Hochschild cohomology algebras of algebras of quaternion type

A. A. Ivanov

2012

TAME ALGEBRAS

An **Algebra of finite representation type** is a finite-dimensional algebra such that there are only finitely many indecomosable modules.

An **Algebra of infinite representation type** is such a finite-dimensional algebra that is not of finite type.

A **Tame algebra**, roughly speaking, is a finite-dimensional algebra of infinite type, such that for any $d \geqslant 1$ almost all indecomposalble modules of dimension d belong to finitely many 1-parameter families.

The precise definition of the tame algebra

A finite-dimensional algebra Λ is of tame representation type provided Λ is not of finite type, whereas for any dimension $d \geqslant 1$, there are a finite number of $K[T] - \Lambda$-bimodules M_i which are as left $K[T]$-modules such that all but a finite number of indecomposable Λ-modules of dimension d are isomorphic to $N \otimes_{K[T]} M_i$ for some i and some simple $K[T]$-module N.

Dichotomy Theorem (Drozd)

A finite-dimensional K-algebra Λ is **wild** if there is a finitely generated $K\langle X, Y\rangle - \Lambda$-bimodule B which is free as a left $K\langle X, Y\rangle$-module such that the functor

$$- \otimes_{K\langle X,Y\rangle} B : \ mod\ K\langle X, Y\rangle \to mod\ \Lambda$$

preserves indecompsability and reflects isomorphisms.

Theorem (Drozd)

Any finite-dimensional algebra over an algebraically closed field which is of infinite representation type is either tame or wild.

THE CASE OF GROUP-ALGEBRAS

QUESTION: When a group algebra is tame?

Definition (BLOCKS AND THEIR DEFECT GROUPS)

Let G be a finite group and K be an algebraically closed field of char p. The group algebra is a direct sum of indecomposable algebras, $KG = B_1 \oplus ... \oplus B_n$, and the B_i are the **blocks** of KG. And the identity of KG is sum of orthogonal centrally primitive idempotents e_i.

Block is symmetric algebra. And when $p \mid |G|$, then blocks of KG are usually not semisimple. A **principal block** is the block B such that trivial KG-module $K \in Ob\ mod\ B$

A **defect group** of a block B is a minimal subgroup D of G such that B is isomorphic to a direct summand of $B \otimes_{kD} B$ as B-B-bimodule.

How representation type of a group algebra depends on its defect group

Theorem

Lets consider the group algebra KG of finite group G over a field K of characteristic p; or a block B of KG.
Suppose D is Sylow p-subgroup of G; or a defect group of B.
Then the representation type of KG; or of B, is
(i) finite if D is cyclic;
(ii) tame if $p = 2$ and D is dihedral or semidihedral or generalized quaternion;
(iii) wild, otherwise.

Defenitions of dihedral, semidihedral and generalised quaternion groups

$$D_{2n} = \langle x, y \mid x^{2^n} = 1 = y^2,\ yxy = x^{-1} \rangle$$

$$SD_{2^n} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}-1} \rangle$$

$$Q_{4n} = \langle x, y \mid x^n = y^2,\ xyx = y \rangle$$

Classification of tame group blocks up to Morita-equivalence

In order to solve the problem of classification of all group blocks having tame type up to Morita-equivalence Karin Erdmann suggested definitions of three new families of algebras, containing group block algebras, for which this problem could be naturally reformulated.

These families are the classes of algebras of Dihedral, Semidihedral and Quaternion types.

Now any tame block with a defect group of any type is contained in the class of algebras of the corresponding type: diheral, semidihedral or quaternion.

Algebras of the quaternion type

Definition

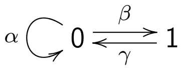
An algebra Λ is of **quaternion type:** if
(i) Λ is symmetric, indecomposeble and tame;
(ii) the stable Auslander-Reiten quiver of Λ consists only of tubes of rank $\leqslant 2$;
(iii) the Cartan matrix of Λ is nonsingular.

In 1991 K. Erdman finished the description of several series of algebras defined by quivers and relations, which were tame and contain all possible examples of algebras of dihedral, semidihedral and quaternion types up to Morita-equivalence.

In particular it was shown that algebras of the quaternion type in the classification have at most 3 simple modules.

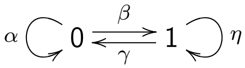
List of an algebras of the quaternion type with two simple modules

$Q(2\mathcal{A})_1(k, c),\ k \geqslant 2,\ c \in\ K$



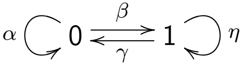
$$\gamma\beta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha,\ \ \beta\gamma\beta = (\alpha\beta\gamma)^{k-1}\alpha\beta,$$
$$\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k,\ \ \alpha^2\beta = 0$$

$Q(2\mathcal{B})_1(k, s, a, c),\ k \geqslant 1,\ s \geqslant 3,\ a \in K^*,\ c \in K$



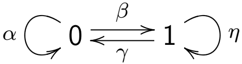
$$\eta\beta = \beta\alpha(\gamma\beta\alpha)^{k-1},\ \ \gamma\eta = \alpha\gamma(\beta\alpha\gamma)^{k-1},\ \ \beta\gamma = \eta^{s-1},$$
$$\alpha^2 = a\gamma\beta(\alpha\gamma\beta)^{k-1} + c(\alpha\gamma\beta)^k,\ \ \beta\alpha^2 = 0,\ \ \alpha^2\gamma = 0$$

$Q(2\mathcal{B})_2(s, a, c; p(x)), s \geqslant 3, a \in K^*, c \in K, p(x) \in K[x], p(0) = 1$



$$\alpha\beta = \beta\eta,\ \ \eta\gamma = \gamma\alpha\ \ \beta\gamma = \alpha^2 p(\alpha),$$
$$\gamma\beta = \eta^2 p(\eta) + a\eta^{s-1} + c\eta^s,\ \ \alpha^{s+1} = 0,$$
$$\eta^{s+1}\gamma = 0,\ \ \gamma\alpha^{s-1} = 0,\ \ \alpha^{s-1}\beta = 0$$

$Q(2\mathcal{B})_3^t(a, c, d)), t \geqslant 3,\ a \in K^*, c, d \in K,\ p(x) \in K[x],\ p(0) = 1$



$$\alpha\beta = \beta\eta,\ \ \eta\gamma = \gamma\alpha\ \ \beta\gamma = \alpha^2 + c\alpha^3,$$
$$\gamma\beta = \alpha\eta^{t-1} + d\eta^t,\ \ \alpha^4 = 0,\ \ \eta^{s+1} = 0,$$
$$\gamma\alpha^2 = 0,\ \ \alpha^2\beta = 0$$

Further questions

- **Are the algebras from the list pairwise Morita-inequivalent or not?**

- **Are the algebras from the list pairwise derived-inequivalent or not?**

Theorem (Holm)

Let Λ be an algebra of quaternion type with two simple modules. Then there exist $k \geqslant 1$, $s \geqslant 3$, $a \in K^*$ and $c \in K$ such that Λ is derived equivalent to $Q(2\mathcal{B})_1(k, s, a, c)$.

- **Are the algebras from the family $Q(2\mathcal{B})_1(k, s, a, c)$ derived-inequivalent or not?**

Hochschild cohomology could give some information of this kind

HOCHSCHILD COHOMOLOGY

Definition

Let R be a finite-dimensional algebra over a field K, let $R^{\mathrm{e}} = R \otimes_K R^{\mathrm{op}}$ be its enveloping algebra, then

$$\mathrm{HH}^*(R) = \bigoplus_{n \geqslant 0} \mathrm{HH}^n(R) = \bigoplus_{n \geqslant 0} \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R)$$

is named the Hochschild cohomology algebra of an algebra R.

Theorem

Let R_1 and R_2 be two finite-dimensional K-algebras, such that

$$D^b(R_1) \sim D^b(R_2)$$

as triangulated categories, then

$$\mathrm{HH}^*(R_1) \simeq \mathrm{HH}^*(R_2)$$

as graded algebras.

Corollary

The graded algebra $\mathrm{HH}^*(R)$ *is Morita-invariant.*

Corollary

Dimensions of the cohomology groups $\dim \mathrm{HH}^n(R)$ *for all* $n \geqslant 0$ *are Derived- and Morita-invariants.*

Hochschild cohomology of cyclic block and of algebras of Dihedral and Semidihedral types

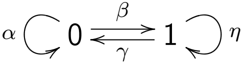
• **S. Siegel, S. Witherspoon** (2000) calculated the multiplicative structure of HH^* in the case of cyclic block (finite type)
• **Th. Holm** (2002) calculated the additive structure of HH^* of group blocks with one and three vertices and for one serie of algebras of semidihedral type with two vertices
• **A. Generalov** (2004, 2010) calculated the multiplicative structure of HH^* of the local algebras of dihedral type and algebras of dihedral type from the family $D(3\mathcal{K})$
• **A. Generalov** (2009-2010) calculated multiplicative structure of HH^* in the case of local and group algebras of semidihedral type

Hochschild cohomology of algebras of quaternion type

• **A. Generalov** (2006) calculated Hochschild cohomology algebra of local algebras of quaternion type
• **K. Erdmann, A. Skowronski** (2006) constructed bimodule resolutions of representatives of classes of Morita-equivalence for all algebras of generalised quaternion type except the case of small parameters
• **A. Generalov, S. Ivanov, A.I.** (2007) calculated Hochschild cohomology algebra of algebras from the family $Q(2\mathcal{B})_1(k, s, a, c)$ over a field of characteristic 2
• **A. Generalov** (2008) calculated Hochschild cohomology algebra of algebras from the family $Q(2\mathcal{B})_1(k, s, a, c)$ in the case of small parameters
Remark
In the paper of K. Erdmann and A. Skowronski (2006) mentioned above in describtion of the resolution of the algebras of the family under cosideration: $Q(2\mathcal{B})_1$, in the case when characteristic of the base field equals 2, there is an inacuracy, which makes the complex be not exact.

Definition of algebras from the family $Q(2\mathcal{B})_1(k, s, a, c)$

Algebras from the family $Q(2\mathcal{B})_1$ (over an algebraically closed field K of any characteristic) are described in terms of a quiver and relations:



$$\eta\beta = \beta\alpha(\gamma\beta\alpha)^{k-1},\ \ \gamma\eta = \alpha\gamma(\beta\alpha\gamma)^{k-1},\ \ \beta\gamma = \eta^{s-1},$$
$$\alpha^2 = a\gamma\beta(\alpha\gamma\beta)^{k-1} + c(\alpha\gamma\beta)^k,\ \ \beta\alpha^2 = 0,\ \ \alpha^2\gamma = 0,$$

where $k, s \in \mathbb{N},\ s \geqslant 3,\ a, c \in K,\ a \neq 0$ (we write the compositon of paths from right to left).

BIMODULE PROJECTIVE RESOLUTION

Let us denote by e_i, $i = 0, 1$, full set of orthogonal primitive idempotents of an algebra $R = Q(2\mathcal{B})_1(k, s, 1, c)$, corresponding to vertices of the quiver $Q(2\mathcal{B})$. In this notation

$$P_{ij} = Re_i \otimes e_j R,\ i, j \in \{0, 1\},$$

form a full set of representstives of a principle indecomposable left R^{e}-modules. Let

$Q_0 := Q_3 := P_{00} \oplus P_{11}$,
$Q_1 := Q_2 := P_{00} \oplus P_{10} \oplus P_{01} \oplus P_{11}$ и
$Q_{n+4} := Q_n$ for $n \geqslant 0$.

For any $i \in \mathbb{N}_0$ let us concider a collection of homomorphisms $d_i \in \mathrm{Hom}(Q_{i+1}, Q_i)$:

$$R \xleftarrow{\mu} Q_0 \xleftarrow{d_0} Q_1 \xleftarrow{d_1} Q_2 \xleftarrow{d_2} Q_3 \xleftarrow{d_3} Q_0 \xleftarrow{d_0} \dots$$

defined by the following matrices:

$$d_0 = \begin{pmatrix} \alpha \otimes e_0 - e_0 \otimes \alpha & \beta \otimes e_0 & e_0 \otimes \gamma & 0 \\ 0 & -e_1 \otimes \beta & -\gamma \otimes e_1 & \eta \otimes e_1 - e_1 \otimes \eta \end{pmatrix}$$

$$d_1 = \begin{pmatrix} \star & \sum_{i=1}^{k} \beta a^{i-1} \otimes g^{k-i} & \sum_{i=1}^{k} a^{i-1} \otimes \gamma b^{k-i} & 0 \\ \star & \sum_{i=1}^{k} b^{i-1} \otimes \alpha g^{k-i} - \eta \otimes e_0 & \sum_{i=1}^{k-1} \alpha\gamma b^{i-1} \otimes \alpha\gamma b^{k-i-1} & -e_1 \otimes \gamma \\ \star & -\sum_{i=1}^{k-1} \beta\alpha g^{i-1} \otimes \beta\alpha g^{k-i-1} & e_0 \otimes \eta - \sum_{i=1}^{k} \alpha g^{i-1} \otimes b^{k-i} & \beta \otimes e_1 \\ c\gamma \otimes \beta & -e_1 \otimes \beta & -\gamma \otimes e_1 & \sum_{i=1}^{s-1} \eta^{i-1} \otimes \eta^{s-i-1} \end{pmatrix},$$

where

$$(d_1)_{11} = \alpha \otimes e_0 + e_0 \otimes \alpha - \sum_{i=1}^{k-1} \gamma\beta a^{i-1} \otimes \gamma\beta a^{k-i-1} + c\sum_{i=1}^{k} \gamma\beta a^{i-1} \otimes g^{k-i} + c\sum_{i=1}^{k} a^{i-1} \otimes \gamma\beta a^{k-i},$$

$$(d_1)_{21} = -\sum_{i=1}^{k} \gamma b^{i-1} \otimes a^{k-i} + c\sum_{i=1}^{k} \gamma b^{i-1} \otimes \alpha g^{k-i} + c\sum_{i=1}^{k-1} \alpha\gamma b^{i-1} \otimes a^{k-i},$$

$$(d_1)_{31} = \sum_{i=1}^{k} g^{i-1} \otimes \beta a^{k-i} + c\sum_{i=1}^{k} \alpha g^{i-1} \otimes \beta a^{k-i} + c\sum_{i=1}^{k-1} g^{i} \otimes \beta\alpha g^{k-i-1};$$

$$d_2 = \begin{pmatrix} \alpha\otimes e_0 - e_0\otimes\alpha & 0 \\ -\gamma\otimes e_0 + c\gamma\otimes\alpha & -e_1\otimes\gamma \\ e_0\otimes\beta + c\alpha\otimes\beta & \beta\otimes e_1 \\ 0 & \eta\otimes e_1 - e_1\otimes\eta \end{pmatrix};$$

d_3 is a 2×2-matrix, which has

$$(d_3)_{11} = \sum_{i=1}^{k} a^{i-1}\otimes a^{k-i+1} + \sum_{i=1}^{k}\gamma\beta a^{i-1}\otimes\alpha g^{k-i} + \sum_{i=1}^{k} g^i\otimes g^{k-i} + \sum_{i=1}^{k}\alpha g^{i-1}\otimes\gamma\beta a^{k-i} + c\sum_{i=1}^{k-1} g^i\otimes\alpha g^{k-i}$$

$$+c\sum_{i=1}^{k-1}\alpha g^i\otimes a^{k-i} + c\sum_{i=1}^{3}\alpha^i\otimes\alpha^{4-i} + c\gamma\beta a^{k-1}\otimes\gamma\beta a^{k-1} + c\alpha g^{k-1}\otimes\alpha^3 + c\alpha^3\otimes\alpha g^{k-1},$$

$$(d_3)_{12} = \sum_{i=1}^{k}\beta\alpha g^{i-1}\otimes\gamma b^{k-i} + \sum_{i=1}^{k}\beta a^{i-1}\otimes\alpha\gamma b^{k-i} + c\sum_{i=1}^{k}\beta\alpha g^{i-1}\otimes\alpha\gamma b^{k-i} + c\beta\alpha g^{k-1}\otimes\alpha\gamma b^{k-1},$$

$$(d_3)_{21} = -\sum_{i=1}^{k}\alpha\gamma b^{i-1}\otimes\beta a^{k-i} - \sum_{i=1}^{k}\gamma b^{i-1}\otimes\beta\alpha g^{k-i} + c\alpha\gamma b^{k-1}\otimes\beta\alpha g^{k-1},$$

$$(d_3)_{22} = -\sum_{i=1}^{k} b^{i-1}\otimes b^{k-i+1} - \sum_{i=1}^{s}\eta^i\otimes\eta^{s-i} + c\sum_{i=1}^{s}\eta^i\otimes\eta^{s-i+1};$$

and for all $n\in\mathbb{N}\cup\{0\}$

$$d_{n+4} = d_n.$$

Hochschild cohomology of $Q(2\mathcal{B})_1(k,s,a,c)$ over a field of caracteristic 2

Let us concider a set

$$\mathcal{Z}_1 = \{p_1, p_2, p_3, p_4, u_1, \widetilde{u}_2, v_1, v_3, w_1, t\} \tag{0.1}$$

and on the algebra $\mathcal{K}[\mathcal{Z}_1]$ we define grading, such that

$$\deg p_i = 0\ (i = 1,2,3,4), \deg u_1 = \deg \widetilde{u}_2 = 1,$$
$$\deg v_1 = \deg v_3 = 2,\ \ \deg w_1 = 3,\ \ \deg t = 4.$$

Let us define an algebra $\mathcal{R}_1 = K[\mathcal{Z}_1]/J_1$, where the ideal J_1 of the algebra $K[\mathcal{Z}_1]$ is generated by following elements

$$p_i p_j \text{ for } 1 \leqslant i < j \leqslant 4; \;\; p_1^k + p_2^s, \; p_3^2, \; p_4^2; \tag{0.2}$$

$$p_1 u_1 + p_2^{s-1}\widetilde{u}_2, \; p_2 u_1 + p_1^{k-1}\widetilde{u}_2, p_1^k\widetilde{u}_2, \; p_4\widetilde{u}_2, \; p_4 u_1 + p_3\widetilde{u}_2; \tag{0.3}$$

$$p_3 v_1, \; p_4 v_1, \; p_3 v_3, \; p_4 v_3, p_1 v_1 + p_2 v_3, \; p_1 v_1 + p_3 u_1 \widetilde{u}_2; p_3 u_1^2 + p_2^{s-1} v_1,$$

$$p_2 v_3 + p_4 u_1^2, p_2^{s-1} v_1 + p_1^{k-1} v_3, \widetilde{u}_2^2 + \mathrm{th}_{k+s}\, p_1^{k-1} v_3,$$

$$p_3 w_1, \; p_4 w_1, p_1 w_1 + \widetilde{u}_2 v_3, \; p_2 w_1 + \widetilde{u}_2 v_1, \; u_1 v_1 + p_1^{k-2}\widetilde{u}_2 v_3, u_1 v_3 + p_2^{s-1} w_1,$$

$$u_1^4, \; u_1^3\widetilde{u}_2 + p_1^k t, v_1^2 + p_2^2 t, \; v_3^2 + p_1^2 t, \; v_1 v_3, u_1 w_1 + p_3 t, \; \widetilde{u}_2 w_1 + (p_4 + \theta_{k+s} p_1^k) t,$$

$$v_1 w_1 + p_2 \widetilde{u}_2 t, \; v_3 w_1 + p_1 \widetilde{u}_2 t,$$

$$w_1^2 + \theta_{k+s} p_1^{k-1} v_3 t,$$

where for n even we define a constant from the base field K

$$\theta_n = \sum_{i=1}^{n-1} i = \begin{cases} 1, & \text{if } n \equiv 2 \pmod 4, \\ 0, & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{0.4}$$

The ideal J_1 of algebra $\mathcal{R}_1$ is homogeneous and therefore the algebra $\mathcal{R}_1$ inherits the grading of the algebra $K[\mathcal{Z}_1]$.

Let us concider a set

$$\mathcal{Z}_2 = \{p_1, p_2, p_3, p_4, u'_2, u_3, u_4, u_5, v_0, v_1, v_2, v_3, w_0, w_1, w_4, t\} \tag{0.5}$$

and on the algebra $\mathcal{K}[\mathcal{Z}_2]$ we define grading, such that

$$\deg p_i = 0 \ (i = 1, 2, 3, 4), \ \deg u'_2 = \deg u_3 = \deg u_4 = \deg u_5 = 1,$$
$$\deg v_j = 2 \ (j = 0, 1, 2, 3), \ \deg w_0 = \deg w_1 = \deg w_4 = 3, \ \deg t = 4.$$

Let us define an algebra $\mathcal{R}_2 = K[\mathcal{Z}_2]/J_2$, where the ideal J_2 of the algebra $K[\mathcal{Z}_2]$ is generated by elements from (0.2) and in addition by following elements

$$p_1 u_2', \; p_3 u_2', \; p_4 u_2', \; p_2^{s-1} u_2', p_2 u_3, \; p_3 u_3, \; p_4 u_3, \; p_1^{k-1} u_3,$$
$$p_1 u_4, \; p_2 u_4, \; p_3 u_4, \; p_4 u_4, \; p_1 u_5, \; p_2 u_5, \; p_3 u_5, \; p_4 u_5;$$
$$p_1 v_0, \; p_2 v_0, \; p_3 v_0, \; p_4 v_0, p_1 v_1, \; p_3 v_1, \; p_4 v_1, \; p_2^{s-1} v_1,$$
$$p_1 v_2, \; p_2 v_2, \; p_3 v_2, \; p_4 v_2, p_2 v_3, \; p_3 v_3, \; p_4 v_3, \; p_1^{k-1} v_3,$$
$$(u_2')^2, \; u_3^2, \; u_4^2, \; u_5^2, u_4 u_5, \; u_4 u_3, \; u_4 u_2', \; u_5 u_3, \; u_5 u_2', \; u_3 u_2';$$
$$p_1 w_0, p_2 w_0, p_3 w_1, p_4 w_1, p_1 w_4, p_2 w_4, p_4 w_4, p_4 w_0 + p_3 w_4, cp_3 w_4 + p_3 w_0,$$
$$p_3 w_0 + cp_1^k w_1, cp_1^k w_1 + u_4 v_0, \; u_4 v_0 + cu_5 v_0, u_5 v_0 + u_4 v_2, u_2' v_0, \; u_3 v_0, \; u_2' v_2,$$
$$u_3 v_2, \; u_5 v_2, u_3 v_1, \; u_4 v_1, \; u_5 v_1, \; u_2' v_3, \; u_4 v_3, \; u_5 v_3, u_2' v_1 + p_2^2 w_1, \; u_3 v_3 + p_1^2 w_1,$$
$$v_1 v_2, v_3 v_2, v_3 v_1, v_3 v_0, v_1 v_0, v_0^2 + cv_0 v_2, \; v_0 v_2 + p_1^k t, \; v_1^2 + (p_2^2 + cp_2^3)t, v_3^2 + p_1^2 t,$$
$$v_2^2, \; u_2' w_0, \; u_3 w_0, u_4 w_0, u_5 w_0, u_2' w_1, u_3 w_1, \; u_4 w_1, \; u_5 w_1, u_2' w_4, \; u_3 w_4, u_4 w_4, u_5 w_4,$$
$$v_0 w_0, \; v_1 w_0, \; v_2 w_0, \; v_3 w_0, \; v_0 w_4, \; v_1 w_4, \; v_2 w_4, \; v_3 w_4,$$
$$v_0 w_1 + u_4 t, \; v_1 w_1 + (1 + cp_2) u_2' t, \; v_2 w_1 + u_5 t, \; v_3 w_1 + u_3 t,$$
$$w_0^2, \; w_1^2, \; w_4^2, \; w_0 w_1, \; w_0 w_4, \; w_1 w_4.$$

The algebra inherits $\mathcal{R}_2$ natural grading from algebra $K[\mathcal{J}_2]$.

Theorem

Let K be algebraically closed field of characteristic 2, and let $R = Q(2\mathcal{B})_1(k, s, 1, c)$, where $k, s \geqslant 3$.

$1)$ *If k and s are odd and $c = 0$, then $\mathrm{HH}^*(R) \simeq \mathcal{R}_1$ as graded K-algebras;*

$2)$ *If k and s are odd and $c \neq 0$, then $\mathrm{HH}^*(R) \simeq \mathcal{R}_2$ as graded K-algebras;*

$3)$ *If k even and s is odd, then $\mathrm{HH}^*(R) \simeq \mathcal{R}_3$ as graded K-algebras;*

$4)$ *If k odd and s is evens, then $\mathrm{HH}^*(R) \simeq \mathcal{R}_4$ as graded K-algebras;*

$5)$ *If k and s are even, then $\mathrm{HH}^*(R) \simeq \mathcal{R}_5$ as graded K-algebras.*

Proposition

Let $R = Q(2\mathcal{B})_1(k, s, 1, c)$ *and* k, s *be odd. Then:*

a) $\dim_K \mathrm{HH}^0(R) = k + s + 2$;

b) *for* $t \geqslant 0$

$$\dim_K \mathrm{HH}^{4t+1}(R) = \dim_K \mathrm{HH}^{4t+2}(R) = \begin{cases} k + s + 2, & \text{if } c = 0, \\ k + s, & \text{if } c \neq 0; \end{cases}$$

c) *for* $t \geqslant 0$

$$\dim_K \mathrm{HH}^{4t+3}(R) = \dim_K \mathrm{HH}^{4t+4}(R) = k + s + 2.$$

From the proposition, b) immediately follows next statement, improving classification of tame blocks.

Corollary

Let k and s be odd, $k \geqslant 2$, $s \geqslant 3$. Then the algebra $Q(2\mathcal{B})_1(k, s, 1, c)$, where $c \neq 0$, is not derived-equivalent (and, in particular, it is not Morita-equivalent) to the algebra $Q(2\mathcal{B})_1(k, s, 1, 0)$.

Proposition

Let k, s be even and such that $\theta_k = \theta_s = 1$. Then

$$\dim_K(\mathrm{HH}^1 \cdot \mathrm{HH}^1) = \begin{cases} 2, & \text{if } c = 0, \\ 3, & \text{if } c \neq 0; \end{cases}$$

Proposition

a) *Let k, s be even and such that $\theta_k = \theta_s = 1$. Then $Q(2\mathcal{B})_1(k, s, 1, c)$, where $c \neq 0$ is not derived equivalent to $Q(2\mathcal{B})_1(k, s, 1, 0)$.*

b) *Let $k + s$ be odd $Q(2\mathcal{B})_1(k, s, 1, c)$. Then $Q(2\mathcal{B})_1(k, s, 1, c)$, where $c \neq 0$ is not derived equivalent to $Q(2\mathcal{B})_1(k, s, 1, 0)$.*

Theorem

Let K be algebraically closed field of caracteristic 3, and let $R = Q(2\mathcal{B})_1(k, s, 1, 0)$, where $s \geqslant 3$ and $k \geqslant 2$. Then $\forall\, m \in \mathbb{N} \cup \{0\}$

1) $\dim_K \mathrm{HH}^0(R) = k + s + 2$;

2) $$\dim_K \mathrm{HH}^{4m+1}(R) = \begin{cases} k+s-1, & \text{if } s \neq 0, k \neq 0 \\ k+s, & \text{if } s = 0, k \neq 0 \text{ или } s \neq 0, k = 0 \\ k+s+1, & \text{if } s = 0, k = 0; \end{cases}$$

3) $$\dim_K \mathrm{HH}^{4m+2}(R) = \begin{cases} k+s-1, & \text{if } s \neq 0, k \neq 0 \\ k+s, & \text{if } s = 0, k \neq 0 \text{ или } s \neq 0, k = 0 \\ k+s+1, & \text{if } s = 0, k = 0; \end{cases}$$

4) $$\dim_K \mathrm{HH}^{4m+3}(R) = \begin{cases} k+s, & \text{if } s \neq 0, k \neq 0 \\ k+s+1, & \text{if } s = 0, k \neq 0 \text{ или } s \neq 0, k = 0 \\ k+s+2, & \text{if } s = 0, k = 0; \end{cases}$$

5) $$\dim_K \mathrm{HH}^{4m+4}(R) = \begin{cases} k+s, & \text{if } s \neq 0, k \neq 0 \\ k+s+1, & \text{if } s = 0, k \neq 0 \text{ или } s \neq 0, k = 0 \\ k+s+2, & \text{if } s = 0, k = 0. \end{cases}$$

Theorem

Let K be an algebraically closed field of characteristic not equal to 2 *and* 3, *and let* $R = Q(2\mathcal{B})_1(k,s,1,0)$, *where* $s \geqslant 3$ *and* $k \geqslant 2$. *Then*
$\forall\, m \in \mathbb{N} \cup \{0\}$ 1) $\dim_K \mathrm{HH}^0(R) = k+s+2$;

$$2)\ \dim_K \mathrm{HH}^{4m+1}(R) = \begin{cases} k+s-1, & \text{if } 4ks-3k-3s \neq 0 \\ k+s, & \text{if } 4ks-3k-3s = 0; \end{cases}$$

$$3)\ \dim_K \mathrm{HH}^{4m+2}(R) = \begin{cases} k+s-1, & \text{if } 4ks-3k-3s \neq 0 \\ k+s, & \text{if } 4ks-3k-3s = 0; \end{cases}$$

$$4)\ \dim_K \mathrm{HH}^{4m+3}(R) = \begin{cases} k+s, & \text{if } s \neq 0, k \neq 0 \\ k+s+1, & \text{if } s = 0, k \neq 0 \text{ или } s \neq 0, k = 0 \\ k+s+2, & \text{if } s = 0, k = 0; \end{cases}$$

$$5)\ \dim_K \mathrm{HH}^{4m+4}(R) = \begin{cases} k+s, & \text{if } s \neq 0, k \neq 0 \\ k+s+1, & \text{if } s = 0, k \neq 0 \text{ или } s \neq 0, k = 0 \\ k+s+2, & \text{if } s = 0, k = 0. \end{cases}$$

Remark

Using results of Th. Holm (1997), we can apply obtained answers to describe Hochschild cohomology groups of algebras from the family $Q(2\mathcal{A})^k(c)$ from Erdmann's list.

Remark

The proposition and theorems partialy complete analogous result due to Th. Holm (2002), where he calculates additive structure of Hochschild cohomology algebra of algebras of dihedral, semidihedral and quaternion types with one or three simple modules and for several families of semidihedral type with two simple modules over a field of caracteristic 2.

Hochschild cohomology of $Q(2\mathcal{B})_1(k, s, a, c)$ over a field of characteristic 3

To describe Hochschild cohomology algebra $\mathrm{HH}^*(Q(2\mathcal{B})_1)$ of algebras $Q(2\mathcal{B})_1$ over a field of characteristic 3 we will consider several graded algebras.

Let

$$\mathcal{X}_1 = \{p_1, p_2, p_3, u_1, u_2, v_0, v_1, v_3, w_0, w_1, t\}. \tag{0.6}$$

On the algebra $K[\mathcal{X}_1]$ define grading such that

$$\begin{gathered} \deg p_i = 0\ (i = 1, 2, 3),\ \ \deg u_j = 1\ (j = 1, 2), \\ \deg v_l = 2\ (l = 0, 1, 3),\ \ \deg w_q = 3\ (q = 1, 2),\ \ \deg t = 4. \end{gathered} \tag{0.7}$$

Define graded K-algebra $\mathcal{A}_1 = K[\mathcal{X}_1]/I_1$, where the ideal I_1 of algebra $K[\mathcal{X}_1]$ is generated by following homogeneous elements

$$p_i p_j \text{ для } 1 \leqslant i < j \leqslant 3;\ p_1^{k+1},\ p_2^{s+1},\ p_3^2; \tag{0.8}$$

$$p_2 u_1,\ p_1^k u_1, \tag{0.9}$$

$$p_3 u_2,\ p_2^s u_2, p_1 u_1 - p_1 u_2;$$

$$p_1 v_0,\ p_2 v_0,\ p_1 v_1,\ p_3 v_1,\ p_2 v_3,\ p_3 v_3; \tag{0.10}$$

$$p_3 v_0 + p_2^{s-1} v_1 - p_1^{k-1} v_3;\ u_1^2,\ u_2^2,\ u_1 u_2;$$

$$p_2 w_0,\ p_3 w_0,\ p_3 u_1 v_0 - p_1^{k-1} u_1 v_3,\ p_1 w_0 + u_1 v_3, \tag{0.11}$$

$$p_3 w_1,\ p_1 w_1 + u_1 v_3,\ p_2 w_1 - u_2 v_1,\ u_1 v_3 + u_2 v_3, u_1 v_1,\ u_2 v_0,$$

$$v_0^2 + p_3 t,\ v_3^2 + p_1^2 t,\ v_1^2 + p_2^2 t, v_0 v_1,\ v_0 v_3,\ v_1 v_3; \tag{0.12}$$

$$u_1 w_0,\ u_2 w_0,\ u_1 w_1,\ u_2 w_1;$$

$$v_0 w_0,\ v_1 w_0,\ v_3 w_0 - p_1 u_1 t, \tag{0.13}$$

$$v_0 w_1,\ v_1 w_1 + p_2 u_2 t,\ v_3 w_1 + p_1 u_1 t,$$

$$w_0^2,\ w_1^2,\ w_0 w_1.$$

On algebra $\mathcal{A}_1$ we define a grading induced by the grading from $K[\mathcal{X}_1]$.

Now let

$$\mathcal{X}_2 = \Big(\mathcal{X}_1 \setminus \{u_2, w_1\}\Big) \cup \{u_2', w_2\}, \tag{0.14}$$

and on algebra $\mathcal{K}[\mathcal{X}_2]$ we define a grading coinsiding with the grading (0.7) for elements from $\mathcal{X}_1 \setminus \{u_2, w_1\}$ and such that $\deg u_2' = 1$, $\deg w_2 = 3$.

Let $\mathcal{A}_2 = K[\mathcal{X}_2]/I_2$, where the ideal I_2 of algebra $K[\mathcal{X}_2]$ is generated by the elements (0.8),(0.9),(0.10), (0.11),(0.12),(0.13) and by elements

$$p_1 u_2', \; p_3 u_2', \; p_2^{s-1} u_2'; \tag{0.15}$$

$$p_2^{s-1} v_1, \; p_1^{k-1} v_3 - p_3 v_0, \; u_1 u_2', \; (u_2')^2, \; u_1^2;$$

$$p_1 w_2, \; p_3 w_2, \; p_2 w_2 - u_2' v_1, \; u_2' v_0, \; u_2' v_3, \tag{0.16}$$

$$p_2^{s-2} u_2' v_1 + p_1^{k-1} u_1 v_3, \; u_1 v_1;$$

$$p_2^{s} t; \tag{0.17}$$

$$u_2' w_i, \; u_1 w_i, \; \text{для} \;\; i \in \{0, 2\}; \tag{0.18}$$

$$v_0 w_2, \; v_3 w_2, \; v_1 w_2 + p_2 u_2' t; \tag{0.19}$$

$$w_0^2, \; w_2^2, \; w_0 w_2. \tag{0.20}$$

Since I_2 is homogeneous, algebra $\mathcal{A}_2$ inherits grading from algebra $K[\mathcal{X}_2]$.

Let

$$\mathcal{X}_3 = \Big(\mathcal{X}_1 \setminus \{u_1, u_2, w_1, w_0\}\Big) \cup \{u_{12}, w_{01}, w_3\}, \qquad (0.21)$$

and a grading on the algebra $\mathcal{K}[\mathcal{X}_3]$ coinsiding with (0.7) for elements from $\mathcal{X}_1 \setminus \{u_1, u_2, w_1, w_0\}$ and such that $\deg u_{12} = 1$, $\deg w_{01} = \deg w_3 = 3$. Let us define an algebra $\mathcal{A}_3 = K[\mathcal{X}_3]/I_3$, where the ideal I_3 of the algebra $K[\mathcal{X}_3]$ is generated dy the elements from (0.8), (0.10), (0.12), and also by following elements

$$p_1^k u_{12}; \qquad (0.22)$$

$$u_{12}^2;\ v_3 p_1^{k-1},\ v_0 p_3 - v_1 p_2^{s-1};$$

$$p_1 w_{01},\ p_3 w_{01},\ p_2 w_{01} + u_{12} v_1,\ p_4 w_{01} - u_{12} v_1 p_2^{s-1};$$

$$u_{12} v_3;\ p_1^{k-1} w_3 + u_{12} v_1 p_2^{s-1},$$

$$p_2 w_3,\ p_3 w_3; \qquad (0.23)$$

$$p_1^k t; \qquad (0.24)$$

$$u_{12} w_{01},\ u_{12} w_3;$$

$$v_0 w_3,\ v_1 w_3, \qquad (0.25)$$

$$v_3 w_3 + p_1^2 u_{12} t;\ v_0 w_{01},\ v_3 w_{01},\ v_1 w_{01} - p_2 u_{12} t;$$

$$w_{01}^2,\ w_3^2,\ w_{01} w_3.$$

The agebra $\mathcal{A}_3$ inherits grading from $K[\mathcal{X}_3]$.

Let us consider a set

$$\mathcal{X}_4 = \{p_1, p_2, p_3, u_2', u_3, u_4, v_0, v_1, v_3, w_2, w_3, w_4, t\} \qquad (0.26)$$

and define a grading on the algebra $\mathcal{K}[\mathcal{X}_4]$, such that

$$\begin{aligned}
&\deg p_i = 0\ (i = 1, 2, 3), \deg u_2' = \deg u_3 = \deg u_4 = 1, \\
&\deg v_0 = \deg v_1 = \deg v_3 = 2, \ \deg w_2 = \deg w_3 = \deg w_4 = 3, \ \deg t = 4.
\end{aligned}$$

Let $\mathcal{A}_4 = K[\mathcal{X}_4]/I_4$, where the ideal I_4 of the algebra $K[\mathcal{X}_4]$ is generated by the elements from (0.8), (0.15), (0.10), (0.16), (0.23),(0.12), (0.17), (0.24), (0.19), (0.25) and in addition by elements

$$\begin{gathered}
p_2 u_3, \ p_3 u_3, \ p_1^{k-1} u_3, \ p_i u_4 \ \text{для} \ 1 \leqslant i \leqslant 3; \\
(u_2')^2, \ u_3^2, \ u_4^2, \ u_2' u_3, \ u_2' u_4, \ u_3 u_4; \\
v_0 p_3, \ v_3 p_1^{k-1}, \ v_1 p_2^{s-1}, \\
u_3 v_0, \ u_3 v_1, \ u_3 v_3 - p_1 w_3, \ u_4 v_1, \ u_4 v_3, \ u_4 v_0 - p_3 w_4, \\
p_1 w_4, \ p_2 w_4, \ p_3 w_4 + p_2^{s-1} w_2, \ w_3 p_1^{k-1} + w_2 p_2^{s-1}; \\
u_2' w_i, \ u_3 w_i, \ u_4 w_i, \ \text{где} \ 2 \leqslant i \leqslant 4; \\
v_0 w_4, \ v_1 w_4, \ v_3 w_4, \ v_3 w_3 - p_1 u_3 t; \\
w_2^2, \ w_3^2, \ w_4^2, \ w_2 w_3, \ w_2 w_4, \ w_3 w_4.
\end{gathered} \qquad (0.27)$$

The algebra $\mathcal{A}_4$ inherits grading from the algebra $K[\mathcal{X}_4]$.

Theorem

Let K be algebraically closed field of characteristic 3 and let $R = Q(2\mathcal{B})_1(k, s, 1, 0)$, where $k \geqslant 2$.
1) *If* $k, s \equiv 0\,(3)$, *then Hochschild cohomology algebra* $\mathrm{HH}^*(R)$ *is isomorphic to the algebra* $\mathcal{A}_1$ *as graded* K*-algebra.*
2) *If* $k \equiv 0\,(3)$ *and* $s \not\equiv 0\,(3)$, *then* $\mathrm{HH}^*(R) \simeq \mathcal{A}_2$ *as graded* K*-algebras.*
3) *If* $k \not\equiv 0\,(3)$, *and* $s \equiv 0\,(3)$, *then* $\mathrm{HH}^*(R) \simeq \mathcal{A}_3$ *as graded* K*-algebras.*
4) *If* $k, s \not\equiv 0\,(3)$, *then* $\mathrm{HH}^*(R) \simeq \mathcal{A}_4$ *as graded* K*-algebras.*

Remark

As above we can use obtained results to describe Hochschild cohomology algebras for algebras from the family $Q(2\mathcal{A})^k(c)$.

An algorithm

• Construct a bimodule projective resolution $(P_\bullet \to R, \{d_i\}_{i \geqslant 0})$ of the bimodule R

• Apply the functor $\mathrm{Hom}_{R^{\mathrm{e}}}(\text{ - }, R)$ to the resolution and get a complex

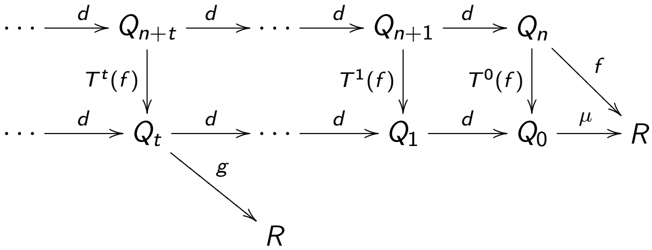
$$(\mathrm{Hom}_{R^{\mathrm{e}}}(Q_\bullet,\ R),\ \{\delta^i = Hom_{R^{\mathrm{e}}}(d_i, R)\}_{i \geqslant 0})$$

• Find bases of the groups $H^n(\mathrm{Hom}_{R^{\mathrm{e}}}(Q_\bullet, R))$

• Use the fact, that $\cup$-product coincides with Yoneda-product on Ext-algebra

• Notice that for any cocycle $f \in \mathrm{Ker}(\delta_n)$, $f : Q_n \to R$ one can construct a chain map $\{T^i : Q_{n+i} \to Q_i\}_{i \geqslant 0}$

• To compute the product of cocycles $f \in \mathrm{Ker}(\delta^n)$ and $g \in \mathrm{Ker}(\delta^t)$ use the formula $cl(g) \cdot cl(f) = gT^t(f)$



• Compute translations $T^t(f)$ of cocycles and find relations defining the multiplicative structure

• Use Groebner bases to prove, that all relations are found