On Normalized Integral Table Algebras Generated by a Faithful Non-real Element of Degree 3

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Table algebras

Definition

Let $B = \{b_1 = 1, ..., b_k\}$ be a distinuished basis of an associative commutative complex algebra A. A pair (A, B) is called a table algebra if it satisfies the following conditions

1. $b_i b_j = \sum_{m=1}^{k} \lambda_{ijm} b_m$ with λ_{ijm} being non-negative reals;
2. there exists a table algebra automorphism $x \mapsto \bar{x}$ of A whose order divides two such that $\overline{B} = B$ ($^-$ defines a permutation on $[1, k]$ via $\overline{b_i} = b_{\bar{i}}$);
3. there exists a coefficient function $g : B \times B \to \mathbb{R}^+$ such that $\lambda_{ijm} = g(b_i, b_m)\lambda_{\bar{j}mi}$

An element b_i is called real if $i = \bar{i}$. For any $x = \sum_{i=1}^{k} x_i b_i$ we set $Irr(x) := \{b_i \in B \,|\, x_i \neq 0\}$.

Table subsets

Definition

A non-empty subset $T \leq B$ is called a table subset if $Irr(T\overline{T}) \subseteq T$. In this case a linear span $S := \langle T \rangle$ of T is a subalgebra of A. The pair (S, T) is called table subalgebra of (A, b).

Faithful elements

Since an intersection of table subsets is a table subset by itself, one can define a table subset generated by an element $b \in B$, notation B_b, as the intersection of all table subsets of B containing b. An element $b \in B$ with $B_b = B$ is called faithful.

Isomorphism between table algebras

Rescaling

Given a table algebra (A, B) one can replace its table basis $B = \{b_1, ..., b_k\}$ by $B' = \{\beta_1 b_1, ..., \beta_k b_k\}$ where β_i's are positive real numbers with $\beta_1 = 1$. A table algebra (A, B') is called a rescaling of (A, B).

Isomorphisms between TA

Two table algebras (A, B) and (A', B') are called isomorphic, notation $(A, B) \cong (A', B')$, if there exists an algebra isomorphism $f : A \to A'$ such that $f(B)$ is a rescaling of B'. In the case of $f(B) = B'$, the algebras are called exactly isomorphic, notation $(A, B) \cong_x (A', B')$.

Degree homomorphism

Theorem (Arad, Blau)

Let (A, B) be a table algebra. Then there exists a unique algebra homomorphism $a \mapsto |a|, a \in A$ onto $\mathbb{C}$ such that $|b| = |\bar{b}| > 0$ holds for all $b \in B$. The number $|b|$ is called the degree of b.

Normalized and standard TAs

An element $b_i \in B$ is called standard (normalized) if $\lambda_{i\bar{i}1} = |b_i|$ ($\lambda_{i\bar{i}1} = 1$). A table algebra is called standard (normalized) if all the elements of its table basis are standard (normalized). Notice that any table algebra may be rescaled to a standard or normalized one. If (A, B) is normalized, then $g(b_i, b_j) = 1$. For standard table algebras $g(b_i, b_j) = |b_i|/|b_j|$.

The order of a TA

Definiion

The number

$$o(B) := \sum_{i=1}^{k} \frac{|b_i|^2}{\lambda_{i\bar{i}1}}$$

does not depend on a rescaling of (A, B) and is called the order of (A, B). If (A, B) is standard, then $o(B) = \sum_{i=1}^{k} |b_i|$. If (A, B) is normalized, then $o(B) = \sum_{i=1}^{k} |b_i|^2$.

Definition

A table algebra is called integral if all its degrees and structure constants are non-negative integers.

Examples: character algebra of a finite group

Let G be a finite group and $Ch(G)$ denote the algebra of all complex valued class functions on G with pointwise mutliplication. This algebra has a natural basis $Irr(G)$ consisting of irreducible characters of G. The pair $(Ch(G), Irr(G))$ satisfies the axioms of a table algebra. In this case $\bar{\chi}, \chi \in Irr(G)$ is a complex conjugate character and the degree function of χ is a usual degree of an irreducible character - $\chi(1)$. The algebra $(Ch(G), Irr(G))$ is a normalized integral table algebra (NITA, for short).

Examples: the center of a finite group algebra

Let G be a finite group and $Z(\mathbb{C}[G])$ denote the center of a group algebra. $Z(\mathbb{C}[G])$ is a subalgebra of $\mathbb{C}[G]$. Let $C_1 = \{1\}, C_2, ..., C_k$ be a complete set of conjugacy classes of G. Denote $b_i := \sum_{g \in C_i} g$, $Cla(G) := \{b_1, ..., b_k\}$. Then $Z((\mathbb{C}[G]), Cla(G))$ satisfies the axioms of a table algebra with $\overline{b_i} = \sum_{g \in C_i} g^{-1}$ and degree function $|b_i| = |C_i|$. The algebra $Z((\mathbb{C}[G]), Cla(G))$ is a standard integral table algebra (SITA, for short).

Table algebras classification results

Minimal degree

A minimal degree $m(B)$ of an ITA (A, B) is $\min\{|b_i| \mid i > 1\}$. ITAs containing a faithful element of degree 2 with $m(B) = 2$ were classified by Blau.

Homogeneous ITAs

HITAs of degrees $1, 2, 3$ were completely classified in a series of papers by Arad, Blau, Fisman, Miloslavsky and Muzychuk.

Standard ITAs

SITAs containing a faithful non-real element of minimal degree 3 and 4 were classified in a series of papers by Arad, Arisha, Blau, Fisman and Muzychuk.

Normalized integral table algebras

Let (A, B) be a NITA containing a faithful element b of minimal degree m. If $m = 1$, then (A, B) is exactly isomorphic to the character algebra of a cyclic group.If $m = 2$, then the classification of such algebras follows from Blau's result. In this talk we present the results obtained for $m = 3$ under additional assumption that b is non-real.

Normalized integral table algebras

Theorem (Arad, Chen)

Let (A, B) be a NITA of minimal degree 3 containing a faithful element b_3 of minimal degree 3. Then $b_3\overline{b_3} = 1 + b_8$ where $b_8 \in B$ is real of degree 8 and one of the following holds.

1. $(A, B) \cong_x ((Ch(G), Irr(G)), G \cong PSL(2, 7);$
2. $b_3^2 = b_4 + b_5$ where $b_4, b_5 \in B$;
3. $b_3^2 = c_3 + b_6$ where $c_3, b_6 \in B$, $c_3 \neq b_3, \bar{b}_3$;
4. $b_3^2 = \bar{b}_3 + b_6, b_6 \in B$ is non-real;

Theorem (Arad, Xu)

The second case cannot occur.

The third case

Theorem (Arad, Cohen, Arisha)

Assume that

$$b_3^2 = c_3 + b_6, c_3 \neq b_3, \bar{b}_3.$$

Then $(b_3b_8, b_3b_8) = 3, 4$. If $(b_3b_8, b_3b_8) = 3$ and c_3 is real, then there exists a unique NITA of dimension 22. If c_3 is not real, then there exists a unique NITA of dimension 32 satisfying these conditions. Both NITAs are not induced from character tables of finite groups.

Problem

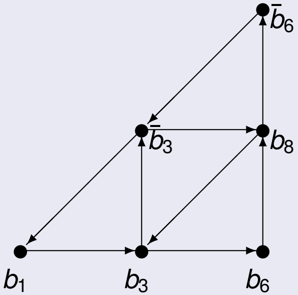
Classify the NITAs in the title with $(b_3b_8, b_3b_8) = 4$.

The fourth case

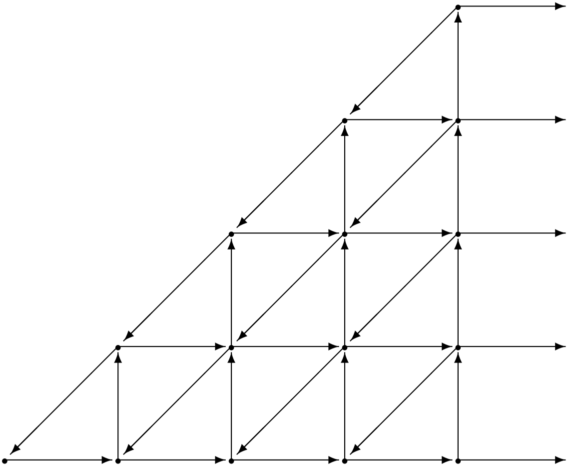
A representation graph

A representation graph of $b_i \in B$ is a weighted graph on B in which two vertices b_j and b_k are connected by an edge of weight λ_{ijk}.

A representation graph of b_3 at distance two



Graph C_n



Fourth case: $b_3^2 = \bar{b}_3 + b_6$

Definition

A NITA (A, B) in the title satisfies C_n -condition if the representation graph at distance n is isomorphic to C_n. We say that n is a stopping number for (A, B) if n is a maximal number for which (A, B) satisfies C_n-condition. In the case when (A, B) satisfies C_n-condition for each n, we say that its stopping number is ∞. In the latter case (A, B) is infinite dimensional algebra with $|B| = \aleph_0$,

Theorem (Arad, Cohen)

There exist only two algebras of fourth type with stopping number at most three, namely $(Ch(PSL(2,7)), Irr(PSL(2,7))$ and $(Ch(3 \cdot A_6), Irr(3 \cdot A_6))$.

Fourth case: $b_3^2 = \bar{b}_3 + b_6$

Theorem (Arad, Cohen)

There exists no NITA of fourth type with stopping number at least 43.

Theorem (Arad, Cohen, Muzychuk)

There exists a unique infinite dimensional NITA of fourth type with stopping number ∞. This is the NITA of polynomial characters of $SL_3(\mathbb{C})$.

Open Problem

Classify all NITAs of fourth type with stopping number in the range $[4, 42]$.