Dimensions of triangulated categories with respect to subcategories

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$\operatorname{mod} A$: the category of finitely generated A-modules.
$\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$: the bounded derived categury of $\operatorname{mod} A$.

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(3) The *triangle dimension* of $\mathcal{T}$ is defined as

$$\mathrm{tri.dim}\, \mathcal{T} := \inf\{n \geq 0 \mid \mathcal{T} = \langle M \rangle_{n+1},\ \exists M \in \mathcal{T}\}.$$

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In general, let A be an artinian ring, then we have

$$\mathrm{tri.dim}\, \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A) < \ell\ell(A).$$

Here, $\ell\ell(A) := \inf\{\ l \mid (\mathrm{rad}\, A)^l = 0\ \}$ is the Loewy length of A.

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If Λ is of finite representation type, then we have

$$\mathrm{tri.\,dim}\,\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) = 0.$$

Known results

Proposition 3

(1) (Krause-Kussin (2006)) $\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leq \operatorname{gl.dim} R$.

(2) (Rouquier (2008)) If R is essential of finite type over a field, then $\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) < \infty$.

(3) (Aihara-Takahashi (2011)) If R is a complete equi-characteristic local ring with perfect residue field, then $\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) < \infty$.

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Let Λ be a hereditary algebra. Then we have

$$(\operatorname{mod}\Lambda)\text{-}\operatorname{tri.dim}\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) = 0.$$

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We call T *cotilting* if it satisfies the following three conditions.

(1) The injective dimension of the A-module T is finite.

(2) $\operatorname{Ext}^i_A(T, T) = 0$ for all $i > 0$ (i.e., $T \in \mathcal{X}_T$).

(3) For any $X \in \mathcal{X}_T$, there exists an exact sequence $0 \to X \to T' \to X' \to 0$ in $\operatorname{mod} \Lambda$ with $T' \in \operatorname{add} T$ and $X' \in \mathcal{X}_T$.

Applications

Proposition 7 (Iyama (2007))

Let T be a cotilting module of Λ. Then one has

$$\mathrm{gl.dim}(\mathrm{mod}\,\mathcal{X}_T) \leq \max\{2, \mathrm{inj.dim}\,T\}.$$

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$$\mathrm{CM}(R)\text{-}\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leq \max\{1, \dim R\}.$$

THANK YOU !