts in computational geometry at efficient algorithms for linear programming, their abstract view, and how this led to an elegant proof of a difficult Helly-type result in convexity, or how a combination of algebraic techniques with results on the number of incidences of points and curves provides a better understanding of geometric configurations, where the straightness of straight lines (finally) matters. In the accompanying paper, a few other areas will also be surveyed, and basic bibliography will be provided.

1991 MS Classification: 05, 11, 14, 94

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Nets, (t,s)-sequences, and algebraic curves over finite fields with many rational points

Nets and (t,s)-sequences are finite point sets, respectively infinite sequences, satisfying strong uniformity properties with regard to their distribution in the s-dimensional unit cube $I^s = [0,1]^s$. The general theory of these combinatorial objects was first developed by the speaker [2], and they have attracted a lot of interest in recent years because of their applications to difficult problems of scientific computing, such as high-dimensional numerical integration. For a given dimension $s \geq 1$ and integers $b \geq 2$ and $0 \leq t \leq m$, a (t,m,s)-*net in base* b is a point set P consisting of b^m points in I^s such that every subinterval J of I^s of the form $J = \prod_{i=1}^{s}[a_i b^{-d_i}, (a_i+1)b^{-d_i})$ with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ and with $\mathrm{Vol}(J) = b^{t-m}$ contains exactly b^t points of P. Furthermore, a sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ of points in I^s is a (t,s)-*sequence in base* b if for all integers $k \geq 0$ and $m > t$, the point set consisting of the $\mathbf{x}_n$ with $kb^m \leq n < (k+1)b^m$ is a (t,m,s)-net in

base b.
It was realized early on (see [2], [3]) that (t,m,s)-nets are closely linked with classical combinatorial objects such as MOLS and orthogonal arrays. In fact, there is a combinatorial equivalence between (t,m,s)-nets and certain generalized orthogonal arrays. This implies combinatorial obstructions to the existence of (t,m,s)-nets and (t,s)-sequences and leads to the lower bounds in the recent tables [1] for the quality parameter t.
Powerful methods of constructing (t,m,s)-nets and (t,s)-sequences can be based on algebraic geometry, and the currently best results are achieved by using algebraic curves over finite fields with many rational points (see [4], [5]). For a prime power q and an integer $g \geq 0$, let $N_q(g)$ be the maximum number of $\mathbf{F}_q$-rational points that a smooth, projective, absolutely irreducible algebraic curve over $\mathbf{F}_q$ of genus g can have. Then it has been shown, for instance, that for every q and every dimension $s \geq 1$ there exists a $(V_q(s), s)$-sequence in base q, where $V_q(s) = \min\{g \geq 0 : N_q(g) \geq s+1\}$.
In view of this connection, the speaker and C. P. Xing have embarked on a systematic search for algebraic curves over $\mathbf{F}_q$ with many rational points, in order to obtain good lower bounds for $N_q(g)$; upper bounds can be derived by a known technique from Weil's explicit formula for the number of rational points in terms of the zeta function of an algebraic curve over $\mathbf{F}_q$. This research has led to new principles for the construction of algebraic curves over $\mathbf{F}_q$ with many rational points, among them methods based on Hilbert class fields and Drinfeld modules of rank 1. The full paper will contain an up-to-date table of lower and upper bounds on $N_q(g)$ for $q = 2, 3, 4, 5, 8, 9, 16, 27$ and all $g \leq 50$.
Related work of the speaker and C. P. Xing [6] used algebraic curves over $\mathbf{F}_q$ with many rational points, or equivalently global function fields over $\mathbf{F}_q$ with many rational places, to produce an improvement on the classical Gilbert-Varshamov bound for the existence of good linear codes over $\mathbf{F}_q$ in the case where q is a sufficiently large composite nonsquare. This extends the well-known result in [7] for squares q.

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1991 MS Classification: 52C17
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Recent developments on the sphere-packing problem.

A report on recent work (mostly by J. H. Conway, E. M. Rains and myself, but also mentioning C. Bachoc, J. Bierbrauer, A. Bonnecaze, A. R. Calderbank, Y. Edel, Y. Kitaoka, G. Nebe, H. -G. Quebbemann, P. Solé, A. Vardy and others) on the sphere-packing problem, proceeding from dimension 3 to 128. For example: a little-known 3-D analogue of the planar lattice A_2; a putative list of *all* the best packings in dimensions 3 to 8; an astonishing property of the densest packings known in 9 dimensions; codes over $\mathbb{Z}/4\mathbb{Z}$ and the E_8 and Leech lattices; new nonlattice packings in dimensions below 32; shadow lattices and new bounds for unimodular and N-modular lattices; tensor products; new record kissing numbers.

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Finite Geometries, Varieties and Codes

In recent years there has been an increasing interest in finite projective spaces, and important applications to practical topics such as coding theory and design of experiments have made the field even more attractive. In 1935, Yates, in a lecture to the Royal Statistical Society, introduced the notion of design, thus showing how finite geometries can be applied to the planning of experiments. A few years later the statistician Bose started with his investigations on graph theory, design theory and finite projective spaces, using mainly pure combinatorial arguments. Another great pioneer in finite projective geometry was the Italian geometer Beniamino Segre. His celebrated result of 1954 stating that in the projective plane $\mathrm{PG}(2,q)$ over the Galois field $\mathrm{GF}(q)$ with q odd, every set of $q+1$ points, no three of which are collinear, is a conic stimulated the enthusiasm of many young geometers. Finally, the fundamental and deep work in the last four decades on polar spaces, generalized polygons, and, more generally, incidence geometry, in the first place by Tits, but also by Buekenhout, Kantor, Shult, Veldkamp and others, gave a new dimension to finite geometry. It is my intention to mention a few important and elegant theorems, to say something about the used techniques and the relation with other fields, and to mention some conjectures.

First I will speak on the geometry of $\mathrm{PG}(2,q)$, that is, the projective plane over the finite field $\mathrm{GF}(q)$. To begin with it is my purpose to show how classical algebraic curves in $\mathrm{PG}(2,q)$ can be characterized in pure combinatorial terms. I will illustrate this with one theorem on conics and one on Hermitian curves. Next, I will mention a characterization of Hermitian curves in terms of algebraic geometry, that is, from the start we assume that our pointset in $\mathrm{PG}(2,q)$ is an algebraic curve. In the proofs of these three theorems classical projective geometry, finite algebraic geometry, finite field theory and counting arguments were used. Relying on completely different techniques Ball, Blokhuis and Mazzocca proved in 1997 the longstanding conjecture saying that for q odd pointsets P of $\mathrm{PG}(2,q)$ with the property that each line of $\mathrm{PG}(2,q)$ either contains 0 or m points of P, m a constant with $1 < m < q$, do not exist.

Next I will speak about the geometry of $\mathrm{PG}(n,q)$. If $\mathcal{V}$ is a "classical" algebraic variety in $\mathrm{PG}(n,q)$, e. g. a quadric, a Hermitian variety, a Veronese variety, then a first approach is to characterize $\mathcal{V}$ either as a subset of $\mathrm{PG}(n,q)$ which intersects certain subspaces of $\mathrm{PG}(n,q)$ in sets with cardinalities in some range or as a subset of $\mathrm{PG}(n,q)$ whose points satisfy certain linear independence conditions. One characterization of the first type will be given, while in the section on arcs and codes we will show that (subsets) of normal rational curves can be characterized by one simple independence condition on the points. A second approach is that where the object is described as an incidence structure satisfying certain properties; here the object is not a priori embedded in a projective space, even the finite field is in many cases a priori absent. This will be illustrated with some beautiful theorems on inversive (or Möbius) planes, polar spaces and Shult spaces.

Finally there will be a section on k-arcs in $\mathrm{PG}(n,q)$ and on linear Maximum Distance Separable codes (MDS codes), where the interplay between finite projective geometry, coding theory and algebraic geometry is particularly present. A k-arc K in $\mathrm{PG}(n,q), n \geq 2$, is a set of k points of $\mathrm{PG}(n,q)$ with $k \geq n+1$ such that no $n+1$ points of K lie in a hyperplane of $\mathrm{PG}(n,q)$. Many deep theorems on k-arcs were and are still obtained using finite projective and algebraic geometry. A linear MDS code is a linear code for which we have equality in the Singleton bound; in their famous book on coding theory MacWilliams and Sloane [1977] introduce their chapter on MDS codes as "one of the most fascinating in all of coding theory". Now it appears that MDS codes of dimension at least three and k-arcs are equivalent objects, so that all results on k-arcs can be immediately translated in terms of MDS codes.

In an appendix, as an example of brand new research in the field, I will mention a problem on representing finite projective planes into the lineset of $\mathrm{PG}(n,q)$, its complete solution, and an application to generalized hexagons.

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1991 MS Classification: 5, 14, 20
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Multisegment duality, canonical bases and total positivity

The traditional interaction of representation theory, algebraic geometry and algebraic combinatorics involves:

- finite-dimensional representations of symmetric and general linear groups;
- geometry of flag varieties and Schubert varieties;
- combinatorics of Young tableaux and related algorithms.

New concepts in representation theory such as Lusztig's canonical bases [8] and Kashiwara's crystal bases [5] require new geometric and combinatorial tools. The new triad includes:

- representations of reductive groups over p-adic fields and of their Hecke algebras; quantum groups and their canonical bases;
- geometry of quiver representation varieties and totally positive varieties;
- combinatorics of rational polyhedral convex cones and polytopes, their lattice points, and their piecewise-linear transformations.

We illustrate this interplay with a particular piecewise-linear involution, the multisegment duality. It was introduced in [12, 13] in the context of representations of general linear groups over a p-adic field. It also naturally appears in the geometry of quiver representations, and in the study of canonical bases for quantum groups. On the combinatorial side, it is closely related to Schützenberger's involution on Young tableaux [11], as demonstrated in [2]. In a joint work with H. Knight [6], a closed formula for the multisegment duality was obtained using the machinery of network flows. In this talk, we give a new combinatorial interpretation of the multisegment duality as an intertwining map between two piecewise-linear actions of the Lascoux-Schützenberger plactic monoid [7]; this leads to a generalization and a new proof of the Mœglin-Waldspurger recursive description [10].

The multisegment duality is a special case of Lusztig's piecewise-linear transition maps between various combinatorial parametrizations of the canonical basis for the unipotent part of the quantum general linear group. In a joint work with A. Berenstein and S. Fomin [1], closed formulas for these maps were obtained using a parallelism discovered by Lusztig [9] between canonical bases and total positivity. Combinatorial tools used in [1] include a geometric presentation of the reduced decompositions for the symmetric group by pseudo-line arrangements, and the tropical semiring (or $(\min, +)$-algebra) appearing in combinatorial optimization and control theory. The classical theory of totally positive matrices and its recent generalization to an arbitrary reductive group due to Lusztig [9] give rise to other algebraic and combinatorial problems. If time allows, some results in this direction obtained in [1, 3, 4] will be presented.

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14. Mathematical Aspects of Computer Science

1991 MS Classification: 68-Q15

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Worst-case Complexity, Average-case Complexity and Lattice Problems

The goal of complexity theory is to describe the necessary resources, in terms of time, memory etc. for the solutions of computational problems. For cryptographic applications it would be particularly important to know that certain problems (e. g. finding the prime factors of a large integer) cannot be solved in a reasonable amount of time. (In fact the popular RSA public-key cryptosystem is based on that assumption.) Unfortunately we do not have yet any results of this type. Still we may get some information about the (relative) hardness of such problems if we accept as an axiom the hardness of a well-known computational problem which was attacked for a long time by many mathematicians without success (that is, we accept that there is no polynomial time solution of the problem in the size of the input) and prove from this axiom the hardness of other problems or the security of a cryptographic protocol. E. g. factoring integers, finding the discrete logarithm can be such problems.
Another similar solution would be to accept as an axiom that there is a problem in NP (that is a problem where the correctness of a proposed solution can be checked in polynomial time) which has no polynomial time solution. This is the famous $P \neq NP$ conjecture. There are known problems (namely each NP-complete problem) whose hardness follows from this assumption. E. g. "find a Hamilton-cycle in a given graph" is an NP-complete problem. Unfortunately these methods (either we choose the hardness of a famous problem or an NP-complete problem as an axiom) has an inherent limitation. The problems in both categories are so-called worst-case problems. That is, they are hard in the sense that finding a solution is assumed to be difficult only for some unknown values of the input and can be very easy for other values. E. g. there are integers whose factors are very easy to find. For cryptographic applications we have to present a hard instance of the problem, that is, a particular input where the solution cannot be found easily. The practical solution in the case of factoring e. g. is to pick the integer as the product of two random primes with some additional constraint to make sure that factoring is not made easier by the specific structure of the prime factors. That is, we use an average-case problem instead of a worst-case problem. We assume now that this problem whose input is chosen at random is difficult on the average (or for almost all choice of the input). We gave up however our original requirement namely that we use only simply stated and well-studied problems. The algorithmic theory of average-case propblems are ususally only a few decades long, while the history of certain worst-case problems go back for hundreds of years. The statement of an average-case problem is also generally less clear-cut because of the many possible choices of the parameters involved in the randomization. In the

case of the Hamilton cycle problem it is not even clear what would be a good randomization.

There is however a possiblitiy which unites the advantages of the two (worst-case average-case) methods. Namely we need an average-case problem which is just as difficult as a well-known worst-case problem. There are two different worst-case problems concerning short vectors for lattices which has been used in this way to create average-case problems which are at least as difficult as the original worst-case problem and can be used for various cryptographic purposes. It is important that individual random instances of these average-case problems can be created together with a known solution.

To formulate these problems we need some basic defintions about lattices.

A lattice is a subset of the n-dimensional space $\mathbf{R}^n$ over the reals which consist of the integer linear combinations of n fixed lineraly independent vector. Such a set of vectors will be called a basis of the lattice. The history of finding short vectors in lattices goes back to the works of Gauss and Dirichlet. With the fundamental results of Minkowski about a hundred years ago the theory of lattices became a separate branch of number theory with a huge literature. Finding short vectors in a lattice (in various possible senses) was always one of the main goals of this theory.

The two mentioned worst-case problems are the following:

(1) *Find a basis $b_1, ..., b_n$ in the n-dimensional lattice L whose length, defined as $\max_{i=1}^{n} \|b_i\|$, is the smallest possible up to a factor of n^c, where c is constant.*

(2) *Find the shortest nonzero vector in an n dimensional lattice L where the shortest vector v is unique in the sense that any other vector whose length is at most $n^c\|v\|$ is parallel to v, where c is a sufficiently large absolute constant.*

Problem (1) is equivalent to the problem of finding a single vector shorter than a given number in a class of randomly generated lattices, with a positive probability (Ajtai). Therefore finding a short vector in a random lattice is just as difficult (with high probability) as finding a short basis in the worst-case. This random construction also gives a one-way function which leads to some cryptographic tools like pseudo-random number generators. A different cryptographic application namely a collision-free hash function were given by Goldreich, Goldwasser and Halevi. Problem (2) is somewhat weaker than Problem (1), but it seems to be more easily applicable for cryptographic protocols, e. g. its hardness guarantees the security of a public-key cryptosystem (Ajtai and Dwork). Another completely different public-key cryptosystem based on the hardness of lattice problems (without worst-case average-case connection) was proposed by Goldreich, Goldwasser and Halevi.

The talk will also survey results which try to determine the hardness of these (or closely related) problems in the hierarchy of complexity classes. It is unlikely that Problem (2) is NP-hard for $c > \frac{1}{2}$ since it would lead to a collapse in the computational hierarchy (Goldreich and Goldwasser).

However if we drop the uniqueness requirement from the problem, that is, we want to find the shortest vector (under the Euclidean norm) then the problem is NP-hard at least for randomized reductions (Ajtai). The proof of this result uses lattices constructed from logarithms of small primes. This type of lattice construction was originally used by Adleman to reduce factoring to the shortest vector problem (for the proof of correctness he used number theoretical conjectures about the distribution of smooth numbers.) The shortest vector problem is NP-hard even in some approximate sense. Micciancio has proved recently, that the problem "find a vector which is longer than the shortest vector only by a constant factor c, where c is smaller than $\sqrt{2}$", is NP-hard. He also proved that the NP-hardness of the shortest vector problem for deterministic reductions follows from a natural number theoretic conjecture about the existence of square-free smooth numbers in long enough intervals.

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Games, Complexity Classes, and Approximation Algorithms

Game theory provides a framework in which to model and analyze conflict and cooperation among independent decision makers. Many areas of computer science have benefitted from this framework, including artificial intelligence, distributed computing, security and privacy, and lower bounds. Games are particularly important in computational complexity, where they are used to characterize complexity classes, to understand the power and limitations of those classes, and to interpret the complete problems for those classes.

A standard example of the use of games to characterize complexity classes is the well-known result of Chandra, Kozen, and Stockmeyer [CKS] that Alternating Polynomial Time (APTIME) is identical to Polynomial Space (PSPACE). An alternating, polynomial-time machine and an input x of length n determine a zero-sum, perfect-information game for which the game tree has depth polynomial in n. The referee R is a deterministic, polynomial-time machine that reads an entire played game before deciding whether the winner is Player P_1 (who claims that x is in the APTIME set L) or Player P_2 (who claims that x is not in L). P_1 (resp. P_2) has a winning strategy if and only if x is in L (resp. x is not in L). Because P_1, P_2, and R are deterministic, R always decides the winner correctly; in game-theoretic terms, the value of the game is 1 if x is in L, and it is 0 if x is not in L.

Subsequent results of Papadimitriou [P], Babai and Moran [BM], Lund, Fortnow, Karloff and Nisan [LFKN], and Shamir [S] show that the connection between PSPACE and zero-sum, perfect-information, polynomial-depth games is quite robust. Both [P] and [BM] modify the basic computational model of APTIME so that Player P_2 uses a trivial strategy: Whenever it is his turn to move, i. e., to write down a $p(n)$-bit string for some polynomial p, P_2 simply writes down the results of $p(n)$ fair coin-flips. Now if P_1

plays optimally on input string x, there is some probability (computed over the coin tosses of P_2) that R declares P_1 to be the winner; that probability is the value of the game determined by x, P_1, and R. In Papadimitriou's "games against nature, " x is in L if and only if the value of the game is at least 1/2. In Babai and Moran's "Arthur-Merlin games, " x is in L if the value of the game is at least 2/3, and x is not in L if the value is at most 1/3. Under both definitions, the class of sets accepted by such a game system is still PSPACE. (This was proven in [P] for games against nature and in [LFKN] and [S] for Arthur-Merlin games.) Thus we get the same computational complexity class from zero-sum, perfect-information, polynomial-depth games whether P_2 is a strategic player or a random player and whether we demand that the referee compute the value of the game exactly (as in games against nature) or approximately (as in Arthur-Merlin games).

In this talk, I will present two recent characterizations of PSPACE that push its identification with zero-sum, perfect-information, polynomial-depth games even further. The Probabilistically Checkable Debate System model (PCDS) is obtained by modifying the APTIME model in two ways: The polynomial-time referee R is allowed to flip coins, and it is not necessarily allowed to read the entire transcript of a played game before deciding whether the winner is P_1 or P_2. If x is in L, then R must decide in P_1's favor with probability 1, and if x is not in L, then R must decide in P_2's favor with probability at least 2/3. In the Randomized PCDS model (RPCDS), Player P_1 and referee R are exactly as in PCDS, but Player P_2 must play by flipping a sequence of fair coins. Condon, Feigenbaum, Lund, and Shor [CFLS1, CFLS2] show that a set is in PSPACE if and only if it is recognized by a PCDS and by an RPCDS each of which allows its referee to flip $O(\log n)$ coins and to read only a constant number of bits of the game transcript.

If time allows, I will present two related set of results about games and complexity. The first uses these characterizations to show that certain natural maximization and minimization functions, drawn from domains such as propositional logic, graph searching, graph reliability, and stochastic optimization, are as hard to approximate closely as they are to compute exactly. The second is a general framework defined by Feigenbaum, Koller, and Shor [FKS] and further developed by Feige and Kilian [FK] in order to explore relationships between complexity classes and imperfect-information games. Some tight characterizations of exponential-time complexity classes are obtained, but no class of imperfect-information games can be as robustly identified with any complexity class as perfect-information, polynomial-depth games can be with PSPACE.

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On approximating NP-hard optimization problems

In complexity theory algorithms are considered efficient if they run in time polynomial in the size of the input and the set of efficiently solvable problems is denoted by P. A possibly slightly larger class of problems, called BPP, is obtained if we also allow that algorithm to use random coins and only ask the algorithm to give the correct answer with high probability. Another central class of problems is NP, the set of decision problems for which a positive answer has a proof which can be verified efficiently.
Many problems encountered in practice do not seem to allow efficient algorithms and one famous class of such problems is given by the NP-complete problems. These problems belong to NP and have the property that if one problem can be solved efficiently then so can all problems in NP. It is widely believed, but still unproven, that NP is not contained in P, and thus NP-completeness is the best available proof of intractability for problems in NP.
A problem (itself not necessarily contained in NP) is said to be NP-hard if it has the property that if it can solved efficiently so can any problem in NP. In this talk we will be concerned with NP-hard optimization problems.

Examples of such problems are given by graph-coloring[1], and maximum satisfiability[2].

Finding the exact optimum for these problems is computationally infeasible we resort to finding non-optimal solutions by approximation algorithms. A C-approximation algorithm is one which, for every input, the result is off by a factor at most C from optimal. Here C could either be a constant or a function depending on the size of input. The key question now is to determine, for each problem of interest, for what values of C efficient C-approximation algorithms exist.

The problem we concentrate on in this talk is Max-Lin; given an over-determined system of m linear equations in n variables over the field of two elements, find the assignment that satisfies the largest number of equations. Since a random assignment satisfies, on the average, $m/2$ equations and no assignment can satisfy more than all m equations we get an efficient 2-approximation algorithm. The main results we want to discuss is taken from [6] and says that for any $\epsilon > 0$ it is NP-hard to $(2-\epsilon)$-approximate Max-Lin.

The basic technique for proving such inapproximability results was introduced in [3]. While the technique can be described in completely combinatorial terms, the intuitively most compelling way to describe it is through *probabilistically checkable proofs* which we, as customary abbreviate PCP.

The notion of a verifier is central to a number of concepts in computer science and a standard situation would be to verify an NP-statement. A proof of the fact that a Boolean formula φ is satisfiable is a satisfying assignment and it is checked by evaluating φ on the given assignment. This verifier is deterministic, always comes to the correct conclusion but reads a large number of bits. In a PCP the verifier is probabilistic, comes to the correct conclusion with high probability, but reads very few bits. As a culmination of a series of papers it was proven in [1] that any satisfiable formula φ has a proof of polynomial size that can be verified by reading a constant, independent of the size of φ, number of bits. This verifier always accepts a correct proof of a satisfiable formula while any proof for a non-satisfiable formula is rejected with probability 1/2. By reduction, any NP-statement has a similar proof.

The connection to approximation is, in very general terms, given by considering $\max_\pi Acc_\varphi(\pi)$ where $Acc_\varphi(\pi)$ is the probability that the verifier accepts proof π. This is a combinatorial optimization problem and, by the above reasoning, $(2-\epsilon)$-approximating this problem enables us to solve an NP-complete problem and thus this is NP-hard. Different PCPs give results for different optimization problems.

The first results [3, 1] gave in most circumstances, in the quantitative sense rather weak results. The key to improving the quantitative bounds to tight

[1]Given a graph G find the minimum number of colors needed to color the nodes such that each pair of adjacent nodes have different colors.

[2]Given a collection of disjunctive clauses, find the assignment that satisfies the maximal number of clauses.

or almost tight results has been the use of very informative ways of coding binary strings. The code used for the results we describe is the long code introduced by [2] which codes a string of length n by a string of length 2^{2^n}. Strong quantitative results have also been obtained for graph-coloring [4] and max-clique[3] [5]. For both problems, for any $\epsilon > 0$ and graphs with n nodes, one cannot find an $n^{1-\epsilon}$-approximation algorithm unless NP $\subseteq$ BPP. The goal of this talk is to describe the set of ideas that lead to these efficient PCPs and discuss the connection to strong inapproximability results. For illustration we focus on the above mentioned problem Max-Lin.

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Combinatorial and algebraic techniques in proof complexity

An very basic and fundamental question in logic is: How long does a proof of a propositional tautology have to be, as a function of the size of the tautology? This question is equivalent to whether or not NP equals coNP, and has many other connections with computer science and mathematics. In

[3]Given a graph G find the maximal number of nodes that are pairwise connected.

the last 20 years, significant advances have been made in this area, through the introduction of techniques from such areas as circuit complexity, combinatorics, and representation theory.

In this talk, we will survey some of the important advances that have been made recently. In an influential paper in the 70's, Cook and Reckhow began a systematic program of studying and classifying standard proof systems in terms of their relative complexity. (Some standard systems include: resolution, natural deduction, Frege systems, FOL and ZFC.) A key contribution of this work is the realization that the representation complexity of intermediate formulas in a proof (more specifically the complexity of the cut formulas in the proof) is the key property of proof system power. Thus, proof systems can be divided into groups and these groups correspond to well-known complexity classes.

This analogy/connection with complexity classes has led to several beautiful results in proof complexity lower bounds. Here we give a few examples. First, in breakthrough work by Ajtai, it was shown that the propositional pigeonhole principle does not have short bounded-depth Frege proofs. This result makes heavy use of ideas used to obtain lower bounds for bounded-depth circuits.

Secondly, Bonet, Pitassi and Raz, and independendently Razborov developed a new technique for proving lower bounds for various proof systems, including Cutting Planes and systems of bounded arithmetic. This technique, known as feasible interpolation, and first proposed by Krajicek, essentially shows that sometimes a proof system lower bound can be obtained directly from a corresponding circuit complexity lower bound. This technique has been extensively studied and generalized, and has been successfully used to prove many other proof system lower bounds (including work by Pudlak, Cook, and Haken.) Furthermore, feasible interpolation gives a precise tradeoff between proving power and proof search: If a proof system has feasible interpolation, then lower bounds for the proof system can be obtained (sometimes under assumptions). But on the other hand, if a proof system does not possess feasible interpolation, then it can be shown that proofs in the system cannot be searched for efficiently.

Most recently, algebraic proof systems have been studied, and techniques from the representation theory of the symmetric group have come into play in a very important way. This line of research, pioneered in a paper by Ajtai, has led to an important new lower bound for Frege systems with mod p counting gates, by Krajicek. One important aspect in this work is that uniformity of the underlying family of tautologies is exploited in a new and powerful way. We will survey the important developments highlighted above, and conclude by discussing some of the most important problems yet to be resolved.

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Probabilistic checking of proofs

The notion of efficient verification of proofs has been a central theme in the theory of computing. The fundamental complexity class NP arises out of the study of this concept; as does the well-known question "Is P=NP?". Over the years, the class NP has played a crucial role in separating potentially intractable problems from the tractable ones, especially in the study of combinatorial optimization. Research in this decade has seen a revival of the connection between proofs and optimization. This revival is motivated by the proposal of several alternatives to the notion of efficient verifiability of proofs; and results showing that seemingly restrictive notions of verification are surprisingly general. The revived connection shows that for many optimization problems finding a "nearly-optimal" solution is as hard as finding the optimal one. In this talk we will describe some of the early results in this sequence of new developments.
A computational view of the notion of a proof abstracts the semantics of the proof system into a verification procedure or *verifier* i. e., a polynomial time computable Boolean function described by a Turing machine. A purported theorem T and proof π are then just a sequence of bits; π proves T if the verifier accepts the pair (T, π). The purported theorem T is true if such a proof π exists. The class NP [5, 9] represents the class of all theorems with "short" proofs.
The new notions of proof systems proposed during the last ten years consider a probabilistic verifier who is allowed some small probability of making an error. One of these notions, known as a probabilistically checkable proof (PCP), is motivated by the following informally stated question: "How fast can the verifier be compared to the size of the proof?" It is easy to check that a deterministic verifier must at the very least "look" at the whole proof. This however need not be true for probabilistic ones. The notion of "looking at a bit of the proof". can be formalized by providing the verifier with oracle access to the proof, i. e., the verifier can specify the address of a location of the proof and gets back the bit written in that location and this entire process takes only as much time as required to write the address. The number of bits of the proof that are "looked" at is now the number of oracle queries. To contrast such a verifier with the traditional verifier, one also quantifies the amount of randomness used by such a verifier. Thus we define $(r(\cdot), q(\cdot))$-restricted PCP verifier to be a probabilistic verifier with random access to a proof oracle, such that on input x of length n, the verifier tosses at most $r(n)$ coins and accesses the oracle at most $q(n)$ times, where the locations accessed are a function of the random coins. A language L is said to be in class PCP(r, q) if there exists an $(r(\cdot), q(\cdot))$-restricted PCP verifier V satisfying the following: If $x \in L$ there exists an oracle π such that the verifier V accepts x with probability 1 on oracle access to π. If $x \notin L$, for every π, V accepts x with probability at most $1/2$. Notice that

the verifier can make mistakes when $x \notin L$.
It is immediate from the definition that $\mathrm{NP} = \cup_{\mathrm{c}>0}\mathrm{PCP}(0, \mathrm{n}^{\mathrm{c}})$ (the verifier is not randomized, but is allowed unlimited access to the proof). The results of [3, 4, 6] showed that by allowing the verifier small amounts of randomness, the query complexity can be reduced dramatically and in particular $\mathrm{NP} \subseteq \cup_{\mathrm{c}>0}\mathrm{PCP}(\mathrm{c} \log \mathrm{n} \log \log \mathrm{n}, \mathrm{c} \log \mathrm{n} \log \log \mathrm{n})$. Subsequently, [1, 2], showed that it is possible to restrict the verifier even more significantly to just a *constant* number of queries (independent of the theorem, the proof or the system of axioms). They also reduce randomness to strictly logarithmic in input size. Specifically, they show

$$\exists q < \infty \text{ such that } \mathrm{NP} = \cup_{\mathrm{c}>0}\mathrm{PCP}(\mathrm{c} \log \mathrm{n}, \mathrm{q}).$$

The main focus of this talk will be to describe the ingredients of this result.

The consequences to combinatorial optimization may be described informally as follows: The notion of a PCP verifier allows one to formalize the notion of an "approximately" correct proof; and the strong results obtained above show that such a proof exists if and only if a perfectly correct proof exists. Thus the task of finding an approximately correct proof is as hard as the task of finding a perfectly correct proof. The traditional connection between proofs and optimization [8, 9] now indicates that for some optimization problems (unfortunately, not necessarily natural ones) finding near-optimal solutions should be as hard as finding optimal ones. The statement can actually be formalized and made applicable even to natural optimization problems. This connection was discovered by [6] and its applicability was further extended in [2] to apply to a large number of optimization problems considered in [10].

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Halving Point Sets

The talk will discuss a problem that ermerged from combinatorial geometry around 1970, and which is popular and relevant in computational geometry. Consider first the following algorithmic problem. Given n points in $\mathbb{R}^d$, we want to find a hyperplane that minimizes the sum of Euclidean distances to these n points. A glimpse of reflection tells us that an optimal hyperplane cannot have a majority ($\lfloor n/2 \rfloor + 1$ or more) of the points on either side; otherwise a parallel motion towards this side will improve its quality. A hyperplane with at most $\lfloor n/2 \rfloor$ points on either side is called *halving*. How many partitions of a point set (into the points on one side, on the hyperplane, and on the other side) by halving hyperplanes can be realized by an n-point set in $\mathbb{R}^d$? The notions and results mentioned below are closely related to this question. Emphasis in the presentation is on techniques that may be useful elsewhere, and on interconnections to other topics in discrete geometry.

Halving edges and a crossing lemma. Let P be a set of n points in the plane, n even, and no three points on a line. A *halving edge* is an undirected edge between two points, such that the connecting line has the same number of points on either side. Around 1970 L. Lovász, and in a more general setting with P. Erdős, A. Simmons, and E. G. Straus, were the first to investigate the geometric graph of halving edges of a point set, and proved that there cannot be more than $O(n^{3/2})$ such edges. Except for a small improvement to $O(n^{3/2}/\log^* n)$, there was no progress on the problem until T. Dey recently gave an upper bound of $2n^{4/3}$. He shows that the graph of halving edges cannot have more than $O(n^2)$ pairs of crossing edges. Then he employs a *crossing lemma* (due to M. Ajtai et al., 1982, and T. Leighton, 1983), which has a number of other applications: A geometric graph with n vertices and c pairs of crossing edges has at most $O(\max(n, \sqrt[3]{cn^2}))$ edges.

We will present a short probabilistic proof of the crossing lemma, and a proof of the $O(n^2)$ bound on the number of crossings via the following identity observed by B. Aronov and the author (inspired by Dey's proof): $8c + \sum_{p \in P} (\deg p)^2 = n(n-1)$, where $\deg p$ is the number of halving edges incident to point p, and c is the number of crossing pairs of halving edges.

The corresponding problem of bounding the number of *halving triangles* of n points in $\mathbb{R}^3$, n odd, has also been investigated in a sequence of papers with a currently best bound of $n^{8/3}$ due to T. Dey and H. Edelsbrunner, 1994. While this bound still allows for a simple proof, the situation gets more involved in dimensions 4 and higher, where the best results (see Agarwal et al., 1998) are based on a colored version of Tverberg's Theorem by R. T. Živaljević and S. T. Vrećica, 1992.

Lower bounds and oriented matroids. The upper bounds mentioned may be far from optimal. In the plane several constructions of n-point sets with $\Omega(n \log n)$ halving edges are known. If we consider the corresponding problem for oriented matroids of rank 3 (or pseudoline arrangements in the dual), then there is an unpublished lower bound of $n2^{\Omega(\sqrt{\log n})}$ due to M. Klawe, M. Paterson, and N. Pippenger, inspired by a connection to sorting networks. It is open whether this construction is realizable (stretchable) or not.

j-Facets and the Upper Bound Theorem. The following notion generalizes halving edges and triangles. Let P be a set of $n > d$ points in $\mathbb{R}^d$ in general position, i. e., no $d+1$ points on a common hyperplane. A *j-facet of P* is an oriented $(d-1)$-simplex spanned by d points in P that has exactly j points from P on the positive side of its affine hull. The 0-facets are the facets of the convex hull of P. Hence, the Upper Bound Theorem (P. McMullen, 1970) gives us a tight bound on the number of 0-facets, which is attained by the vertices of cyclic polytopes. Using their probabilistic technique, K. L. Clarkson and P. W. Shor, 1989, derive an asymptotically tight bound of $O(n^{\lfloor d/2 \rfloor} j^{\lceil d/2 \rceil})$ for the number of $(\leq j)$-facets (i. e., i-facets with $0 \leq i \leq j$). However, except for dimensions 2 and 3, no exact bounds are known. In particular, it is not known whether the *exact* maximum is attained for sets in convex position or not. (It is still possible that the maximum is attained for points on the moment curve, where the number of $(\leq j)$-facets can be easily counted.)

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15. Numerical Analysis and Scientific Computing

1991 MS Classification: 65, 15, 35

Beylkin, Gregory, University of Colorado at Boulder, USA

On multiresolution methods in numerical analysis

Multiresolution methods have a fairly long history in numerical analysis, going back to the introduction of multigrid methods. A significant new interest in multiresolution methods was generated recently by the development of wavelet bases and other bases with controlled time-frequency localization. The introduction of these new tools allows us to relate numerical analysis with harmonic analysis and signal processing by the fundamental need to have efficient representations of operators and functions.

The task of constructing fast algorithms for evaluating operators on functions or computing functions of operators is somewhat different in signal processing and numerical analysis. Whereas there has been a general acceptance of wavelet methods in signal processing (e. g., wavelet decompositions as a part of efficient compression algorithms), the acceptance in numerical analysis has been somewhat slower.

In numerical analysis it is useful to compare the wavelet approach with the multigrid and the Fast Multipole Method (FMM). For most problems the wavelet approach, FMM, and multigrid methods provide the same asymptotic complexity. The differences are typically in the "constants"of the complexity estimates and these differences will, most likely, diminish in the future.

In order to emphasize several distinct features of the wavelet approach, we consider two topics. First, we describe connections between the multiresolution LU decomposition, multigrid and multiresolution reduction/homogenization for self-adjoint, strictly elliptic operators and, second, the effects of computing exponentials of such operators on numerical properties of time evolution schemes for advection-diffusion equations. We note that both topics are in the area of active research.

The essence of the first topic is that multiresolution LU decomposition (the usual LU decomposition interlaced with projections) is equivalent to what can, perhaps, be called the "direct" multigrid, i. e., a multigrid method without the so-called W-cycles. The reason for the absense of W-cycles is that on every scale we construct equations for the orthogonal projection of the *true* solution on that scale. Once these equations are solved, there is no need to return to a coarser scale to correct the solution (which is the role of W-cycles in multigrid). Equations oftained in this manner on coarser scales are of interest by themselves, since they can be interpreted as "homogenized" or reduced equations, leading to multiresolution homogenization.

The essence of the second topic is that we can significantly improve properties of time evolution schemes for advection-diffusion equations by using exponentials of operators represented in wavelet bases. As it turns out, the exponential $\exp(-t\mathcal{L})$ of a self-adjoint, strictly elliptic operator $\mathcal{L}$ is sparse (for a finite but arbitrary precision) for all $t \geq 0$, which makes algorithms for its construction feasible in two and three spatial dimensions. Given a proper choice of basis and several additional algorithms, we are led to adaptive numerical schemes for the solution of advection-diffusion equations.

1991 MS Classification: 42C05

Deift, Percy, Courant Institute, USA

Uniform asymptotics for orthogonal polynomials

The speaker will present the results of recent joint work with T. Kriecherbauer, K. T-R McLaughlin, S. Venakides and X. Zhou.
We consider asymptotics of orthogonal polynomials with respect to weights $e^{-Q(x)}dx$ on the real line where either
1) $Q(x)$ is a polynomial of even order with positive leading coefficient
or
2) $Q(x) = nV(x)$ where V is real analytic and $\frac{V(x)}{\log(1+x^2)} \to \infty$ as $|x| \to \infty$.
We formulate the orthogonal polynomial problem as a Riemann-Hilbert problem following the work of Fokas, Its and Kitaev. We employ the steepest descent type method introduced by Deift and Zhou, and further developed by Deift, Venakides and Zhou, in order to obtain uniform Plancherel-Rotach type asymptotics in the entire complex plane, as well as asymptotic formulae for the zeros, the leading coefficients and the recurrence coefficients of the orthogonal polynomials. These asymptotics are also used to prove various universality conjectures in the theory of random matrices.

1991 MS Classification:

Bjorn Erik Engquist, Department of Mathematics UCLA, Los Angeles, USA

Wavelet Based Numerical Homogenization

Wavelet Based Numerical Homogenization
In homogenization a differential equation and its solution with multiple scales are replaced by an approximating equation and its corresponding solution with fewer scales. The scales related to the shortest wavelengths are eliminated. This reduction of scales may be needed in order to prepare a problem for a numerical approximation. The analytical homogenization technique is, however, only applicable to a small class of problems.
We shall start from a discretization of the original differential equation, which includes all the scales. The solution and the difference operator will be represented in a wavelet basis and the homogenized discrete operator will correspond to a particular form of an approximative projection onto the coarser scales. We shall show that this new operator inherits many of the properties of the original discrete operator, including sparseness. Comparisons with the analytic homogenization process and different convergence results will be given. We shall also present computational examples and discuss direct coarse grid approximation.

1991 MS Classification: Primary 76D30; Secondary 76C05, 35Q30, 35Q35.
Okamoto, Hisashi, Research Inst. for Math. Sci., Kyoto University, Japan
Bifurcation of Kolmogorov flows with an emphasis on singular limit

Keywords and Phrases: Kolmogorov flows in 2D tori, incompressible fluid, bifurcation, singular perturbation, internal layer, inviscid limit.
We consider a family of stationary Navier-Stokes flows in 2D tori. The flow is driven by an outer force which is of the form $(\sin y, 0)$. Varying the Reynolds number and the aspect ratio of the torus, we numerically compute bifurcating solutions by a path-continuation method. Folds and cusps are obtained in the range where the Reynolds number is < 100. Some solutions are computed until the Reynolds number becomes 10,000. Although every solution is known to be smooth (analytic actually), the vorticity tends, as the Reynolds number $\to \infty$, to a function which is at most C^1. In other words, smooth solutions tend to functions with a finite number of differentiability as the Reynolds number tends to infinity.
From our numerical computation, we can guess some interesting asymptotic behaviors as the Reynolds number tends to infinity. Since the Navier-Stokes equations are defined in a two-dimensional torus, there can not be a boundary layer. However, we can observe the appearance of some internal layers. The layer yields an energy spectrum of k^{-r}, where r is between -7 and -4, depending on the aspect ratio. Analysis as the aspect ratio $\alpha \to 0$ is also given.

1991 MS Classification: 41
Jan-Olov Strömberg, Royal Institute of Technology, Stockholm, Sweden
Apriori estimates for approximation by wavelets in higher dimensions.

In dimension d, a lattice grid of size N has N^d points. The representation of a function by, for instance, splines or the so-called non-standard wavelets with error ϵ would require $O(\epsilon^{-ad})$ lattice point values (resp. wavelet coefficients), for some positive a depending on the spline order (resp. the properties of the wavelet). Unless d is very small, we easily will get a data set that is larger than a computer in practice can handle, even for very moderate choices of N or ϵ.
I will discuss how to organize the wavelets so that functions can be represented with $O((\log(1/\epsilon))^{a(d-1)}\epsilon^{-a})$ coefficients. Using wavelet packets, the number of coefficients may be further reduced.
In the talk, I will only give some basic ideas and basic estimates. To work out these ideas in real applications is a longer project, which I have only begun. Some ideas with details can be found in the Cand. Scient. thesis of my student Øyvind Bjørkås. Also, further work on this theme is planned to be included in his Ph. D. thesis.
Reference: Ø. Bjørkås, Numerical Analysis of Singular Integral Operators with wavelets, Cand. Scient. thesis, University of Tromsø, 1997.

1991 MS Classification: 30C30, 31A05

Lloyd N. Trefethen, Oxford University, UK, and Tobin A. Driscoll, University of Colorado, USA

Schwarz–Christoffel Mapping in the Computer Era

In the past twenty years, because of new algorithms as well as new computers, Schwarz–Christoffel conformal mapping of polygons has matured to a technology that can be used at the touch of a button. Many authors have contributed to this progress, including Däppen, Davis, Dias, Elcrat, Floryan, Henrici, Hoekstra, Howell, Hu, Reppe, Zemach, and ourselves. The principal SC software tools are the Fortran package SCPACK [5] and its Matlab successor, the Schwarz–Christoffel Toolbox [2]. It is now a routine matter to compute an SC map involving a dozen vertices to eight digits of accuracy in a few seconds on a workstation.

With the power to compute has come the ability to explore. The obvious kind of problem—"here is a polygon; map it onto a disk or a half-plane"—is rarely what one encounters in practice. Instead it is variations on the idea of SC mapping that arise. The published paper will describe four examples:

(1) oblique derivative problems on polygons [6]

(2) ideal two-dimensional jets, wakes, and cavities [1]

(3) the CRDT (cross-ratio Delaunay triangulation) algorithm [3]

(4) Green's functions for multiply connected domains with symmetry [4].

References will be given to variants of SC mapping including rectangles, infinite strips, periodic domains, fractals, circular polygons, curved boundaries, gearlike domains, exterior polygons, doubly connected polygons, and polygonal Riemann surfaces. Applications mentioned will include grid generation, Faber polynomials, polynomial approximation, digital filtering, matrix iterations, the KdV equation, electrical resistances and capacitances, magnetostatics, the classical Hall effect, queuing theory, vortex methods in fluid mechanics, and inverse problems.

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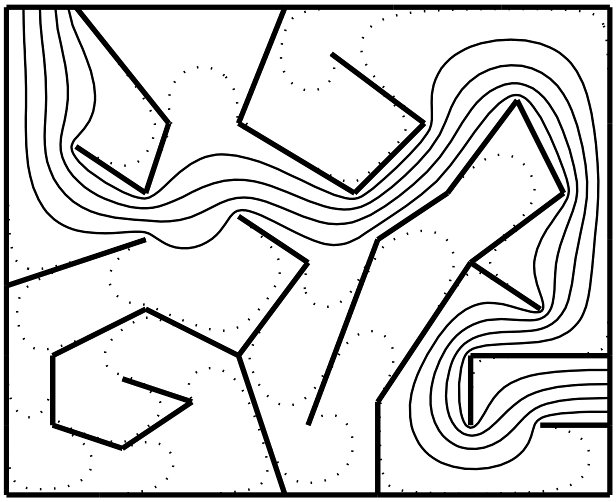

Figure 1. Conformal map of "Emma's maze" onto a rectangle, computed by the CRDT algorithm [3]. The solid curves are the conformal images of straight lines in the rectangle; the dotted curves are the same, but exponentially close to the rectangle boundary. Because of exponential distortions, the numerical computation of maps like this is far beyond the capabilities of ordinary SC algorithms.

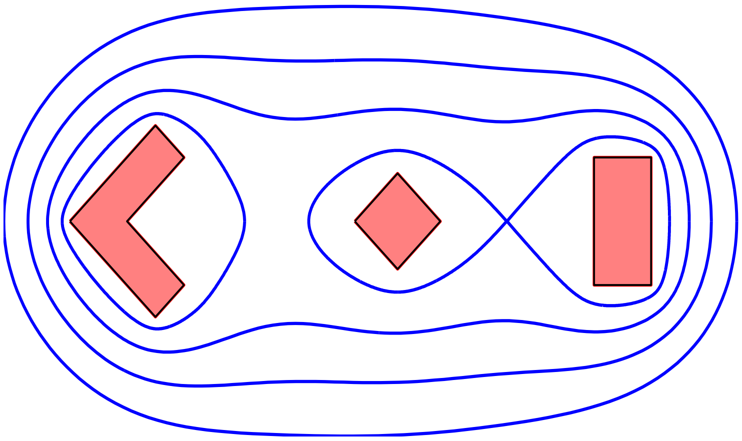

Figure 2. Level curves of the Green's function for a multiply connected domain [4]. Because of the symmetry with respect to an axis, the Green's function can be computed via a conformal map onto the exterior of a disk with two protruding radial slits. The level curves are images of concentric circles.

16. Applications

Section 16 a: Applications of Mathematics in Other Sections.

1991 MS Classification:
Avellaneda, Marco, Courant Institute, New York University, USA
Inverse Problems in Quantitative Financial Modeling

(see section 12)

1991 MS Classification: 62P99
Grenander, Ulf, Brown University, USA
Strategies for Seeing

In order to develop algorithms for machine vision one must adapt the strategy for seeing to the type of scenery that is expected as input to the sensor, and also to what is the purpose of the vision machine at hand. Since both the type of scenery and the purpose can vary from case to case it will be necessary to develop many strategies and we shall present a few.
The crucial step is the mathematical representation of the scenery and its variability. We shall create pattern theoretic representations that describe the scenes in algebraic terms: via concepts like generators, similarity groups, configurations, images and patterns, and then express their variability through a probability measure on the resulting algebraic structure. The physics of the sensor will be given in terms of a likelihood function that relates the true (but unknown) scene to the output of the sensor. We can then derive the posterior probability measure of the scene, given the observation, from Bayes' theorem.
While this is possible in theory, a major obstacle to practical implementation is the fact that the posterior measures live on structures of extremely high dimension and are seldom of analytically tractable form. This implies that the search for maxima will be difficult: there will be lots of local minima and global search will be prohibitively slow.
One way out of this dilemma is to use saccadic search: employing covariants for the similarity group that expresses invariances in the scene structure it is possible to reduce the global problem to a local one, which can then be dealt with by simple gradient methods. While this does not guarantee the finding of the global maximum in all situations, it can be shown in many cases that it provides a computationally feasible way of approximating the solution to the optimization problem.
To explore to what extent this is true some strategies for seeing have been developed and implemented through software that has been used in computer experiments to find the strengths and weaknesses of this methodology. Some results of these experiments will be shown.

1991 MS Classification: 58F39, 58F14, 76B15, 34A47, 76B25,
Iooss, Gérard, Institut Universitaire de France, Valbonne, France
Travelling water-waves, as a paradigm for bifurcations in reversible infinite dimensional "dynamical" systems

First we recall few properties of reversible systems, in finite or infinite dimensional spaces, of the form

$$\frac{dU}{dx} = F(U), \qquad SF(U) = -F(SU), \quad S^2 = \mathbb{I}$$

near a symmetric equilibrium taken at 0. Considering bifurcation problems, i. e vector fields depending on a set of parameters, which are non hyperbolic near 0 for certain parameter values, we give necessary conditions under which *center manifold reduction* applies for the study of solutions staying close to the equilibrium for all $x \in \mathbb{R}$ (see the basic paper [6] for application of center manifold reduction to elliptic problems in a strip). We give the basic example of "*reversible Hopf*" bifurcation and state the results on known small bounded solutions: periodic, quasi-periodic, homoclinic to 0, homoclinic to periodic solutions. The main tool for such a study is Poincaré-Birkhoff normal form theory [1] (revisited following the global characterization given in [2], taking care of reversibility) and a proof of persistence of reversible (symmetric) solutions under higher order terms, not considered in the normal form [5] (see [3], [7] for other typical proofs on a similar reversible bifurcation).
We then present a large class of water-wave problems, where we look for 2D travelling waves in a potential flow, with one or several superposed layers, in the presence or not of surface tension at the free surface and at interfaces. Coordinate x corresponds to the direction of travelling waves. In case of finite depth layers, such a problem of finding small bounded solutions for all $x \in \mathbb{R}$, is shown to be reducible to a finite dimensional symmetric center manifold, on which the system reduces to a reversible ODE. For "*reversible Hopf*" bifurcation, this leads to various kind of travelling waves, for instance to solitary waves with damped oscillations at infinity or to "black" solitary waves, corresponding to solutions homoclinic to periodic waves with a minimum of amplitude at $x = 0$.
If the bottom layer has infinite depth, which appears to be the most physically realistic case, concerning the validity of results in the parameter set, the mathematical problem is far more difficult, because it cannot be reduced to a finite dimensional one, due to the occurence of a continuous spectrum of the linearized operator $DF(0)$ crossing the imaginary axis (at 0). This continuous spectrum may also contain an embedded eigenvalue at 0. We give some ways, via an infinite dimensional normal form reduction, on how to attack this difficulty, specially for periodic and homoclinic solutions with a *polynomial decay* at infinity (see [4] for solitary waves in the case of one layer of infinite depth, in presence of surface tension).

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Section 16 b: (Non-Continuum) Applied Area

1991 MS Classification: Mathematical Biology

Berger, Bonnie, Massachusetts Institute of Technology, USA

Sequence-based Methods for Protein Folding

One of the most important and challenging problems in computational biology is that of predicting the three-dimensional structure or shape of a protein from its amino acid sequence. As a first step to tackling this problem, many researchers have focused on the structural motif recognition problem: given a known local three-dimensional structure, or motif, determine whether this motif occurs in a given amino acid sequence.
In this talk, I will present algorithms that use probabilistic techniques to improve existing methods for recognizing protein structural motifs. These algorithms are particularly effective at eliminating false positives found by previous methods without introducing false negatives.
We have implemented these algorithms and have tested them on 2-stranded and 3-stranded coiled coils. The coiled coil motif has many important biological roles; for example, it is found in some DNA binding proteins and is thought to play a role in the membrane fusion of retroviruses. Our algorithms have been codified in three programs: PairCoil, which is significantly better than existing methods at identifying 2-stranded coiled coils; MultiCoil, which identifies and distinguishes between 3-stranded and 2-stranded coiled coils; LearnCoil, which is particularly effective in scenarios where sufficiently diverse examples of a motif are not known.
Finally, I will talk about some of the biological implications of our work. In particular, we have identified interesting domains in the envelope proteins of retroviruses such as the influenza virus, Moloney murine leukemia virus, and HIV, as well as in a large class of proteins involved in sensory signaling. (Portions of this work are joint with Peter S. Kim, Ethan Wolf, Mona Singh, David Wilson, and Andrea Cochran.)

1991 MS Classification: 90C10 90C27

Cook, William, Rice University, USA

The Traveling Salesman Problem

(see Section 17)

1991 MS Classification: 92, 05, 51

Andreas Dress, Werner Terhalle, Universität Bielefeld, Germany

The Tree of Life and Other Affine Buildings

(1) Given a field K with a valuation v, that is, a map $v : K \to \mathbb{R} \cup \{-\infty\}$ satisfying the conditions $v(x) = -\infty \iff x = 0$, $v(x \cdot y) = v(x) + v(y)$, and $v(x+y) \geq \max(v(x), v(y))$, the points in the associated affine building of $GL(n, K)$ can be identified in a canonical fashion with the maps f from the set $K^n \setminus \{0\}$ of non-zero vectors in K^n into $\mathbb{R}$ satisfying the (including the creation of software implementing the resulting algorithms) following condition for each non-zero vector $\mathfrak{x}$:

$$f(x) = \sup(v(\det(x, x_2, \ldots, x_n)) - f(x_2) - \ldots f(x_n))$$

where $x_2, \ldots, x_n$ range over $K^n \setminus \{0\}$, that is, with the minimal elements in the infinite dimensional convex polytope of all maps f from $K^n \setminus \{0\}$ into $\mathbb{R}$ satisfying the infinitely many linear inequalities

$$f(x_1) + \cdots + f(x_n) \geq v(\det(x_1, \ldots, x_n)$$

($x_1, \ldots, x_n$ ranging again over all non-zero vectors in K^n).

(2) In the special case $n = 2$, the following generalization is possible and of some interest: We may replace the set $K^2 \setminus \{0\}$ by any set X and the map $v \circ \det$ by any map $w : X^2 \to \mathbb{R} \cup \{-\infty\}$ satisfying the (generalized Grassmann-Pl“ucker) relation

$$w(x_1, x_2) + w(y_1, y_2) \leq \max(w(x_1, y_1) + w(x_2, y_2), w(x_1, y_2) + w(x_2, y_1))$$

for all x_1, x_2, y_1, y_2 from X. Then the above construction always leads to an $\mathbb{R}$-tree so that the elements $x \in X$ with $w(x, x) \neq -\infty$ represent points in that tree and the elements $x \in X$ with $w(x, x) = -\infty$ represent ends.

In particular, if $w(x, x) = 0$ holds for all $x \in X$, the pair (X, w) is a metric space, and the above construction identifies the resulting $\mathbb{R}$-tree with its injective hull in the category of metric spaces as defined by Isbell; so, it shows in particular that $\mathbb{R}$-trees can be characterized as the injective objects in that category which are of dimension 1.

In some way, in all of these constructions, a (generally contractable) space is constructed which somehow ‘fills the empty space between the elements from X’.

(3) In evolutionary theory, an important task is to unravel the phylogenetic relationships between present day (and, maybe, also some former) species.

The final outcome of phylogenetic analysis is a phylogenetic tree representing the (putative) branching pattern of species evolution. The task is similar to that described above: we know the present day species, and we want to 'fill the empty space between them' by a tree-like structure representing their joint evolutionary history.
(4) The remarkable observation that will be discussed in the lecture is that the apparent similarity between the two tasks mentioned above is not just a superficial similarity: methods developed in the context of $\mathbb{R}$-trees and affine buildings have inspired methods for phylogenetic analysis which have proved helpful in replacing black-box type square-mallet computer programs by much more sensitive algorithms which allow an interactive and visually supported exploration of kinship relations -not just between species, but also between, say, proteins or other types of bio-molecules, and which can also be applied to analyze other types of (dis)similarity data regarding, say, languages, archaeological artifacts, or manuscripts.

1991 MS Classification:
Hoppensteadt, Frank C., Arizona State University, Tempe AZ 85287-7606, USA
Canonical Models in Mathematical Neuroscience

Experiments have shown that there are phase relationships between inputs and outputs in neurons [1]. To study these, we model neural systems in the frequency domain. Neural systems are sometimes at rest and sometimes oscillating, and there is a hierarchy of canonical models describing how they can change from being at rest to oscillating [2]. For example, the Saddle-Node on Limit Cycle (SNLC), the Andronov-Hopf (AH) and the Saddle-Loop (SL) bifurcations describe such phenomena. The first two have co-dimension 1 and the second co-dimension 2. In SNLC and SL an oscillation of (almost) constant amplitude but increasing frequency appears, and in AH an oscillation of increasing amplitude and (almost) constant frequency appears. Both phenomena are observed in neuroscience experiments [3]. We focus here on SNLC and SL cases using the VCON model (voltage controlled oscillator neuron model [4]), which is based on phase-locked loop electronic circuits. We show that populations of such oscillators self-organize by frequencies [5]; that a same-frequency sub-population of oscillators can interact in the sense that a change in phase deviation in one will be felt by the others in the sub-population [2]; and that oscillators operating at different frequencies do not interact in this way. Therefore, sub-networks are defined by the firing frequency of their constituents. Elements can change their sub-population membership by changing their frequency, much like tuning to a new station on an FM radio. Thus, a rich array of network configurations is possible without rewiring. Specifically, the dynamical system

$$\tau\ddot{\theta}_j + \dot{\theta}_j = \omega_j + \cos\theta_j + (1 - \cos\theta_j)\sum_{k=1}^{N} C_{k,j}(1 + \cos\theta_k)$$

for $j, k = 1, \cdots, N$, describes a network of N oscillators having center frequencies ω_j, voltage outputs proportional to $1 + \cos\theta_j$, and connection strengths between oscillators $C_{k,j}$, indicating the strength of connection from site k to site j. Each of these phase variables has an asymptotic limit of the form $\theta_j \to \rho_j t + \phi_j$ where ρ_j is the asymptotic frequency (rotation number) and ϕ_j is the asymptotic phase deviation. A sub-network comprises the set of oscillators having the same value for ρ_j, and for this set information is carried by the phase deviation ϕ_j of each. This model, an analysis and computer simulations of it, and some related issues will be described.

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1991 MS Classification: 92C05
Peskin, Charles S., Courant Institute of Mathematical Sciences, New York University, USA

How to Catch and Transport a Chromosome

This talk is concerned with the mathematical modeling and computer simulation of four related processes involving biomolecular motors, all four of which play a role in chromosome transport during cell division. These processes are: (1) dynamic instability of microtubules, (2) the biased random walk of kinesin along a microtubule, (3) depolymerization driven transport, and (4) the role of elasticity in the function of Brownian ratchet molecular motors.

Dynamic instability is the process by which a microtubule repeatedly grows and shrinks at its free end (known as the "plus" end), as though casting into the intracellular sea to catch a chromosome. Once a chromosome has been caught, the process of dynamic instability terminates, to be replaced by systematic degradation (depolymerization) of the microtubule at its plus end. This ratchets the chromosome along towards the minus end of the microtubule which is anchored in the daughter cell. During depolymerization

driven transport, the chromosome is linked to the microtubule by kinesin, which paradoxically drives it towards the depolymerizing plus end, against the overall direction of transport. Theory and in vitro experiment explain how this facilitates the process of depolymerization driven transport.
The Brownian ratchet mechanism of depolymerization driven transport depends for its operation on the Brownian motion of the transported body, so it is bound to be slow for an object the size of a chromosome. We shall see, however, that the speed of transport is enhanced by the elasticity of the chromosome that is being transported.

Section 16 c: Materials/Hydrodynamics

1991 MS Classification: 65R10 , 44A35, 35K05
Greengard, Leslie, Courant Institute, New York University, USA
A new version of the fast Gauss transform

The evaluation of the sum of N Gaussians at M points in space arises as a computational task in diffusion, fluid dynamics, finance, and, more generally, in mollification. The work required for direct evaluation grows like the product NM as N and M increase, rendering large-scale calculations impractical. We present an improved version of the fast Gauss transform [L. Greengard and J. Strain, *SIAM J. Sci. Stat. Comput.* **12**, 79 (1991)], which evaluates the sum of N Gaussians at M arbitrarily distributed points in $O(N+M)$ work, where the constant of proportionality depends only on the precision required. The new scheme is based on a diagonal form for translating Hermite expansions and is significantly faster than previous implementations. Numerical examples and applications will be presented.
The work has been carried out jointly with Xiaobai Sun, Duke University.

1991 MS Classification: 65M12, 76B15
Hou, Thomas Yizhao, California Institute of Technology, USA
Numerical Study of Free Interface Problems Using Boundary Integral Methods

Many physically interesting problems involve propagation of free interfaces. Water waves, boundaries between immiscible fluids, vortex sheets, Hele-Shaw cells, thin-film growth, crystal growth and solidification are some of the better known examples. Numerical simulations for interfacial flows play an increasingly important role in understanding the complex interfacial dynamics, pattern formations, and interfacial instabilities. Many numerical methods have been developed to study these interfacial problems, including phase field models, volume of fluids, level set methods, front tracking methods, and boundary integral/element methods. Here we will focus on boundary integral methods.
Numerical study of fluid interfaces is a difficult task due to the presence of high frequency numerical instabilities. Small perturbations even at the round-off error level may lead to rapid growth. This makes it very difficult

to distinguish the numerical instability from the physical one. In our study, we first establish the well-posedness of the linearized motion far from equilibrium. This involves careful analysis of singular integral operators defined on free interfaces. The continuous well-posedness analysis provides a critical guideline for our numerical analysis of the discrete system. We found that there is a corresponding compatibility condition between the discrete singular operators and the discrete derivative operator. Violation of this compatibility condition will lead to numerical instability. We completely eliminate the numerical instability by introducing an effective filtering. The amount of filtering is determined by enforcing a discrete compatibility condition. The resulting scheme is shown to be stable and convergent. The corresponding 3-D problem is considerably more difficult since the singular operators have non-removable branch point singularities and there is no spectrally accurate discretization. A new stabilizing technique is introduced to overcome this difficulty. This technique is very general and effective. It also applies to non-periodic problems with rigid boundaries.

Surface tension also introduces a severe time step stability constraint. We overcome this difficulty by reformulating the problem in tangent angle/arclength (curvature in 3-D) variables and using a small scale decomposition technique. Using this reformulation, the leading order singular operators become linear and have constant coefficient. Thus semi-implicit discretizations can be easily applied and can be inverted efficiently. This greatly improves the time step stability constraint for a number of interfacial problems. Applications to Hele-Shaw flows, 3-D vortex filaments, and the Kirchhoff rod model for protein folding all give very impressive results. The improved method enables us to perform a careful numerical study of the stabilizing effect of surface tension for fluid interfaces. Water waves with small surface tension are shown to form singular capillary waves dynamically. The mechanism for generating such capillary waves is revealed and the zero surface tension limit is studied. In another study, surface tension is shown to regularize the early curvature singularity induced by the Rayleigh-Taylor instability in a unstably stratified two-fluid interface. However, a pinching singularity is observed in the late stage of the roll-up. The interface forms a trapped bubble and self-intersects in finite time.

Motivated by our analysis, we propose a new model equation for interface problems in both two and three dimensions. This model equation is nonlocal and nonlinear. But the nonlocal operator is given by the Hilbert transform in 2-D, and by the Rietz transform in 3-D. Consequently they can be evaluated fast with $NlogN$ operations by Fast Fourier Transform. Our numerical experiments have demonstrated convincingly that our model equations can captures effectively the small scale features even for interfaces with curvature or corner singularities.

1991 MS Classification: 73K20, 73B27, 73S10
Milton, Graeme, The University of Utah, USA
Exact relations for composites: towards a complete solution

This is joint work with Yury Grabovsky and Dan Sage. Typically, the elastic and electrical properties of composite materials are strongly microstructure dependent. So it comes as a nice surprise to come across exact formulae for (or linking) effective moduli that are universally valid no matter what the microstructure. Such exact formulae provide useful benchmarks for testing numerical and actual experimental data, and for evaluating the merit of various approximation schemes.
For example, consider the effective thermal expansion coefficient and effective bulk modulus of a composite of two isotropic phases. When one heats the composite the phases expand at different rates and the microgeometry is distorted. Similarly when one pressurizes the composite the microgeometry is distorted because of the differences in the bulk moduli of the phases. However one can imagine applying a judicious combination of pressure and temperature increase so the two phases, and hence the composite, expand at exactly the same rate. By this argument one obtains Levin's exact formula linking the effective bulk modulus and effective thermal expansion coefficient. As a second example, consider electrical conduction in a two dimensional medium. The electric field is curl free while the current field is divergence free, and they are linked by a constitutive relation which incorporates the conductivity of the medium. Now in two dimensions rotation of a divergence free field at each point by 90 degrees produces a curl-free field and vice-versa. Such a rotation of the electric field and current field produces, respectively, a new current field and new electric field which solve the conductivity equations in a new "dual" medium with a different conductivity. One easily obtains the formulae of Keller, Dykhne and Mendelson linking the effective conductivity tensors of the original medium and its dual. These formalae lead to exact expressions for the effective conductivity of self-dual media such as checkerboards and two-dimensional polycrystals. As a third example consider a two-phase thermoelectric medium where the constitutive law gives the heat current and electrical current as linear combinations of the temperature gradient and electric field. Now one is free to take a new pair of potentials which are linear combinations of the temperature and electric potential and one is free to take a new pair of divergence free fields which are linear combinations of the heat current and electrical current. By choosing approriate combinations one eliminates the cross-coupling terms. In this way one obtains the formulae of Straley and Milgrom and Shtrikman linking the effective theroelectric coefficient with the effective electrical and thermal conductivities.
There are many more known exact relations, not all of which are so simple. Traditionally they have been derived by a variety of different arguments, as the above examples suggest. One might wonder if there is a more general approach. Indeed there is! Here we present a systematic theory of exact

relations embracing the known exact relations and establishing new ones. The search for exact relations is reduced to a search for tensor subspaces satisfying certain simple algebraic conditions. Previously Grabovsky and Sage found an algebraic condition which was necessary for an exact relation to hold. It derives from the observation that an exact relation must hold at least for laminate microgeometries. Here we present a sufficient condition which guarantees that an exact relation will hold for all composites. It is derived from an analysis of certain convergent series expansions for the effective tensor. One of many new exact relations we derive is for the effective shear modulus of a class of three-dimensional polycrystalline materials.
Although the necessary conditions and sufficient conditions look quite similar, a gap remains between them. It is an open and challenging question as to what condition is both necessary and sufficient.

1991 MS Classification:

Rannacher, Rolf, Inst. für Angewandte Mathematik, Universität Heidelberg, Germany

Mathematics in Numerical Flow Simulation

The numerical computation of fluid and gas flow is a rich field asking for more mathematical analysis. The key issues today are “accuracy”, “efficiency” and “reliability” in terms of quantitative error control and economic grid generation. New developments in these directions make it possible to run simulations routinely on workstations. Reaching this goal requires input from the theory of differential equations, multiscale techniques and optimization. The impact of these theoretical aspects on practical problems is exemplarily illustrated by the computation 1) of drag and lift forces of blunt bodies and 2) of concentration profiles in chemical flow reactors.

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17. Control Theory and Optimization

1991 MS Classification: 90C10 90C27
Cook, William, Rice University, USA

The Traveling Salesman Problem

In 1954, G. Dantzig, R. Fulkerson, and S. Johnson demonstrated that large instances of the TSP could be solved using the duality theory of linear programming. Their work was inspired by the theoretical studies of J. Robinson and H. Kuhn in the late 1940s and the early 1950s, and it remains the only known method for solving TSP instances having more than several hundred cities. We discuss the evolution of their methodology, including the research of M. Grötschel, S. Hong, M. Padberg and others. The techniques will be illustrated on the solution of a 13,509-city example. This talk is based on joint work with David Applegate, Robert Bixby, and Vašek Chvátal.

1991 MS Classification: 90, 05, 68
Goemans, Michel, Université catholique de Louvain, Belgium, and Massachusetts Institute of Technology, USA

Semidefinite programming and combinatorial optimization

Convex optimization, and more specifically semidefinite programming, has found many applications in combinatorial optimization. The LP bound in coding, the theta function for the stability number, or spectral bounds on graph partitioning and sparsest cut can all be interpreted in terms of semidefinite programs, among many other results. Recently, other applications have been found ranging from the automatic generation of valid inequalities for integer programs, to randomized approximation algorithms for the maximum cut and other combinatorial problems such as the bandwidth problem. In this talk, I will survey some of these developments and applications, and their limitations.

1991 MS Classification: 65K05 90C30
Nocedal, Jorge, Northwestern University, Evanston, USA

Nonlinear Optimization: The Interplay Between Mathematical Characterizations and Algorithms

We discuss several fundamental questions concerning the problem of minimizing a nonlinear function subject to a set of inequality constraints. We begin by asking: What makes the problem intrinsically difficult to solve, and which characterizations of the solution make its solution more tractable? This leads to a discussion of the two main methods of solution: transformation and approximation.
Interior point methods make use of transformations. The nonlinear programming problem is recast a series of (more nonlinear) subproblems in which the constraints have been relegated to a secondary role. The key of their success lies not only in the proper subproblem formulation, but in

exploiting their geometry to locate an approximate solution rapidly. Approximation methods, as represented by active-set sequential quadratic programming, take the opposite view and generate a series of mildly nonlinear subproblems. We do not yet know which of the two approaches will prove to be more powerful, but we present numerical results illustrating the strengths of each method.

1991 MS Classification: 65Y20

Pulleyblank, William R., T. J. Watson Res. Center, IBM, New York, USA

Advances in Solving Airline Crew Pairing Optimization Problems

Airline Crew Scheduling Problems provide an important class of very large set packing problems. The problem is to determine a set of *pairings*, legal sequences of duties for airline crews, which minimize the cost of the associated schedule.

Part of the difficulty of this problem arises from the complexity of both the feasibility rules and the cost structure. Part is also due to the very large size of the set P of feasible pairings. Traditionally the problem is formulated as the following integer program:

$$\begin{array}{ll} \text{Minimize} & \sum_{j \in P} c_j x_j \\ \text{subject to} & x_j \in \{0,1\} \text{for all } j \in P \\ & \sum_{j \in P(d)} x_j = 1 \text{ for all flights } \; d \end{array}$$

where for each scheduled flight d, $P(d)$ is the set of feasible pairings that cover that flight.

Successfully solving these problems require specialized integer programming methods, combined with methods for generating feasible pairings which can be combined to yield the required low cost solution.

We discuss the general problem as well as the differences between domestic and short haul versions. After surveying previous approaches, we describe how subgradient methods for approximating the solutions of the underlying linear programs yield approximate dual solutions which greatly improve the performance of the column generator and actually enable Dantzig-Wolfe decomposition to become an effective solution procedure.

We also present computational experience using these methods on problems arising from several airlines.

This is joint work with Ranga Anbil and John Forrest of IBM Research.

1991 MS Classification: 05C85, 05C90, 90B06, 90B10, 90B35

Schrijver, Alexander, CWI, Amsterdam, and University of Amsterdam, The Netherlands

Routing and timetabling by topological search

We discuss how decomposing the search space into homotopy classes can help in finding solutions to combinatorial optimization problems. Searching any homotopy class then amounts to finding a group function ψ on the arcs of a directed graph such that ψ is cohomologous to a given function ϕ and such that ψ has values in a prescribed range.

We describe applications to two specific classes of NP-complete problems: routing wires on a chip (where the main tool is solving the cohomology problem in a free group, and a main result the polynomial-time solvability of the wire-routing problem for any fixed number of modules), and finding a periodic timetable (applied to the Dutch railway timetable, where liftings of the period group C_{60} to the integers give the classes to be searched).
The methods also imply a characterization of the existence of an isotopy of a compact surface S that brings a given set of disjoint closed curve on S to a given undirected graph embedded on S.

1991 MS Classification: 93B05, 93B07, 93B36, 93B51, 93C05, 93C15
Willems, Jan C., University of Groningen, The Netherlands
Open Dynamical Systems and their Control

Dynamical systems usually interact with their environment. Such systems are called *open systems.* It is often essential to incorporate this feature in the mathematical model, for example, in application areas such as signal processing and control engineering. The aim of this talk is to describe the mathematical structures that emerge in modelling open dynamical systems, to discuss two central concepts, controllability and observability, and to apply these ideas to the design of robust controllers. For ease of exposition, we will concentrate on a rather specific class of systems, linear time-invariant differential systems.
We now describe our formulation of the control problem in a bit more detail. I n a typical control problem we have a given plant that relates three (vector-valued) time-signals: disturbances d, to-be-controlled variables z, and control variables c. Control is viewed as interconnecting a controller to the plant, with the controller acting only through the control variables. Before the controller is attached to the plant, the variables $w = (d, z)$ define a set $\mathcal{P}$ of realizable time-signals; $\mathcal{P}$ is, for example, a linear shift-invariant subspace of $\mathfrak{L}_2^{\ell oc}(\mathbb{R}, \mathbb{R}^q)$. When the controller is attached to the plant, it restricts $\mathcal{P}$ to $\mathcal{K} \subseteq \mathcal{P}$: $\mathcal{K}$ is called the *controlled behavior.* The fact that the controller can act on the control variables only, translates into the requirement that $\mathcal{K}$ must contain a given subspac e $\mathcal{N}$ of $\mathcal{P}$. The control problem is thus to find, for a given
$\mathcal{N}$ and $\mathcal{P}$, a controlled behavior $\mathcal{K}$ with $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ such that $\mathcal{K}$ meets certain specifications. For example, in robust control, $\mathcal{K}$ must leave the exogenous disturbances d free, it must be stable (appropriately defined), and it must be attenuate the disturbances ($|| \ z \ || \leq || \ d \ ||$ in suitable norms).
The solution to this robust control problem involves the theory of linear dissipative systems with quadratic functionals. We will present this solution and sketch algorithms that allow to compute the controlled behavior and the controller in terms of a representation of the plant in terms of a differential equation.

1991 MS Classification:

Zowe, Jochem, Institut für Angewandte Mathematik, Universität Erlangen, Germany

What is the Best Material? — The Answer of a Mathematician

The talk deals with a central question of structural optimization which is formulated as the problem of finding the stiffest structure which can be made when both the distribution of material as well as the material itself can be freely varied. Our emphasis is on a mathematical formulation which leads to numerical procedures for the problem, and we show that the problem after discretization can be rewritten as a mathematical programming problem of special form. We propose iterative optimization algorithms based on interior-point methods and on semidefinite programming and show a broad range of numerical examples that demonstrates the efficiency of our approach.

18. Teaching and Popularization of Mathematics

1991 MS Classification: 97A15 - 98A15

Bartolini Bussi, Maria G., University of Modena, Italy

Drawing instruments: the theory of a practice from history to didactics

Didactics of mathematics as a scientific discipline is fairly young compared to other sciences, yet deeply rooted in the perennial effort of mathematicians to advance human understanding of mathematics and to transmit mathematics knowledge to future generations. In recent years several efforts have been done at the international level (detailed references will be given in the paper) to clarify the objects, the aims, the research questions, the methodologies, the results and the criteria to evaluate the results of research in didactics of mathematics. It has become clear that analytical tools are needed from different disciplines (such as epistemology, history, psychology) to obtain results that can increase the knowledge of the teaching and learning processes in the classroom, produce effective innovation in schools or understand why some designed innovation does not work, and, at a larger level, influence the development of school systems. Anyway, the starting point of every discussion is the nature of mathematics knowledge. One of the distinctive features of mathematics is theoretical organisation. Unfortunately the production and even the understanding of mathematical proofs is supposed to be very difficult for the students all over the world. As a matter of fact, in some countries proofs are being eliminated from mathematics curriculum. This is the scenery where a collective project has been initiated some years ago by a group of Italian researchers. The project highlights the permanent value of proof in mathematics and mathematics education and aims to design, implement and analyse effective teaching experiments, that can introduce students into the theoretical dimension of mathematical culture. Experiments in different fields have been carried on at very different school levels, from primary school to university. In my lecture I shall focus on some experiments that concern activity with linkages and other drawing instruments at secondary and university levels. The main thesis of my lecture is the following: By exploring, with suitable tasks and under the teacher's guidance, linkages and other drawing instruments, secondary and university students can 1) be enriched with a well-balanced image of mathematics, where theoretical aspects and applications are strictly intertwined, yet not confused; 2) be involved in the generation of 'new' (for the learners) pieces of mathematical knowledge by taking active part in the production of statements and the construction of proofs in a reference theory, living an experience similar to the one of professional mathematicians; 3) be acquainted with a set of exploring strategies and representative tools that nurture the creative process of statement production and proof construction and are transferable to other sets of problems. This thesis will be defended by referring to research studies already published or in progress. Linkages and other drawing instruments belong to the historical phenomenology of geometry. A part of the lecture will be devoted to a sketchy reconstruction of this history from the time of Euclid to the present studies about algebraic and differential geometry of linkages and applications to robotics. Two red threads go through the whole history and concern the organisation in theories of individual linkages on the one side and of individual curves (later algebraic curves) on the other side. Linkages and other drawing instruments have a twofold nature: they are material objects that can be used for practical purposes and they are abstract objects that belong to a mathematical theory. Their material feature makes them good candidate for popularization of mathematics, by means of concrete or virtual copies. Yet the mathematical theory is hidden and not easy to be communicated. This shifts the problem to a correct didactical use of drawing instruments as far as the theoretical dimension of mathematics is concerned. In other words, it is necessary to pay attention to the choice of suitable tasks and to the teacher's role in structuring classroom activity. In this very part of the study, the function of analytical tools from epistemology of mathematics and psychology of mathematics education appears to be relevant.

1991 MS Classification: 00-XX General
Niss, Mogens, Roskilde University, Denmark

Aspects of the nature and the state of research in mathematical education

During the past three decades, a rapidly growing body of research on a multitude of aspects of mathematics education has contributed to establish what today is an academic discipline sometimes referred to as 'the didactics of mathematics', sometimes simply as mathematics education research. What are the issues and research questions in this discipline, what is its methodology, what sorts of findings does it offer, and what sorts of evidence does it provide, in particular with respect to matters of importance to university mathematics?

In this lecture, attempts will be made to characterise the didactics of mathematics as a discipline, and a number of major findings will be presented and discussed. More specifically, it will be explained why such findings only relatively seldom take the shape of empirical or experimental results in the traditional sense (and never, of course, of mathematical theorems). Nevertheless, research findings in mathematics education resulting from thorough theoretical or empirical analyses give rise to solid insights of considerable significance for our understanding of processes and outcomes of mathematical teaching and learning, and hence for the ways in which mathematics may, or may not, be successfully taught and learnt. As far as major findings are concerned, the ones to be presented in this lecture have been selected after consultation of a fair number of the leading researchers in mathematics education in the world.

One major finding may be formulated as follows, *the key role of domain specificity*: For a student engaged in learning mathematics, the specific nature, content and extent of a mathematical concept that he or she is acquiring or building up are to a large part determined by the set of specific domains in which that concept has been concretely exemplified and embedded for that particular student. At first sight, this may seem to be little more than a reformulation of a well known observation belonging to the experience of any observant and reflective teacher of mathematics at whichever level. (If this is true, which it may well sometimes be, it is remarkable, however, how often the finding remains unobserved in actual teaching practice.). But at closer inspection, the range and depth of the instances of this finding have far-reaching bearings on the teaching and learning of mathematics. Thus, not only are most 'usual' students unable to grasp an abstract concept in and of itself unless it has been elucidated by multiple examples (which is well known), but, more importantly, the scope of the notion a student forms is often barred by the very examples given to support that notion. This implies that it is a non-trivial matter of teaching and learning to establish abstract mathematical concepts so as to be both sufficiently general and sufficiently concrete in their scopes. Research further suggests that for this to happen several different representations (e. g. verbal, symbolic, graphical, diagrammatical, etc,) of concepts and phenomena are essential, as are the links and transitions between these representations.

The above-mentioned finding becomes particularly intricate with those mathematical concepts which are formed as objects generated by and encapsulating certain processes. Simple examples of this are the concept of function (arising from mechanisms that produce the values of the function) and the concept of derivative (arising from the process of differentiating). This process-object duality is referred to in the research literature by different terms, including 'procept' (an amalgamation of process-concept), 'tool-object' or 'reification'. This constitutes another major finding in the didactics of mathematics, *the obstacles produced by the process-object duality*. This duality gives rise to fundamental learning (and teaching) hurdles that are difficult to overcome for students (and teachers). Many students have substantial difficulty in leaving the process level and enter the object level. For example, many students conceive an equation as signifying a request to perform certain operations, without dis- tinguishing these operations from the concept of equation as such. To them, an equation simply does not constitute a mathematical entity. Research even shows that quite a few of those who can actually solve a given equation correctly are unable to subsequently tell correctly whether or not a given element is a solution to the equation or not. Other students are able to establish notions of both the processes underlying a certain concept and that concept as an object, but have difficulty in establishing links between the two. The theory of so-called 'conceptual fields' offers general perspectives on these matters.

The findings sketched above are connected to yet other major findings, such as *the mathematical knowledge of the individual learner is not necessarily hiearchical* (it is possible to possess correct knowledge of a certain segment of mathematics without possessing correct knowledge that may be considered a logical or a conceptual prerequisite for the former knowledge), *students'*

conceptions of basic aspects of mathematics are often rational, even if they are incorrect or insufficient, a phenomenon which is sometimes explained by the fact that some students' initial, but implicit, conceptual models are so strong and robust that they exert a decisive influence on later learning experiences.
These and other findings (concerning issues such as students' conceptions of proof; problem solving; assessment; the influence of technology; the significance of the cultural and social environment, of the context, and of the students' activities, on the learning outcomes) will be considered in the lecture.

1991 MS Classification:
Stigler, Jim, UCLA Psychology Department, Los Angeles, USA
Classroom Mathematics Instruction in Three Cultures: An Introduction to the TIMSS Video Study

Classroom Mathematics Instruction in Three Cultures: An Introduction to the TIMSS Video Study
This talk will introduce the methods and findings of a unique and unprecedented video survey of eighth-grade mathematics instruction in Germany, Japan, and the United States. Part of the Third International Mathematics and Science Study (TIMSS), this study marks the first time that videotaped records of classroom instruction–in any subject–have been collected from nationally representative samples of teachers. Methods for overcoming the considerable logistical challenges posed by the study will be discussed. Then, teaching practices in each culture will be described using both quantitative indicators and video examples. Finally, implications of the study for improving classroom mathematics instruction will be discussed.

A Debate on Teaching the First Years of University Mathematics

1991 MS Classification:
Lewis, Donald J, National Science Foundation, USA
Andrews, George E., Pennsylvania State University, USA
Smith, David A., Duke University, USA
Collegiate Mathematics Instruction for the 21st Century – A Panel Discussion

In the last quarter century two developments have occurred that have had great impact on beginning collegiate instruction in North America: a changed constituency and technology. Today a far larger percentage of 17-20 years olds attend university and nearly every program of study requires knowledge of the calculus, linear algebra and statistics; but students arrive with weak backgrounds and are not well motivated. Modern technology has provided cheap graphical calculators and computers with which the students have had exposure; television and computer games have reduced reading, weakened the ability to visualize from the written word, and have led to the expectation of immediate response and satisfaction. Deferred gratification, once common among physical science and engineering students, has disappeared. These developments have led to considerable dissatisfaction with instruction in introductory mathematics by all concerned. These attempts have led to many experiments and much debate. Since other areas of the globe are affected by these developments it is hoped that other than North Americans will participate in the panel's discussion of learning environments for the 21st Century.

Donald J. Lewis
dlewisnsf.gov

For decades there has been the widespread observation that undergraduate mathematics education is falling short. This has created an atmosphere in which all feel compelled to do something, and the world looks to mathematicians to fix the problem. Accompanying this pressure has been the explosion of technological gimmicks from graphics calculators to super computers. In response, university administrators and government agencies are supplying money to solve our problems, and numerous calculator, computer and software manufacturers are hoping to assist, and in doing so, open vast new markets.

In my presentation, I hope to consider the original problem, the public pressures, the funding issues and the industry contributions in order to examine whether this mix of concerns and solutions can improve mathematics education.

Finally I shall try to call attention to what I believe are the major, mostly unaddressed problems facing us. I shall suggest that a positive first step in reform might be the elimination of technology from calculus instruction and from all instruction prior to calculus.

George E. Andrews, Pennsylvania State University
andrewsmath.psu.edu

Renewal in Collegiate Mathematics Education

My presentation will start with a list of skills and attributes that prospective employers expect of college graduates – not because we are engaged in vocational training, but because these employers speak for or heavily influence all the major sources of support for the educational enterprise.

To identify problems with American collegiate education, I will contrast the desired outcomes with some of the features of our enterprise, based on demographics, ideology, unsupported beliefs and fears, lack of standards, and a professoriate highly knowledgable about subject matter, but less so about pedagogy.

In that context, I will describe the primary achievements of the "Calculus Reform Movement": getting "educators" to talk about and learn about education, shifting discussions from teacher-centered instruction to student-centered learning, and demonstrating that new knowledge can be translated into effective learning programs. In particular, I will note how recent research on the biology of the brain confirms earlier research on the psychology of learning, and how both support emerging "best practice" in pedagogy.

Finally, what is the role of technology in educational renewal, especially in mathematics? Technology is neither "the problem" nor "the solution". Rather, it is a fact of our students' lives – before, during, and after college. For the faculty, this fact presents opportunities (a) to rethink a mathematics curriculum heavily weighted (in lower-division courses) with pre-computer techniques, (b) to create learning environments that accord with best practice, and (c) to shift the primary focus in our courses from manipulation to thinking.

David A. Smith, Duke University
dasmath.duke.edu

1991 MS Classification: 01-02

Artigue, Michèle, IUFM de Reims et Université Paris 7, France

De la compréhension des processus d'apprentissage à la conception de processus d'enseignement

La recherche en didactique des mathématiques, par ses travaux théoriques et expérimentaux nous donne aujourd'hui les moyens de mieux comprendre les processus d'apprentissage en mathématiques, des premiers apprentissages de l'école élémentaire à ceux en jeu à l'université. La question de savoir comment tirer parti de cette connaissance pour améliorer les stratégies d'enseignement reste cependant largement ouverte. A travers elle, se pose celle de l'utilité des recherches didactiques mais aussi celle des cadres théoriques adéquats pour penser le fonctionnement de systèmes complexes comme le sont les systèmes d'enseignement. C'est à ces questions que nous nous intéresserons dans notre exposé, en privilégiant au niveau des exemples l'enseignement des mathématiques à l'université ou à la transition lycée/université.

Dans une première partie, nous nous centrerons sur la question de l'apprentissage. Nous l'aborderons d'abord sous un angle épistémologique, en soulignant la diversité des réponses susceptibles d'être apportées à des questions comme : "Qu'est-ce qu'apprendre des mathématiques ?" ou "Comment apprend-t-on des mathématiques ?" suivant les présupposés épistémologiques que l'on adopte, qu'il s'agisse des mathématiques ou des théories de l'apprentissage.

Nous nous centrerons ensuite sur deux caractéristiques de l'apprentissage des mathématiques, attestées par les recherches didactiques mais qui semblent peu prises en compte dans les stratégies d'enseignement usuelles, en particulier à l'université.

1. L'apprentissage des mathématiques n'est pas un processus linéaire ou continu. Il suppose des ruptures, des reconstructions et restructurations successives, qui ne peuvent être laissées à la seule responsabilité des étudiants.
2. L'apprentissage des mathématiques n'est pas seulement le résultat de processus d'abstractions successives. Dans la conceptualisation mathématique, l'articulation de points de vue, de cadres de fonctionnement, de registres de représentation et de traitement des concepts joue un rôle tout aussi fondamental.

Nous illustrerons ces deux caractéristiques par des exemples précis issus de l'enseignement de l'analyse et de l'algèbre linéaire.
Dans une deuxième partie, nous aborderons la question de l'exploitation de ces connaissances au niveau de l'action sur le système d'enseignement. Nous montrerons, en nous référant à des exemples de travaux expérimentaux ou de réformes de l'enseignement, les problèmes posés par des transpositions hâtives des résultats didactiques. Nous soulignerons la nécessité, pour aborder efficacement ces questions, de dépasser les cadres épistémologiques et cognitifs qui ont largement fondé les recherches sur l'apprentissage des mathématiques, au profit d'approches qui permettent de prendre en charge théoriquement les caractéristiques institutionnelles et culturelles de l'enseignement et de l'apprentissage des mathématiques.
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Duval, R. (1995) *Sémiosis et Pensée Humaine*, Peter Lang, Paris.
Sierpinska, A., Lerman, S. (1996) Epistemologies of Mathematics and of Mathematics Education, in A. .J. Bishop et al. (Eds), *International Handbook of Mathematics Education*, p. 827–876, Kluwer Academic Publishers.

1991 MS Classification: 01 - 02
Steiner, Hans-Georg, Institute for Didactics of Mathematics (IDM), University of Bielefeld, Germany

Response to Michèle Artigue's lecture titled "From the Understanding of Learning Processes to the Design of Teaching Processes"

Having been asked by the Program and the Organizing Committee to give a response to Michèle Artigue's lecture, reflecting different currents of thinking, I'll refer to the following aspects and sources:
(1) Philosophical and Epistemological Aspects of Mathematics and their Interaction with Theory and Practice in Mathematics Education. H. G. Steiner. For the Learning of Mathematics 7 (1987), No 1, 7 - 13.
(2) Investigating and Bridging the Teaching-Learning-Gap. Proc. 2nd Conf. Int. Group on Theory of Mathematics Education (TME). Ed. by H. G. Steiner & A. Vermandel. Antwerp-Bielefeld 1988.
(3) Mathematization Processes in Class as Collective Higher Order Learning Processes: The Role of the Teacher and Reflections on Teacher Education. H. G. Steiner. Proc. 4th Central-American and Caribic Conference in Math.

Ed. Acapulco 1990, 155 - 160.
(4) The Nature of Theoretical Concepts in Physics and Mathematics Implications for the Fragility of Knowledge in the Educational Context. H. G. Steiner. Proc. 2nd. Jerusalem Conference on Education. Ed. by Shlomo Vinner. Hebrew Univ. Jerusalem 1989, 387 - 396.
(5) R. Biehler, R. W. Scholz, R. Sträßer, and B. Winkelmann (Eds.): Didactics of mathematics as a Scientific Discipline. Kluwer Academic Publ. Dordrecht-Boston-London. 1994

Round Table

1991 MS Classification: 0

de Guzmán, Miguel, Universidad Complutense de Madrid, Spain;
Hodgson, Bernard R., Université Laval, Québec, Canada;
Robert, Aline, IUFM de Versailles, France;
Villani, Vinicio, Università di Pisa, Italy

Difficulties in the Passage from Secondary to Tertiary Education

For an important part of those students who take mathematics courses at the tertiary level, the transition from secondary to tertiary education presents major difficulties. This is true whether students considered are specialising in mathematics or are inscribed in a program for which mathematics is a service subject. The purpose of this round table is to identify possible causes for these difficulties and to suggest actions which could help to improve the situation.

The age at which students pass from secondary to tertiary education ranges in different countries from 17–18 to 19–20 years. But the difficulties met by the students involved in mathematics courses seem to depend more on other elements than on their age.

Here are some of the most widespread sources of these difficulties.

- *The lack of knowledge of specific topics by students.* Students who enter a university class may have very heterogeneous backgrounds, while university teachers often do not care to make sure from the beginning that each one of them masters the basic notions and skills required for an understanding of their course.
- *The sudden change in the ways of approaching the mathematical activity.* Emphasis in secondary school is often on "drill and practice", so that students entering university may lack conceptual understanding. University mathematics, on the other hand, may rest right from the beginning on abstraction, generalisation, reasoning, capability to read, write and speak about mathematical topics. There is a sudden change in the expectations regarding the students' abilities, but this change is often not even made clear to the students.
- *The underestimation of the role of mathematics with respect to future careers.* Many students have chosen a field of specialisation in which they were not expecting to have to study mathematics. The instructor needs to address this issue and convince students of the importance of mathematics

for their career.

• *The distorted image of students by teachers.* Many university teachers tend to identify their "average" student with an ideal student who has successfully attended a highly scientific track in one of the best secondary schools. But in an actual class this kind of student may be only a very small minority, or even not exist at all.

• *The lack of pedagogical awareness of some teachers.* In most cases, university teachers have received their professional training as if their main (or only) occupation in mathematics is research. Thus they need to develop by themselves the pedagogical and communication capabilities they will need with their students, which is an arduous task.

Is it possible to overcome such difficulties? And how? The presentations and the discussion during the round table will include identification of possible actions and examination of their merits. Time will be allowed for exchange of views with participants in the audience.

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19. History of Mathematics

1991 MS Classification:
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History of Mathematics in China: A factor in World History and a Source for New Questions

In the last decades much research has been devoted to the *Jiuzhang suanshu* or *The nine chapters on mathematical procedures* (hereafter abbreviated to *The nine chapters*), a book which played a crucial part in the mathematical traditions which expressed themselves in Chinese, quite comparable to the one played by Euclid's *Elements of geometry* in the West. Compiled around the beginning of the common era, after the unification of the Chinese empire, the book was soon to become a "Classic" from which most subsequent Chinese mathematicians drew inspiration. As for all the other writings granted the status of Classics in China, commentaries were composed on *The nine chapters* , some of which were selected by the tradition to be handed down together with the book. This is how commentaries ascribed to Liu Hui (third century) and Li Chunfeng (seventh century) survived until today.

The talk will present some recent results obtained on the book itself and its commentaries. It will then discuss various aspects of the transmission of mathematical knowledge between China and the West. Finally, the talk will c onsider some new questions for the study of mathematical activity raised by research on *The nine chapters* . In conclusion, it is argued that only an international approach to history of mathematics can provide an adequate framework to capture the historical processes which have constituted mathematical lores around the world.

I. Recent Developments in the History of Mathematics in Early Imperial China

The nine chapters are constituted of problems and general algorithms with which to solve them. Inspired by Donald Knuth, who suggested reading Old-Babylonian clay-tablets from the point of view of algorithmic theory, Wu Wenjun initiated a new approach to ancient Chinese mathematical sources along similar lines. Algorithms found in *The nine chapters* such as root extractions, rules of false double position, or algorithms to solve systems of simultaneous linear equations, evidence the use of sophisticated resources: assignment of variables, conditionals, iterations. They demonstrate the development, wit hin an algorithmic framework, of an algebra that I shall characterize. In contrast with *The nine chapters* themselves, the commentaries explicitly set out to prove the correctness of the algorithms provided by the Classic. This practice of proof will be described. In particular, it will be argued that here we find, again within an algorithmic context, the development of a form of algebraic proof. The commentator Liu Hui stresses the connection between this form of proof and the various kinds of numbers introduced by the Classic (fractions and quadratic irrationals). How both aspects relate to a philosophical outlook on mathematics developed within the commentaries will also be outlined.

II. A Factor in World History

The earliest extant Chinese and Indian mathematical sources demonstrate enough basic common features to state that, from early on, the communities practising mathematics in both areas must have been part of a common intellectual world. Later on, Arab scholars became interested in this scientific world through India. This is documented historically. However, several elements common to Chinese and Arabic sources from the 9th century onwards, and so far not found in the known Indian sources, seem to indicate that there were direct contacts between the Chinese- and the Arabic-speaking intellectual communities. Using *The nine chapters*, t he talk will hint at several cases of circulation of knowledge, for the different perspectives they provide on the processes thus constituting mathematical knowledge, and on the situations they created.

III. New questions

Not only do ancient Chinese sources contain original results, some of which are now part of common knowledge in world mathematics. They also evidence a specific way of practising mathematics, which can be recovered through historical analysis. For instance, the counting board on which Chinese mathematicians computed was handled according to strict and particular rules. So me recurring features of mathematical notations in China can be traced back to that practice of computing, and the nature of some basic concepts seems to originate from observing mathematical reality as given shape through computations on the board. It will be argued that all the elements entering into the practice of mathematics d́1problems, visual aids such as figures, literary techniquesd́1 demonstrate specific features in the way mathematicians used them in ancient China. However, this is not peculiar to China: more generally, the products of mathematical activity that eventually become universal are worked out by resorting to specific forms of practice, partly inheri ted, partly reworked according to the problems addressed or the conditions available. Clearly, describing such regimes of mathematical activity and their relations to the mathematical results produced, is history with an agenda, and I shall conclude by discussing its relevance.

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1991 MS Classification:
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Marx, Mao and Mathematics: the Story of Nonstandard Analysis in China and the Cultural Revolution

The "Mathematical Manuscripts" of Karl Marx were first published (in part) in Russian in 1933, along with an analysis by S. A. Yanovskaya. Friedrich Engels was the first to call attention to the existence of these manuscripts in the preface to his *Anti-Duehring* [1885]. A more definitive edition of the "Manuscripts" was eventually published, under the direction of Yanovskaya, in 1968, and subsequently numerous translations have also appeared. Marx was interested in mathematics primarily because of its relation to his ideas on political economy, but he also saw the idea of variable magnitude as directly related to dialectical processes in nature. He regarded questions about the foundations of the differential calculus as a "touchstone of the application of the method of materialist dialectics to mathematics. "
Nearly a century later, Chinese mathematicians explicitly linked Marxist ideology and the foundations of mathematics through a new program interpreting calculus in terms of nonstandard analysis. During the Cultural Revolution (1966-1976), mathematics was suspect for being too abstract, aloof from the concerns of the common man and the struggle to meet the basic needs of daily life in a still largely agrarian society. But during the Cultural Revolution, when Chinese mathematicians discovered the mathematical manuscripts of Karl Marx, these seemed to offer fresh grounds for justifying abstract mathematics, especially concern for foundations and critical evaluation of the calculus. At least one study group in the Department of Mathematics at Zhejiang Teachers College issued its own account of "'The Brilliant Victory of Dialectics – Notes on Studying Marx's 'Mathematical Manuscripts', " [1975].
Inspired by nonstandard analysis, introduced by Abraham Robinson only a few years previously, some Chinese mathematicians adapted the model Marx had laid down a century earlier in analyzing the calculus, and especially the nature of infinitesimals in mathematics, from a Marxist perspective. But they did so with new technical tools available thanks to Robinson but unknown to Marx when he began to study the calculus in the 1860s. As a result, considerable interest in nonstandard analysis has developed subsequently in China, and almost immediately after the Cultural Revolution was officially over in 1976, the first all-China conference on nonstandard analysis was held in Xinxiang, Henan Province, in 1978.

1991 MS Classification: 1, 14, 30
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The Riemann-Roch Theorem and Geometry, 1854-1914

In this talk I shall consider aspects of the history of the Riemann-Roch The-

orem, from its discovery by Riemann and Roch to its use by Castelnuovo and Enriques. The theorem offers one of the most instructive examples in the history of mathematics of how a result stays alive by admitting many interpretations: in complex function theory, and in both the algebraic and geometric styles of algebraic geometry. This profusion of approaches, at times incompatible and not always firmly grounded, demonstrates the challenges and the pleasures involved in doing mathematics, and challenges the historian to capture the richness of the subject.
I shall consider three main developments:

1) The original discovery and the uses to which Riemann put it;
2) The algebraic reformulation by Brill and Noether after Riemann's death;
3) The geometric twist Castelnuovo and Enriques gave to the Brill-Noether approach, and the role of the Riemann-Roch Theorem in their eventual classification of algebraic surfaces.

Brill and Noether deliberately rejected Riemann's function-theoretic formulation of the Theorem, but in so doing they also blurred his distinction between intrinsic geometrical properties and properties of embeddings. Italian mathematicians, unlike their French counter-parts, also kept away from complex function theory, but they adapted the techniques of Brill and Noether to give a deliberately intrinsic theory of algebraic surfaces.

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