Tame algebras of semiregular tubular type

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Notation

K – algebraically closed field

algebra – basic, indecomposable, finite-dimensional K-algebra

A algebra

mod A – category of finite-dimensional right A-modules

Γ_A – Auslander Reiten quiver of A

Definition

$\mathcal{C}$ a component of Γ_A

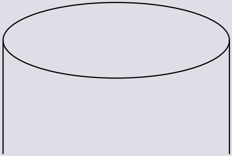
- $\mathcal{C}$ is **regular** if $\mathcal{C}$ contains neither a projective module nor an injective module
- $\mathcal{C}$ is **semiregular** if $\mathcal{C}$ does not contain both a projective module and an injective module

Theorem (Liu, Zhang)

*A algebra, $\mathcal{C}$ regular component of Γ_A.
Then $\mathcal{C}$ contains an oriented cycle $\iff$ $\mathcal{C}$ is a stable tube.*

Definition

$\mathcal{C}$ **stable tube** if $\mathcal{C}$ is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$.



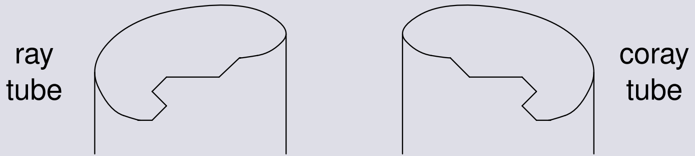
Theorem (Liu)

*A algebra, $\mathcal{C}$ semiregular component of Γ_A.
Then $\mathcal{C}$ contains an oriented cycle $\iff$ $\mathcal{C}$ is a semiregular tube (ray tube or coray tube).*

Definition

$\mathcal{C}$ **ray tube** if $\mathcal{C}$ is obtained from a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ by a finite number (possible empty) of ray insertions.

$\mathcal{C}$ **coray tube** if $\mathcal{C}$ is obtained from a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ by a finite number (possible empty) of coray insertions.



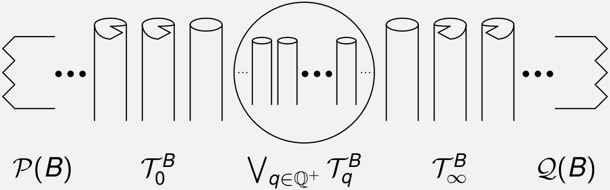
An algebra A is said to be of **semiregular tubular type** if all components in Γ_A are semiregular tubes (ray tubes or coray tubes).

Problem

Describe all algebras A of semiregular tubular type.

Tubular algebra (Ringel) – a tubular (branch) extension B of a tame concealed algebra C of one of the tubular types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$.
B tubular algebra. Then

- $\operatorname{gl.dim} B = 2$
- $\operatorname{rk} K_0(B) = 6, 8, 9, 10$
- B is triangular, nondomestic of polynomial growth
- Γ_B is of the form



$\mathcal{P}(B)$ a unique preprojective component, $\mathcal{Q}(B)$ a unique preinjective component
$\mathcal{T}_0^B$ a $\mathbb{P}_1(K)$-family of ray tubes containing at least one projective module
$\mathcal{T}_\infty^B$ a $\mathbb{P}_1(K)$-family of coray tubes containing at least one injective module
$\mathcal{T}_q^B$ a $\mathbb{P}_1(K)$-family of stable tubes, for $q \in \mathbb{Q}^+$ (the set of positive rational numbers)
The tubular families $\mathcal{T}_q^B$, $q \in \mathbb{Q}^+$, are of the same tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$, denoted by $t(B)$.

B is a tubular algebra $\overset{\text{Ringel}}{\Longleftrightarrow}$ B is a cotubular algebra

B **cotubular algebra** – a tubular (branch) coextension B of a tame concealed algebra C of one of the tubular types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$

An algebra A is called **quasitilted** (Happel-Reiten-Smalø) if $\operatorname{gl.dim} A \leq 2$ and $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$ for any indecomposable module X in $\operatorname{mod} A$

An algebra A is called **tame** (Drozd) if, for any dimension d, there exists a finite number of $K[x]$-A-bimodules M_i, $1 \leq i \leq n_d$, which are free of finite rank as left $K[x]$-modules and all but finitely many isomorphism classes of indecomposable modules in $\operatorname{mod} A$ of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \dots, n_d\}$

Theorem (Skowroński)

Let A be a quasitilted algebra. The following conditions are equivalent.

(i) *A is tame.*

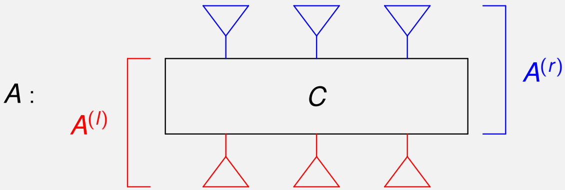
(ii) *The Euler form χ_A of A is weakly nonnegatiwe.*

(iii) *A is a tame tilted algebra or a tame semiregular branch enlargement of a tame concealed algebra.*

A tame semiregular branch enlargement of a tame concealed algebra $C \iff A$ a tame quasitilted algebra of canonical type

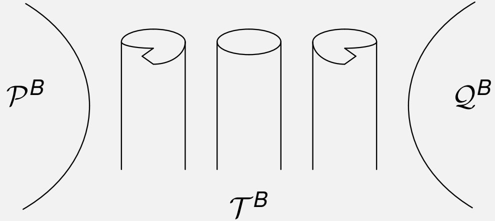
$A^{(l)}$ the maximal branch coextension of C inside A (**left part** of A)

$A^{(r)}$ the maximal branch extension of C inside A (**right part** of A)



$A^{(l)}$ (respectively, $A^{(r)}$) is a representation-infinite algebra of Euclidean type or a tubular algebra

B a tame quasitilted algebra of canonical type. Then Γ_B is of the form



$\mathcal{T}^B$ a $\mathbb{P}_1(K)$-family of semiregular tubes separating $\mathcal{P}^B$ from $\mathcal{Q}^B$

$\mathcal{P}^B = \mathcal{P}^{B^{(l)}}$ is one of the forms

- $\mathcal{P}^{B^{(l)}} = \mathcal{P}(B^{(l)})$, if $B^{(l)}$ is tilted of Euclidean type
- $\mathcal{P}^{B^{(l)}} = \mathcal{P}(B^{(l)}) \vee \mathcal{T}_0^{B^{(l)}} \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B^{(l)}} \right)$, if $B^{(l)}$ is a tubular algebra

$\mathcal{Q}^B = \mathcal{Q}^{B^{(r)}}$ is one of the forms

- $\mathcal{Q}^{B^{(r)}} = \mathcal{Q}(B^{(r)})$, if $B^{(r)}$ is tilted of Euclidean type
- $\mathcal{Q}^{B^{(r)}} = \left(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B^{(r)}} \right) \vee \mathcal{T}_\infty^{B^{(r)}} \vee \mathcal{Q}(B^{(r)})$, if $B^{(r)}$ is a tubular algebra

Periodic sequences of tame quasitilted algebras of canonical type

$\mathbb{B} = (B_1, B_2, \dots, B_{n-1}, B_n)$ periodic sequence of tame quasitilted algebras of canonical type

- B_i a tame quasitilted algebra of canonical type, for any $i \in \{1, \dots, n\}$.
- $B_i^{(r)} = B_{i+1}^{(l)}$ a tubular algebra for any $i \in \{1, \dots, n\}$, where $B_{n+1}^{(l)} = B_1^{(l)}$
- $B_i \ncong B_j$ for any $i \neq j \in \{1, \dots, n\}$

$t(\mathbb{B})$ tubular type of $\mathbb{B}$

$$t(\mathbb{B}) = \left(t(B_1^{(l)}), t(B_2^{(l)}), \dots, t(B_n^{(l)})\right) = \left(t(B_n^{(r)}), t(B_1^{(r)}), \dots, t(B_{n-1}^{(r)})\right)$$

n-tuple of sequences from $\{(2,2,2,2), (3,3,3), (2,4,4), (2,3,6)\}$

$\mathbb{B} = (B_1, B_2, \ldots, B_{n-1}, B_n)$ a periodic sequence of tame quasitilted algebras of canonical type

$\mathbb{B} \rightsquigarrow R(\mathbb{B})$ locally bounded K-category

$R(\mathbb{B})$ is an infinite periodic pushout glueing to the algebras $B_1, \ldots, B_n$

$$R(\mathbb{B}) = \bigcup_{m \in \mathbb{Z}^+} R(\mathbb{B})_m$$

$$R(\mathbb{B})_1 = B_1 \bigsqcup_{B_1^{(r)}=B_2^{(l)}} B_2 \bigsqcup_{B_2^{(r)}=B_3^{(l)}} \ldots \bigsqcup_{B_{n-2}^{(r)}=B_{n-1}^{(l)}} B_{n-1} \bigsqcup_{B_{n-1}^{(r)}=B_n^{(l)}} B_n$$

$$R(\mathbb{B})_{m+1} = B_1 \bigsqcup_{B_1^{(r)}} B_2 \bigsqcup_{B_2^{(r)}} \ldots \bigsqcup_{B_{n-1}^{(r)}} B_n \bigsqcup_{B_n^{(r)}} R(\mathbb{B})_m \bigsqcup_{B_n^{(r)}} B_1 \bigsqcup_{B_1^{(r)}} \ldots \bigsqcup_{B_{n-1}^{(r)}} B_n$$

$\mathbb{B} = (B_1, B_2, \ldots, B_{n-1}, B_n)$ a periodic sequence of tame quasitilted algebras of canonical type
$\mathbb{B} \rightsquigarrow R(\mathbb{B})$ locally bounded K-category

$B_1, B_2, \ldots, B_{n-1}, B_n$ are convex subcategories of $R(\mathbb{B})$

$R(\mathbb{B})$ admits a K-linear automorphism $g_{\mathbb{B}}$ such that $g_{\mathbb{B}}(B_1^{(l)}) = B_n^{(r)}$ and $g_{\mathbb{B}}$ acts freely on the objects of $R(\mathbb{B})$

G a group of K-linear automorphisms of $R(\mathbb{B})$ is called **admissible** if G acts freely on the objects of $R(\mathbb{B})$ and has finitely many orbits

Proposition

Let $\mathbb{B} = (B_1, B_2, \ldots, B_{n-1}, B_n)$ be a periodic sequence of tame quasitilted algebras of canonical type and G a group of K-linear automorphisms of $R(\mathbb{B})$. The following statements are equivalent.

(i) *G is an admissible group of automorphisms of $R(\mathbb{B})$.*
(ii) *$G = (\varphi g_{\mathbb{B}}^m)$ for some $m \geq 1$ and a rigid automorphism φ of $R(\mathbb{B})$.*

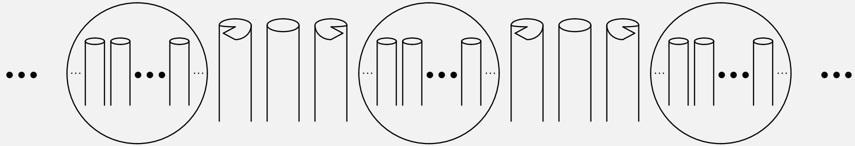
(An automorphism φ of $R(\mathbb{B})$ is said to be **rigid** if $\varphi(B_i) = B_i$ for any $i \in \{1, \ldots, n\}$.)

$\mathbb{B} = (B_1, B_2, \ldots, B_{n-1}, B_n)$ periodic sequence of tame quasitilted algebras of canonical type

$R(\mathbb{B})$ locally support-finite (Dowbor-Skowroński) locally bounded K-category

$\Gamma_{R(\mathbb{B})} = \bigvee_{q\in\mathbb{Q}} \mathcal{T}_q^{R(\mathbb{B})}$

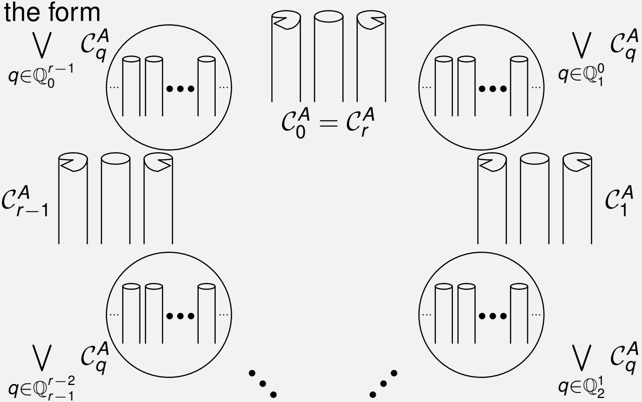
- for $q \in \mathbb{Z}$, $\mathcal{T}_q^{R(\mathbb{B})}$ is a $\mathbb{P}_1(K)$-family of semiregular tubes
- for $q \in \mathbb{Q}\setminus\mathbb{Z}$, $\mathcal{T}_q^{R(\mathbb{B})}$ is a $\mathbb{P}_1(K)$-family of stable tubes



$\bigvee_{q\in\mathbb{Q}_0^{-1}} \mathcal{T}_q^{R(\mathbb{B})}$ $\mathcal{T}_0^{R(\mathbb{B})}$ $\bigvee_{q\in\mathbb{Q}_1^{0}} \mathcal{T}_q^{R(\mathbb{B})}$ $\mathcal{T}_1^{R(\mathbb{B})}$ $\bigvee_{q\in\mathbb{Q}_2^{1}} \mathcal{T}_q^{R(\mathbb{B})}$

$\mathbb{Q}_i^{i-1} = \mathbb{Q} \cap (i-1, i)$

G admissible group of K-linear automorphisms of $R(\mathbb{B})$
$A = R(\mathbb{B})/G$ associated orbit algebra
$F = F^{\mathbb{B},G} : R(\mathbb{B}) \to R(\mathbb{B})/G = A$ Galois covering
F_λ : mod $R(\mathbb{B}) \to$ mod A the push-down functor
$R(\mathbb{B})$ locally support finite implies (by **Dowbor-Skowroński density theorem**) that F_λ is dense and $\Gamma_A = \Gamma_{R(\mathbb{B})}/G$
Hence $A = R(\mathbb{B})/G$ is a tame algebra of semiregular tubular type and Γ_A is of the form



where $\mathcal{C}_q^A = F_\lambda\left(\mathcal{T}_q^{R(\mathbb{B})}\right)$ for any $q \in \mathbb{Q}^+$

Definition

An algebra A is said to be **standard** if A admits a Galois covering $R \to R/G$ with R a simply connected locally bounded category and G an admissible group of K-linear automorphisms of R.

Theorem

Let A be an algebra. The following statements are equivalent.

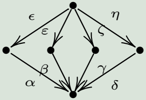
(i) *A is a standard tame algebra of semiregular tubular type.*

(ii) *$A \cong R(\mathbb{B})/G$, for a periodic sequence $\mathbb{B}$ of tame quasitilted algebras of canonical type and an admissible infinite cyclic group G of K-linear automorphisms of the locally bounded category $R(\mathbb{B})$.*

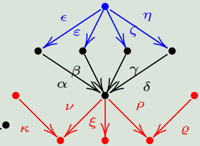
Conjecture

Every tame algebra of semiregular tubular type is a standard tame algebra.

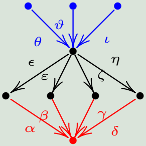
Example



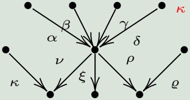
(2, 2, 2, 2)



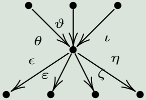
B_5



B_1

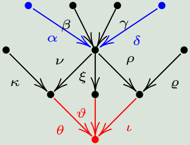


(2, 3, 6)

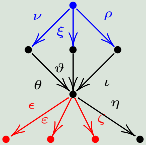


(3, 3, 3)

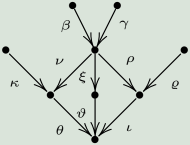
$$\epsilon\alpha + \varepsilon\beta + \zeta\gamma = 0,$$
$$\epsilon\alpha + \lambda\varepsilon\beta + \eta\varrho = 0,$$
$$\theta\epsilon = 0,\ \theta\varepsilon = 0,\ \vartheta\epsilon = 0,$$
$$\vartheta\zeta = 0,\ \iota\varepsilon = 0,\ \iota\zeta = 0,$$
$$\nu\theta + \xi\vartheta + \rho\iota = 0,$$
$$\nu\theta\eta + \mu\xi\vartheta\eta = 0,$$
$$\kappa\theta = 0,\ \varrho\iota = 0,\ \beta\nu = 0,\ \gamma\rho = 0,$$
$$\alpha\nu = 0,\ \alpha\xi = 0,\ \delta\xi = 0,\ \delta\rho = 0,$$
for some $\lambda, \mu \in K \setminus \{0, 1\}$



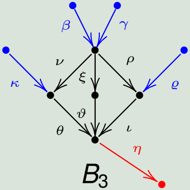
B_4



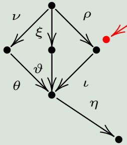
B_2



(2, 4, 4)



B_3



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