Induced pseudofunctors and gluing of derived equivalences

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arXiv:1204.0196 Gluing derived equivalences together
arXiv:1111.3845, arXiv:1111.2239

Contents

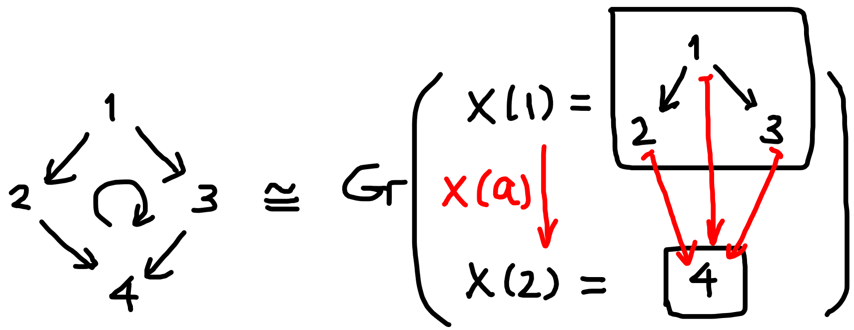
Contents

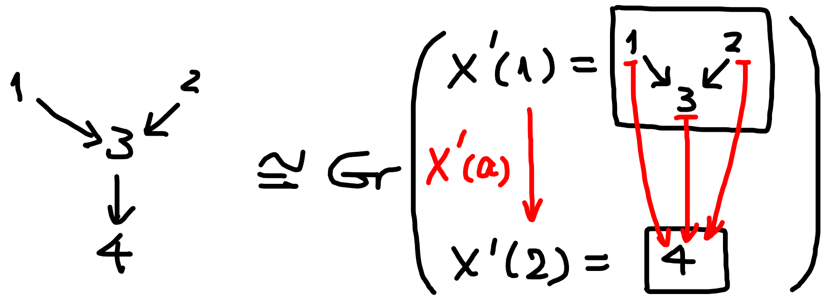
Throughout this talk

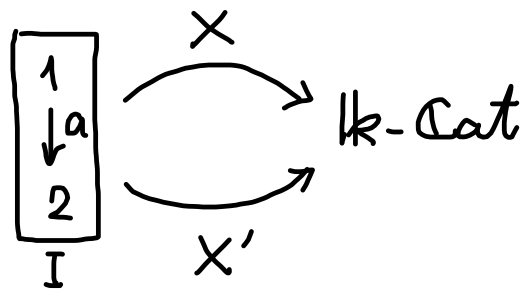
I : a small category,   G : a group.
$\Bbbk$: a commutative ring.
$\Bbbk$-$\mathbb{C}$at := the 2-category of small $\Bbbk$-categories.
$\Bbbk$-$\mathbb{A}$b := the 2-category of small abelian $\Bbbk$-categories.
$\Bbbk$-$\mathbb{T}$ri := the 2-category of small triangulated $\Bbbk$-cats,
where 2-categories := strict 2-categories

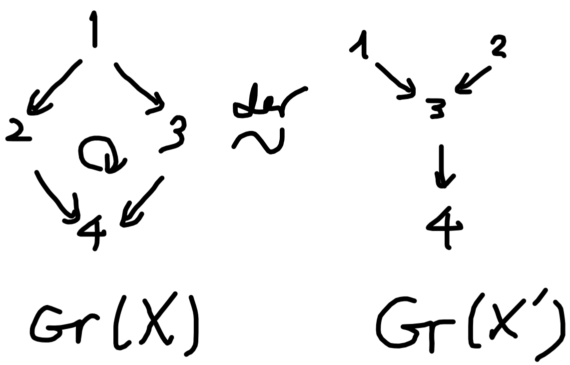
1. Introduction

An easy example.



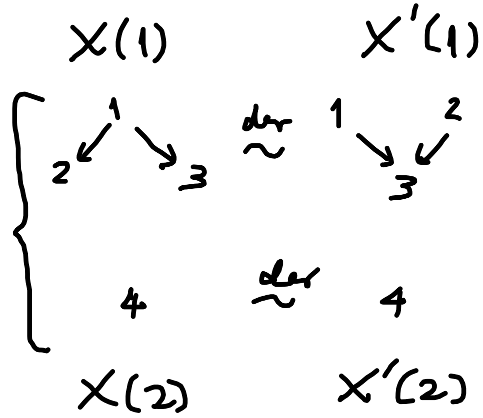






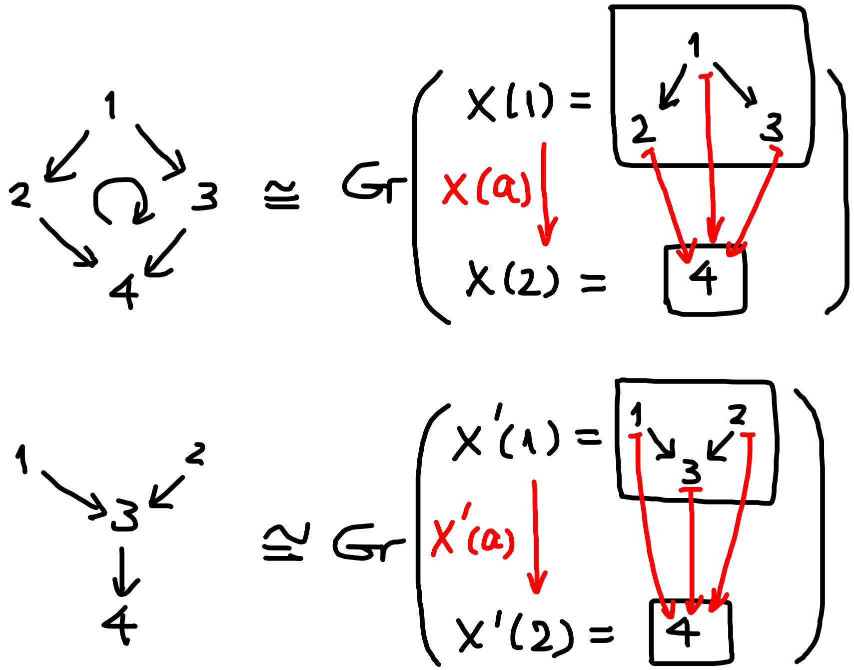
"gluing" of

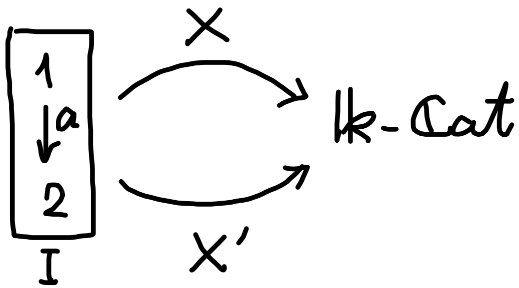
When possible?

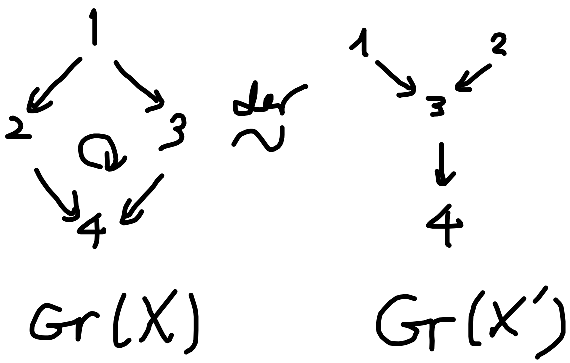


1. Introduction

An easy example.

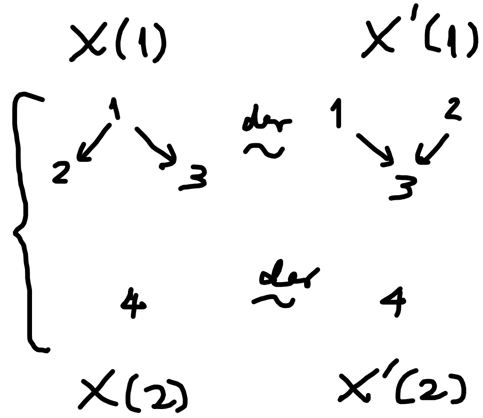






"gluing" of

If $X \overset{\text{der}}{\sim} X'$



2. From group actions to colax functors

- G : a group. Regard G : a cat $(\{\ast\}, G, \cdot)$

  Then "a G-action on a $\Bbbk$-cat $\mathcal{C}$" = "a functor $X : G \longrightarrow \Bbbk\text{-}\mathbb{C}\mathrm{at}$".

  $$\left(\ast \xrightarrow{a} \ast\right) \longmapsto \left(\mathcal{C} \xrightarrow{X(a)} \mathcal{C}\right)$$

  In this case, we have $\mathcal{C}/G = \mathrm{Gr}(X)$.

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  $$\begin{array}{ccc} a \circlearrowright * & \longmapsto & \mathcal{C} \circlearrowleft X(a) \end{array}$$

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- F: an autoeq of a $\Bbbk$-cat $\mathcal{C}$ (eg. $\mathcal{C} = \mathcal{D}^b(\mathrm{mod}\, H)$, $F = \tau^{-1}[1]$) with a q-inv $F^- \dashv F$, $\eta : \mathbb{1}_{\mathcal{C}} \overset{\sim}{\Rightarrow} FF^-$, $\varepsilon : F^- F \overset{\sim}{\Rightarrow} \mathbb{1}_{\mathcal{C}}$. $\langle a \rangle$: infin. cyclic gp with gen. a.

  $\rightsquigarrow$ "a pseudofunctor"

$$\left\{\begin{array}{l}
\bullet\ X : \langle a \rangle \to \Bbbk\text{-}\mathbb{C}\mathrm{at}, \quad * \longmapsto \mathcal{C}, \qquad X(a^n) := \begin{cases} F^n & (n>0) \\ \mathbb{1}_{\mathcal{C}} & (n=0) \\ (F^-)^{|n|} & (n<0) \end{cases} \quad (n \in \mathbb{Z}) \\
\bullet\ \mathbb{1} : X(\mathbb{1}_*) = \underbrace{\mathbb{1}_{X(*)}}_{\mathbb{1}_{\mathcal{C}}} \\
\bullet\ \text{A fam. of nat. isos } X_{n,m} : X(a^n a^m) \overset{\sim}{\Rightarrow} X(a^n)\cdot X(a^m) \text{ defined by } \eta \text{ and } \varepsilon^{-1}.
\end{array}\right.$$

But $X(a^n a^m) \neq X(a^n)\cdot X(a^m)$ in gen.

$\mathbb{1}_{\mathcal{C}} \neq FF^-$.

$X(a \cdot a^{-1}) \neq X(a)X(a^{-1})$.

In this case we have $\underline{\mathcal{C}/\langle F \rangle = \mathrm{Gr}(X)}$.

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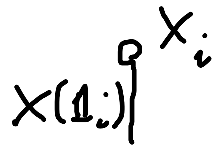
not well-dfn^d if F is not an autom

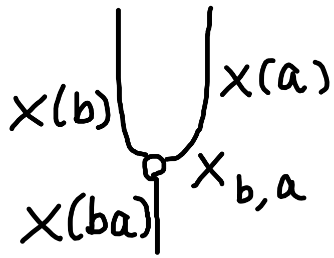
well-dfn^d

Dfn. $\mathbb{C}$: a 2-cat (e.g. $\mathbf{k}$-$\mathbb{C}$at)

A colax functor $X : I \to \mathbb{C}$ consits of data :

- $X : I_0 \longrightarrow \mathbb{C}_0$ a map
- $X : I(i,j) \to \mathbb{C}(X(i), X(j))$ a map $(i,j \in I_0)$
- $X_i : X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ a 2-mor in $\mathbb{C}$ $(i \in I_0)$
- $X_{b,a} : X(ba) \Rightarrow X(b)X(a)$ a 2-mor in $\mathbb{C}$ $(\cdot \xrightarrow{a} \cdot \xrightarrow{b}$ in $I_1)$





that satisfy the axioms :

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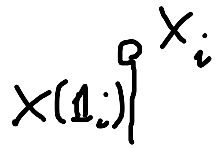

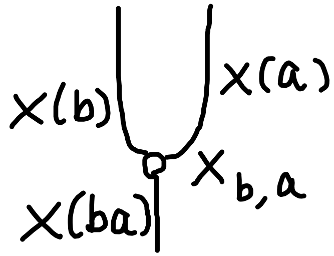

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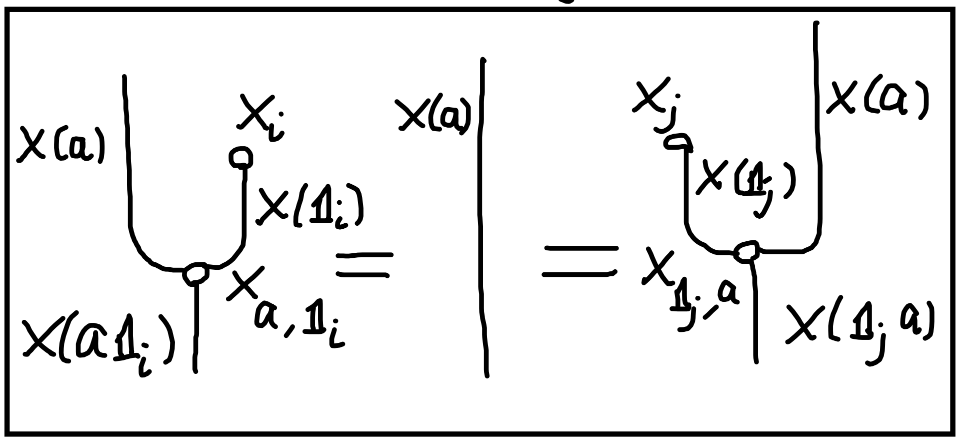
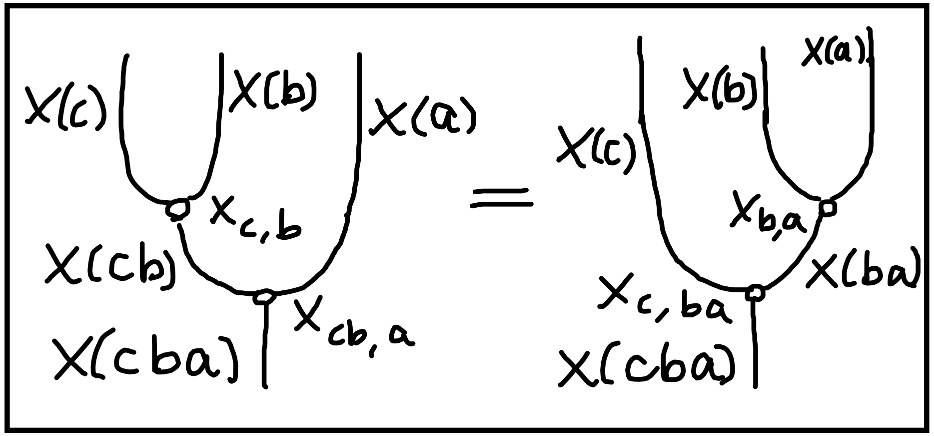


$(\ j \xleftarrow{a} i \quad$ in $I_1\ )$

coassociativity



$(\cdot \xleftarrow{c} \cdot \xleftarrow{b} \cdot \xleftarrow{a} \cdot \quad$ in $I_1\ )$

Exm. A comonad T on a cat $\mathcal{C}$ is a colax fun
$$\begin{aligned} 1 &\to \mathcal{C}at \\ * &\mapsto \mathcal{C} \\ 1_* \circlearrowleft & \quad \circlearrowleft T \end{aligned}$$

Dfn. $\mathbb{C}$: a 2-cat.

$\mathbb{C}^{op}$:= the 2-cat obtained from $\mathbb{C}$ by reversing the 1-morphisms.

$\mathbb{C}^{co}$:= the 2-cat obtained from $\mathbb{C}$ by reversing the 2-morphisms.

$\mathbb{C}^{coop} := (\mathbb{C}^{co})^{op} = (\mathbb{C}^{op})^{co}$.

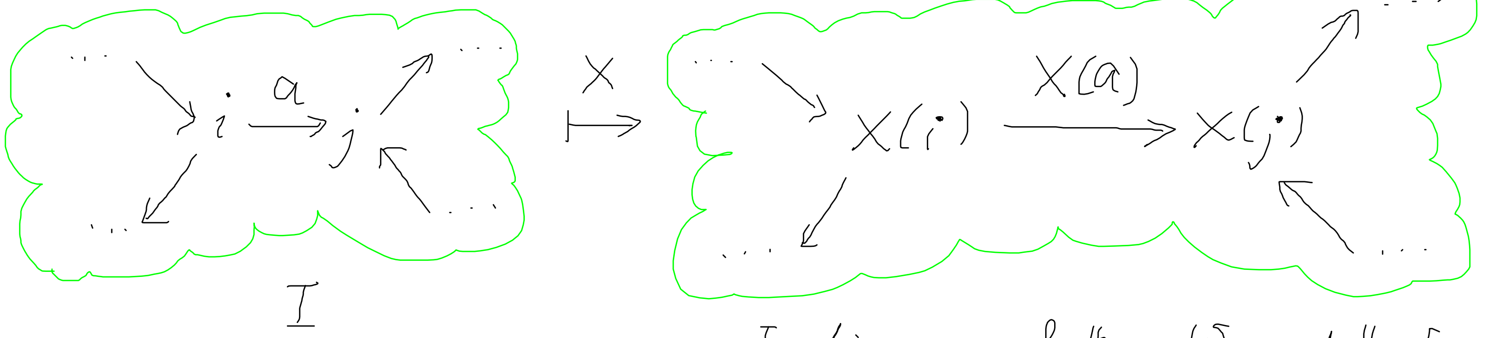
Dfn. A lax functor $I \to \mathbb{C}$ is a colax fun $I \to \mathbb{C}^{co}$.

A pseudofunctor is a colax fun $X : I \to \mathbb{C}$ with $\forall X_i, X_{b,a}$: 2-isos

A functor $I \to \mathbb{C}$ is a colax fun $X : I \to \mathbb{C}$ with $\forall X_i, X_{b,a}$: identities.

3. Grothendieck constructions

$X : I \to \Bbbk\text{-Cat}$ a colax functor.



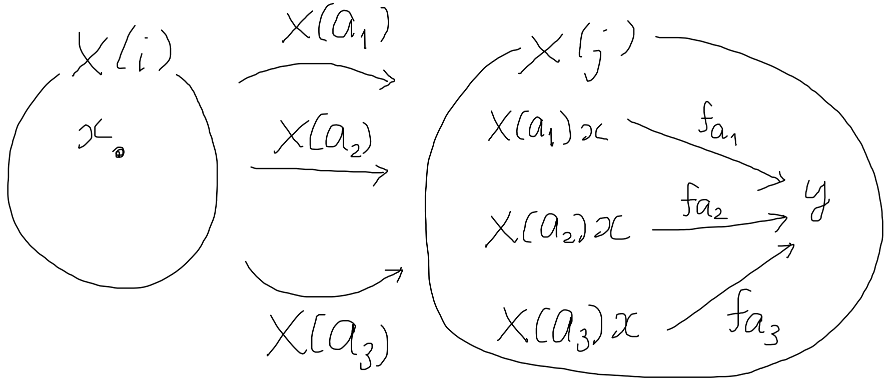
an I-diagram of $\Bbbk$-cats and $\Bbbk$-functors

A category $Gr(X)$ is defined as follows:

<u>Obj</u> $\quad Gr(X)_0 := \coprod_{i \in I_0} X(i)_0 =: \{ (i, x) =: {}_i x \mid i \in I_0,\ x \in X(i)_0 \}$

<u>Mor</u> $\quad \forall\, {}_i x, {}_j y \in Gr(X)_0, \quad Gr(X)({}_i x, {}_j y) := \bigoplus_{a \in I(i,j)} X(j)\big(\underbrace{X(a)\overbrace{x}^{\in X(i)_0}}_{\in X(j)_0}, \underbrace{y}_{\in X(j)_0}\big)$

$$\begin{tikzcd} {}_{i}\bullet \arrow[r, bend left, "a_1"] \arrow[r, "a_2"] \arrow[r, bend right, "a_3"'] & {}_{j}\bullet \end{tikzcd}$$



$$\begin{pmatrix} f_{a_1} \\ f_{a_2} \\ f_{a_3} \end{pmatrix} : {}_{i}x \longrightarrow {}_{j}y$$

<u>Comp</u> $\forall\ {}_{i}x \xrightarrow[\underset{(f_a)_a}{\shortparallel}]{f} {}_{j}y \xrightarrow[\underset{(g_b)_b}{\shortparallel}]{g} {}_{k}z$,

$$\boxed{X(a)x \xrightarrow{f_a} y}$$

$$\downarrow X(b)$$

$$X(ba)x \xrightarrow[X_{b,a}]{} X(b)X(a)x \xrightarrow[X(b)f_a]{} \boxed{X(b)y \xrightarrow[g_b]{} z}$$

$$g \circ f := \left(\sum_{\substack{a \in I(i,j) \\ b \in I(j,k) \\ c = ba}} g_b \circ X(b)f_a \circ X_{b,a} \right)_{c \in I(i,k)}$$

Exm. $X := \Delta(A) : I \to \mathbb{k}\text{-}\mathbb{C}\mathrm{at}$ a functor

$$\boxed{\begin{array}{c} i \\ a \downarrow \\ j \end{array}} \mapsto \boxed{\begin{array}{c} A \\ \downarrow \mathbb{1}_A \\ A \end{array}} \qquad A : \text{a } \mathbb{k}\text{-alg}$$

(1) $I = \mathbb{P}Q$ path-cat of a quiver Q

$\Rightarrow \quad Gr(X) \cong AQ$ path-cat over A, $\quad A \otimes_{\mathbb{k}} \mathbb{k}Q$

(2) $I = S$ a poset

$\Rightarrow \quad Gr(X) \cong AS$ incidence cat of S over A, $\quad A \otimes_{\mathbb{k}} \mathbb{k}S$

(3) $I = G$ a monoid

$\Rightarrow \quad Gr(X) \cong AG$ monoid alg of G over A, $\quad A \otimes_{\mathbb{k}} \mathbb{k}G$

(4) (gen. of (1), (2)) $I = \langle Q \mid R \rangle$, Q : a quiver, R : relations

$\Rightarrow \quad Gr(X) \cong AQ/\langle g-h \mid (g,h) \in R \rangle \cong A \otimes_{\mathbb{k}} (\mathbb{k}Q/\langle g-h \mid (g,h) \in R \rangle)$

4. Induced pseudofunctors

Dfn. $\mathbb{B}, \mathbb{C}$: 2-categories

<u>Obj</u> A colax functor $X : \mathbb{B} \to \mathbb{C}$ is defined similarly by data

- $X : \mathbb{B}_0 \longrightarrow \mathbb{C}_0$: a map
- $X : \mathbb{B}(i,j) \to \mathbb{C}(X(i), X(j))$ a <u>functor</u> $(i,j \in \mathbb{B}_0 \times \mathbb{B}_0)$
- $X_i : X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ a 2-mor in $\mathbb{C}$, $i \in \mathbb{B}_0$
- $X_{b,a} : X(ba) \Rightarrow X(b)X(a)$ a 2-mor in $\mathbb{C}$, $i \xrightarrow{a} j \xrightarrow{b} k$ in $\mathbb{B}_1$ (<u>natural</u> in b, a)

and by the same axioms as before.

<u>1-mor</u> A lax transformation $(F, \psi) : X \longrightarrow X'$

$$\begin{tikzcd} X(i) \arrow[r, "F(i)"] \arrow[d, "X(a)"'] & X'(i) \arrow[d, "X'(a)"] \arrow[dl, phantom, "\swArrow \psi(a)"] \\ X(j) \arrow[r, "F(j)"'] & X'(j) \end{tikzcd}$$

$(\forall a : i \to j$ in $I)$

<u>2-mor</u> A modification $\alpha : (F, \psi) \Rightarrow (F', \psi')$

These form a 2-cateory $\overleftarrow{\mathrm{Colax}}(\mathbb{B}, \mathbb{C})$.

I is regarded as a 2-cat with all 2-mor^s identities: $\overleftarrow{\mathrm{Colax}}(I, \mathbb{C})$.

$\curvearrowright$ possible to define eq $X \sim X'$.

Dfn. A lax functor $\mathbb{B} \to \mathbb{C}$ is a colax fun $\mathbb{B} \to \mathbb{C}^{co}$.

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A 2-functor $\mathbb{B} \to \mathbb{C}$ is a colax fun $X : \mathbb{B} \to \mathbb{C}$ with $\forall X_i, X_{b,a}$: identities.

Exm. $\mathrm{Mod}' : \Bbbk\text{-}\mathbb{C}\mathrm{at} \to \Bbbk\text{-}\mathbb{A}\mathrm{b}^{coop}$ is a 2-functor

$$\begin{array}{ccc} \mathcal{C} & & \mathrm{Mod}\,\mathcal{C} := \Bbbk\text{-}\mathbb{C}\mathrm{at}(\mathcal{C}^{op}, \mathrm{Mod}\,\Bbbk) \\ F \downarrow & \longmapsto & \uparrow (-)\circ F^{op} \\ \mathcal{C}' & & \mathrm{Mod}\,\mathcal{C}' \end{array} \qquad \boxed{\mathrm{Mod}' = \Bbbk\text{-}\mathbb{C}\mathrm{at}((-)^{op}, \mathrm{Mod}\,\Bbbk)}$$

$\mathrm{Mod} : \Bbbk\text{-}\mathbb{C}\mathrm{at} \to \Bbbk\text{-}\mathbb{A}\mathrm{b}$ is a pseudofunctor

$$\begin{array}{ccc} \mathcal{C} & & \mathrm{Mod}\,\mathcal{C} \\ F \downarrow & \longmapsto & \downarrow -\otimes_{\mathcal{C}} \overline{F} \\ \mathcal{C}' & & \mathrm{Mod}\,\mathcal{C}' \end{array} , \qquad \overline{F} := {}_{\mathcal{C}}\mathcal{C}'(-, F(-))_{\mathcal{C}'} : \text{bimod}$$

$\mathcal{D} : \Bbbk\text{-}\mathbb{M}\mathrm{od}\mathbb{C}\mathrm{at} \to \Bbbk\text{-}\mathbb{T}\mathrm{ri}$ is a pseudofunctor.

$$\begin{array}{ccc} \mathcal{A} & & \mathcal{D}\mathcal{A} \\ F \downarrow & \longmapsto & \downarrow \mathbb{L}F \\ \mathcal{A}' & & \mathcal{D}\mathcal{A}' \end{array}$$

Thm. Let $\mathbb{B}, \mathbb{C}, \mathbb{D}$ be 2-cat^s, $\mathbb{C} \xrightarrow{V} \mathbb{D}$: **pseudofun.**

Then

$\overleftarrow{\mathrm{Colax}}(\mathbb{B}, V) : \overleftarrow{\mathrm{Colax}}(\mathbb{B}, \mathbb{C}) \longrightarrow \overleftarrow{\mathrm{Colax}}(\mathbb{B}, \mathbb{D})$ is a **pseudofun.**

$$\begin{tikzcd} X \arrow[d, bend right=60, "{(F,\psi)}"'] \arrow[d, bend left=60, "{(F',\psi')}"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] \\ X' \end{tikzcd} \longmapsto \begin{tikzcd} VX \arrow[d, bend right=60, "{V(F,\psi)}"'] \arrow[d, bend left=60, "{V(F',\psi')}"] \arrow[d, phantom, "\overset{V\alpha}{\Longrightarrow}"] \\ VX' \end{tikzcd}$$

Apply to $\quad I \xrightarrow{X} \mathbb{k}\text{-}\mathbb{C}\mathrm{at} \xrightarrow[\text{pseudofun}]{\mathrm{Mod}} \mathbb{k}\text{-}\mathbb{M}\mathrm{od}\mathbb{C}\mathrm{at} \xrightarrow[\text{pseudofun}]{\mathcal{D}} \mathbb{k}\text{-}\mathbb{T}\mathrm{ri}$

Cor. $\overleftarrow{\mathrm{Colax}}(I, \mathrm{Mod}) : \overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbb{C}\mathrm{at}) \longrightarrow \overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbb{A}\mathrm{b})$ and

$$X \longmapsto \mathrm{Mod}\, X$$

$\overleftarrow{\mathrm{Colax}}(I, \mathcal{D} \circ \mathrm{Mod}) : \overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbb{C}\mathrm{at}) \longrightarrow \overleftarrow{\mathrm{Colax}}(I, \mathbb{k}\text{-}\mathbb{T}\mathrm{ri})$

$$X \longmapsto \mathcal{D}(\mathrm{Mod}\, X)$$

are **pseudofunctors.**

Rmk. $X : I \to \Bbbk\text{-}\mathbb{C}\mathrm{at}$: a colax functor

$\Longrightarrow$

(1) $\mathrm{Mod}\, X : I \to \Bbbk\text{-}\mathbb{A}\mathrm{b}$ a colax functor

$$\boxed{\begin{array}{c} i \\ a \downarrow \\ j \end{array}} \text{ in } I \longmapsto \boxed{\begin{array}{c} \mathrm{Mod}\, X(i) \\ \downarrow\ -\otimes_{X(i)} \overline{X(a)} \\ \mathrm{Mod}\, X(j) \end{array}}$$

$\overline{X(a)}$: $X(i)$-$X(j)$-bimodule

$(x, y) \mapsto X(j)(y, X(a)(x))$

(2) $\mathcal{D}(\mathrm{Mod}\, X) : I \to \Bbbk\text{-}\mathbb{T}\mathrm{ri}$ a colax functor

$$\boxed{\begin{array}{c} i \\ a \downarrow \\ j \end{array}} \text{ in } I \longmapsto \boxed{\begin{array}{c} \mathcal{D}(\mathrm{Mod}\, X(i)) \\ \downarrow\ -\overset{\mathbb{L}}{\otimes}_{X(i)} \overline{X(a)} \\ \mathcal{D}(\mathrm{Mod}\, X(j)) \end{array}}$$

Dfn. Let $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathcal{C}at)$.

$X \overset{\text{der}}{\sim} X' :\Longleftrightarrow \mathcal{D}(\mathrm{Mod}\,X)$ and $\mathcal{D}(\mathrm{Mod}\,X')$ are equivalent in the 2-cat $\overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbb{T}\mathrm{ri})$.

$\overset{\text{Prp.}}{\Longleftrightarrow} \exists\, (F, \psi) : \mathcal{D}(\mathrm{Mod}\,X) \to \mathcal{D}(\mathrm{Mod}\,X')$: a 1-mor st.

$$\begin{cases} \bullet\ F(i) : \text{tri. eq.} \quad \forall i \in I_0, \\ \bullet\ \psi(a) : \text{iso} \quad \forall a \in I_1 \end{cases} \Longleftrightarrow : (F, \psi) : I\text{-equivariant}$$

$$\boxed{\begin{tikzcd} \mathcal{D}(\mathrm{Mod}\,X(i)) \arrow[r, "F(i)", "\sim"'] \arrow[d, "- \overset{\mathbb{L}}{\otimes}_{X(i)} \overline{X(a)}"'] & \mathcal{D}(\mathrm{Mod}\,X'(i)) \arrow[d, "- \overset{\mathbb{L}}{\otimes}_{X'(i)} \overline{X'(a)}"] \arrow[dl, Rightarrow, "\psi(a)"', "\sim"] \\ \mathcal{D}(\mathrm{Mod}\,X(j)) \arrow[r, "\sim", "F(j)"'] & \mathcal{D}(\mathrm{Mod}\,X'(j)) \end{tikzcd}}$$

5. Gluing derived equivalences together

Thm. [A] $X, X' \in \overleftarrow{\mathrm{Colax}}(I, \Bbbk\text{-}\mathbb{C}\mathrm{at})$

(1) $X \overset{\mathrm{der}}{\sim} X'$

$\Downarrow$ $\Uparrow$ if X' : $\Bbbk$-projective (e.g. $\Bbbk$: a field)

(2) $\underset{(I\text{-equivar})}{X' \xrightarrow{\sim}} \underset{I\text{-equivar}}{\mathcal{T} \hookrightarrow} K^b(\mathrm{prj}\, X)$, $\exists\, \mathcal{T}$: tilting colax subcat for X

(i.e. $\mathcal{T}(i)$: tilting $\forall i \in I_0$)

$\Downarrow$

(3) $\mathrm{Gr}(X) \overset{\mathrm{der}}{\sim} \mathrm{Gr}(X')$

Cor. $A, A' \in \Bbbk\text{-}\mathrm{Alg}$ $A \overset{\mathrm{der}}{\sim} A'$

$\Rightarrow \begin{cases} \bullet\ AQ \overset{\mathrm{der}}{\sim} A'Q, & \forall Q : \text{a quiver} \\ \bullet\ AS \overset{\mathrm{der}}{\sim} A'S, & \forall S : \text{a poset} \\ \bullet\ AG \overset{\mathrm{der}}{\sim} A'G, & \forall G : \text{a monoid} \end{cases}$

(Pf) $\forall \mathcal{C}, \mathcal{C}' \in \Bbbk\mathbb{C}\mathrm{at}$

$\mathcal{C} \overset{\mathrm{der}}{\sim} \mathcal{C}'$

$\Rightarrow \Delta(\mathcal{C}) \overset{\mathrm{der}}{\sim} \Delta(\mathcal{C}')$

$\square$

Exm. $3 \leq n \in \mathbb{N}$.

$I := \mathbb{P}Q$, $Q:\ 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} n$

Define functors $X, X' : I \to \Bbbk\text{-Cat}$ as follows.

$$X \left[\begin{array}{l} \forall i \in I_0,\ X(i):\ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \rightleftarrows \cdots \underset{\beta_{i-1}}{\overset{\alpha_{i-1}}{\rightleftarrows}} i \ ; \begin{cases} \alpha_{j+1}\alpha_j = 0,\ \beta_j\beta_{j+1} = 0,\ \alpha_j\beta_j = \beta_{j+1}\alpha_{j+1},\ (j = 1, \dots, i-1) \\ \alpha_1\beta_1\alpha_1 = 0,\ \beta_{i-1}\alpha_{i-1}\beta_{i-1} = 0. \end{cases} \\ \forall a_i : i \to i+1.\ X(a_i) : X(i) \hookrightarrow X(i+1). \end{array}\right.$$

$$\begin{tikzcd}
X(2): \arrow[d, "X(a_2)"'] & 1 \arrow[r, shift left] \arrow[d, mapsto] & 2 \arrow[l, shift left] \arrow[d, mapsto] & & & \\
X(3): \arrow[d, "X(a_3)"'] & 1 \arrow[r, shift left] \arrow[d, mapsto] & 2 \arrow[l, shift left] \arrow[r, shift left] \arrow[d, mapsto] & 3 \arrow[l, shift left] \arrow[d, mapsto] & & \\
X(4): \arrow[d, "X(a_4)"'] & 1 \arrow[r, shift left] \arrow[d, mapsto] & 2 \arrow[l, shift left] \arrow[r, shift left] \arrow[d, mapsto] & 3 \arrow[l, shift left] \arrow[r, shift left] \arrow[d, mapsto] & 4 \arrow[l, shift left] \arrow[d, mapsto] & \\
X(5): \arrow[d] & 1 \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & 5 \arrow[l, shift left] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ \ddots
\end{tikzcd}$$

Exm. $3 \leq n \in \mathbb{N}$.

$I := \mathbb{P}Q$, Q : $2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} n$

Define functors $X, X' : I \to \Bbbk\text{-}\mathcal{C}at$ as follows.

X' $\left[\ \forall i \in I_0,\ X'(i) : \begin{tikzcd} 1 \arrow[r, "\gamma_1"] & 2 \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{i-1}"] & i \arrow[lll, bend left, "\gamma_i"] \end{tikzcd} \ ;\ \gamma_{j+i} \cdots \gamma_{j+1} \gamma_j = 0 \quad (j \in \mathbb{Z}/i\mathbb{Z})\right.$

$$\begin{tikzcd}
X(2) \arrow[d, "X(a_2)"'] : & 1 \arrow[r, shift left] \arrow[d] & 2 \arrow[l, shift left] \arrow[d] & & & \\
X(3) \arrow[d, "X(a_3)"'] : & 1 \arrow[r, shift left] \arrow[d] & 2 \arrow[l, shift left] \arrow[r, shift left] \arrow[d] & 3 \arrow[l, shift left] \arrow[d] & & \\
X(4) \arrow[d, "X(a_4)"'] : & 1 \arrow[r, shift left] \arrow[d] & 2 \arrow[l, shift left] \arrow[r, shift left] \arrow[d] & 3 \arrow[l, shift left] \arrow[r, shift left] \arrow[d] & 4 \arrow[l, shift left] \arrow[d] & \\
X(5) \arrow[d] : & 1 \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & 5 \arrow[l, shift left] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \ddots
\end{tikzcd}$$

$$\begin{tikzcd}
X'(2) \arrow[d, "X'(a_2)"'] : & 1 \arrow[r, shift left] & 2 \arrow[l, shift left] & & \\
X'(3) \arrow[d, "X'(a_3)"'] : & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend left] & & \\
X'(4) \arrow[d, "X'(a_4)"'] : & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[lll, bend left] & \\
X'(5) : & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[llll, bend left] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \ddots
\end{tikzcd}$$

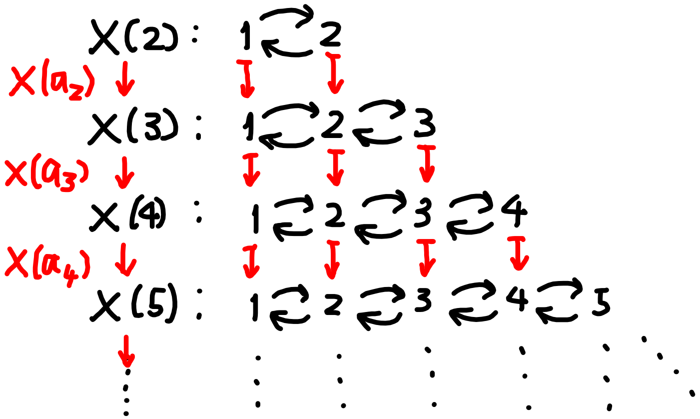
Exm. $3 \leq n \in \mathbb{N}$.

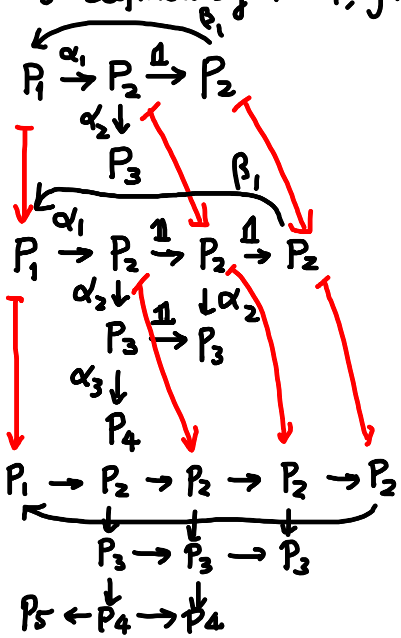
$I := \mathbb{P}Q$, Q: $2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} n$

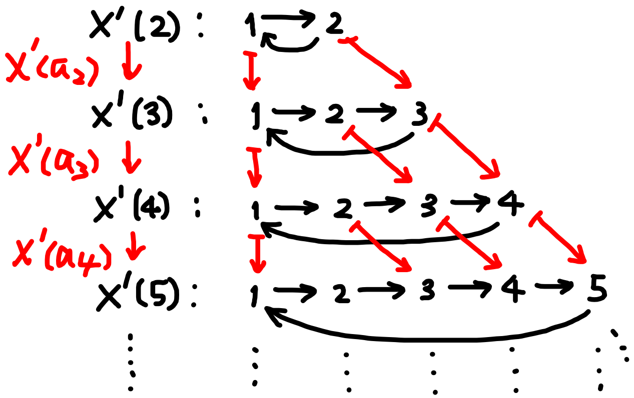
Define functors $X, X' : I \to \Bbbk\text{-Cat}$ as follows.

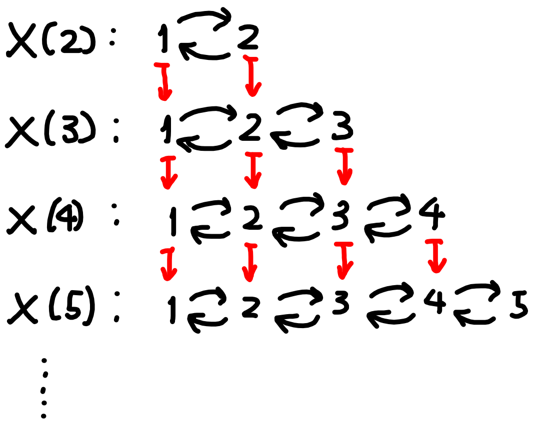
X
- $\forall i \in I_0$, $X(i)$: $1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \rightleftarrows \cdots \underset{\beta_{i-1}}{\overset{\alpha_{i-1}}{\rightleftarrows}} i$; $\begin{cases} \alpha_{j+1}\alpha_j = 0,\ \beta_j\beta_{j+1} = 0,\ \alpha_j\beta_j = \beta_{j+1}\alpha_{j+1},\ (j = 1, \dots, i-1) \\ \alpha_1\beta_1\alpha_1 = 0,\ \beta_{i-1}\alpha_{i-1}\beta_{i-1} = 0. \end{cases}$
- $\forall a_i : i \to i+1$. $X(a_i) : X(i) \hookrightarrow X(i+1)$.

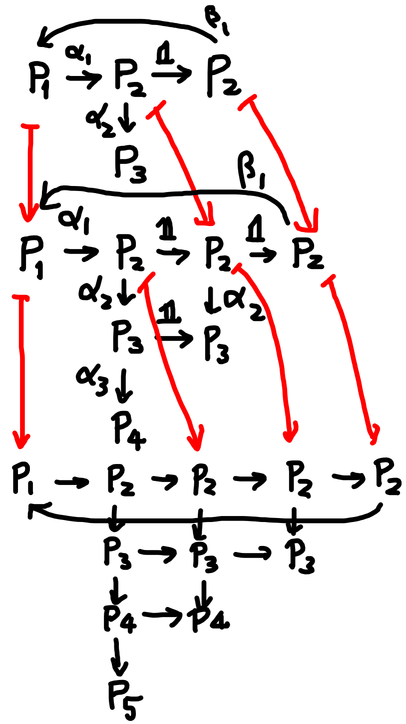
X'
- $\forall i \in I_0$, $X'(i)$: $\begin{tikzcd} 1 \arrow[r, "\gamma_1"] & 2 \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{i-1}"] & i \arrow[lll, bend left, "\gamma_i"] \end{tikzcd}$; $\gamma_{j+i} \cdots \gamma_{j+1}\gamma_j = 0$ $(j \in \mathbb{Z}/i\mathbb{Z})$
- $\forall a_i : i \to i+1$, $X(a_i) : X(i) \to X(i+1)$ defined by $1 \mapsto 1$, $j \mapsto j+1$, $\gamma_1 \mapsto \gamma_2\gamma_1$, $\gamma_j \mapsto \gamma_{j+1}$ $(j = 2, \dots, i)$

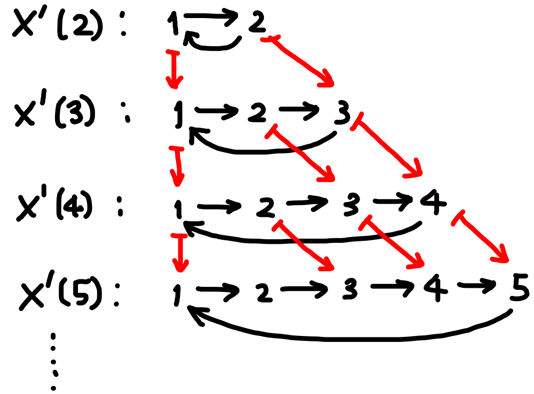


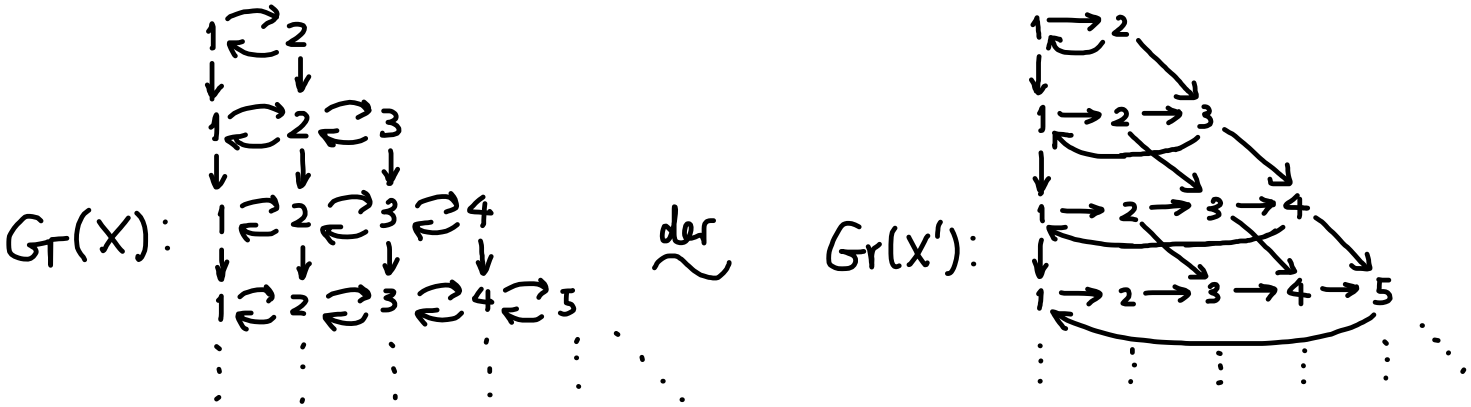












arXiv:1204.0196 Gluing derived equivalences together
arXiv:1111.3845:presentation, 1111.2239:Thm\S5