On selfinjective algebras of finite representation type

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Abstract

Main aim

We describe the structure of finite dimensional selfinjective algebras of finite representation type over a field whose stable Auslander-Reiten quiver has a sectional module which is not the middle of a short chain.

Notation

- algebra = basic indecomposable finite dimensional associative K-algebra with an identity over a fixed field K
- $\operatorname{mod} A$ = category of finite dimensional (over K) right A-modules
- $D = \operatorname{Hom}_K(-, K)\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ standard duality on $\operatorname{mod} A$
- Γ_A Auslander-Reiten quiver of A
- $\tau_A = D\mathrm{Tr}$, $\tau_A^{-1} = \mathrm{Tr}D$

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Notation

- A is called ***selfinjective*** if $A \cong D(A)$ in $\operatorname{mod} A$ (projective right A-module $\Longrightarrow$ injective right A-module).
- A is called ***finite representation type*** if $\operatorname{mod} A$ admits only finitely many indecomposable modules up to isomorphism.
- Sequence $N \longrightarrow M \longrightarrow \tau_A N$ of nonzero homomorphisms in $\operatorname{mod} A$ with N being indecomposable is called a ***short chain***, and M the ***middle*** of this short chain.

Motivation

Theorem

Let A be a nonsimple selfinjective algebra over an algebraically closed field K. The following conditions are equivalent.

1. A is of finite representation type.
2. A is a socle equivalent to an orbit algebra $\widehat{B}/G$, where B is a tilted algebra of Dynkin type $\mathbb{A}_n (n \geqslant 1)$, $\mathbb{D}_n (n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$.

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Motivation

Conjecture (Skowroński-Yamagata)

Let A be a nonsimple selfinjective algebra A over an arbitrary field K. The following conditions are equivalent.

1. A is of finite representation type.
2. A is a socle equivalent to an orbit algebra $\widehat{B}/G$, where B is a tilted algebra of Dynkin type $\mathbb{A}_n (n \geqslant 1)$, $\mathbb{B}_n (n \geqslant 2)$, $\mathbb{C}_n (n \geqslant 3)$, $\mathbb{D}_n (n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ or $\mathbb{G}_2$ and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$.

Repetitive category

B algebra

$1_B = e_1 + \cdots + e_n$ decomposition of the identity 1_B of B into a sum of pairwise orthogonal primitive idempotents

$B \rightsquigarrow \widehat{B}$ selfinjective locally bounded K-category, called the ***repetitive category*** *of* B

- $\operatorname{ob} \widehat{B} = \{e_{r,i} \mid r \in \mathbb{Z}, i \in \{1, \ldots, n\}\}$
- $\widehat{B}(e_{r,i}, e_{s,j}) = \begin{cases} e_j B e_i, & s = r, \\ D(e_i B e_j), & s = r+1, \\ 0, & \text{otherwise.} \end{cases}$

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Compositions: let $r, s, t \in \mathbb{Z}$, $i, j, k \in \{1, \ldots, n\}$

$$\widehat{B}(e_{s,j}, e_{t,k}) \times \widehat{B}(e_{r,i}, e_{s,j}) \longrightarrow \widehat{B}(e_{r,i}, e_{t,k})$$

- $r = s = t :$ $\quad e_k B e_j \times e_j B e_i \longrightarrow e_k B e_i$
- $r = s, t = s + 1 :$ $\quad e_k D(B) e_j \times e_j B e_i \longrightarrow e_k D(B) e_i$
- $s = r + 1, s = t :$ $\quad e_k B e_j \times e_j D(B) e_i \longrightarrow e_k D(B) e_i$
- otherwise: composition is zero

Repetitive category

Definition 2.1

A group G of K-linear automorphisms of $\widehat{B}$ is said to be ***admissible*** if G acts freely on the objects of $\widehat{B}$ ($gx = x \Rightarrow g = 1$ for $x \in \operatorname{ob} \widehat{B}, g \in G$) and has finitely many orbits.

Definition 2.2

Orbit category $\widehat{B}/G$ is defined as follows:

- $\operatorname{ob}(\widehat{B}/G) = (\operatorname{ob} \widehat{B})/G$
- $(\widehat{B}/G)(a,b) =$
  $= \left\{ (f_{y,x}) \in \prod_{(x,y)\in a\times b} \widehat{B}(x,y) | g f_{y,x} = f_{gy,gx}, \forall_{g\in G,(x,y)\in a\times b} \right\}$
  for all objects a, b of $\widehat{B}/G$.

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$\widehat{B}$ is selfinjective locally bounded K-category, G admissible group of K-linear automorphisms of $\widehat{B}$ $\Longrightarrow$ orbit category $\widehat{B}/G$ is selfinjective bounded K-category $\Longrightarrow$

$$\bigoplus(\widehat{B}/G) := = \bigoplus_{x,y \in \mathrm{ob}\,\widehat{B}} \widehat{B}/G(x,y)$$

is finite dimensional basic indecomposable selfinjective K-algebra

K-algebra $\bigoplus(\widehat{B}/G)$ is called the ***orbit algebra*** of $\widehat{B}$ with respect to G

$$\widehat{B}/G \equiv \bigoplus(\widehat{B}/G)$$

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Automorphisms of $\widehat{B}$

An automorphism φ of the K-category $\widehat{B}$ is said to be:

Definition 2.3

positive if for each pair $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geqslant m$ and some $j \in \{1, \dots, n\}$.

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rigid if for each pair $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$, there exists $j \in \{1, \dots, n\}$ such that $\varphi(e_{m,i}) = e_{m,j}$.

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strictly positive if it is positive but not rigid.

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Definition 2.6

Nakayama automorphism of $\widehat{B}$ is an automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ defined by

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$$

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- Automorphisms $\nu_{\widehat{B}}^r$, $r \geqslant 1$, are strictly positive automorphisms of $\widehat{B}$.
- Infinite cyclic group $(\nu_{\widehat{B}}^r)$ generated by the r-th power $\nu_{\widehat{B}}^r$ of $\nu_{\widehat{B}}$ is an admissible group of automorphisms of $\widehat{B}$, $r \geqslant 1$.

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We have the associated selfinjective orbit algebra

$$T(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^{r}) = \left\{ \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ f_2 & b_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & f_3 & b_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_{r-1} & b_{r-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \dots, b_{r-1} \in B, f_1, \dots, f_{r-1} \in D(B) \right\}$$

called the **r-*fold trivial extension algebra*** of B.

In particular, $T(B)^{(1)} \cong T(B) = B \ltimes D(B)$ is the **trivial extension** of B by the injective cogenerator $D(B)$.

Tilted algebras

H hereditary algebra, Q_H valued quiver of H

Definition 2.7

A module T in $\operatorname{mod} H$ is called ***tilting module*** if

- $\operatorname{Ext}^1_H(T,T) = 0$;
- T is a direct sum of n pairwise nonisomorphic indecomposable modules, where $n = \operatorname{rk} K_0(H)$.

Then the endomorphism algebra $B = \operatorname{End}_H(T)$ is called a ***tilted algebra of type*** Q_H.

Remark: All modules $\operatorname{Hom}_H(T,I)$ in $\operatorname{mod} B$, where I is an indecomposable injective module in $\operatorname{mod} H$, form a section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_B. Moreover, $\Delta_T \cong Q_H^{\mathrm{op}}$.

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Sectional module

A nonsimple selfinjective algebra of finite representation type $\Gamma_A^s \cong \mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver and G is an infinite cyclic group of automorphisms of the translation quiver $\mathbb{Z}\Delta$ (by the Riedtmann-Todorov theorem)

A selfinjective algebra of finite representation type $\rightsquigarrow$ $\Delta(A)$ Dynkin graph

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Sectional module

Definition 2.9

- A module M in $\mathrm{mod}\, A$ is called ***sectional*** if M is a direct sum of pairwise nonisomorphic indecomposable nonprojective modules forming a connected full valued subquiver Δ of Γ_A^s, with $\Delta(A)$ as underlying graph.
- Sectional module M in $\mathrm{mod}\, A$ is said to be ***pure*** if no direct summand of M is the radical of a projective module in $\mathrm{mod}\, A$.

Main THEOREM

Theorem

Let A be a nonsimple finite dimensional basic indecomposable selfinjective algebra of finite representation type over a field K. The following statements are equivalent.

(i) $\operatorname{mod} A$ admits a pure sectional module M which is not the middle of a short chain.

(ii) A is isomorphic to a selfinjective orbit algebra $\widehat{B}/(\rho\nu_{\widehat{B}}^2)$, where B is a tilted algebra of the form $B = \operatorname{End}_H(T)$ with H a hereditary algebra of Dynkin type and T is a tilting module in $\operatorname{mod} H$ without indecomposable projective direct summands, and ρ is a positive automorphism of $\widehat{B}$.

Theorem 1.

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Let B be a tilted algebra $\mathrm{End}_H(T)$ of Dynkin type, Δ_T the canonical section of Γ_B given by the images $\mathrm{Hom}_H(T, I)$ of indecomposable injective H-modules I via the functor $\mathrm{Hom}_H(T, -)\colon \mathrm{mod}\, H \to \mathrm{mod}\, B$, and M_T the direct sum of indecomposable B-modules lying on Δ_T. Moreover, let φ be a strictly positive automorphism of $\widehat{B}$, $A = \widehat{B}/(\varphi)$, and $F_\lambda^\varphi\colon \mathrm{mod}\, \widehat{B} \to \mathrm{mod}\, A$ the associated push-down functor. The following statements are equivalent.

(i) $F_\lambda^\varphi(M_T)$ is not the middle of a short chain in $\mathrm{mod}\, A$.

(ii) $\varphi = \rho\nu_{\widehat{B}}^2$ for a positive automorphism ρ of $\widehat{B}$.

(ii)⇒(i)

$M := F_\lambda^\varphi(M_T)$, where $M_T = \mathrm{Hom}_H(T, D(H))$.

Annihilators

A selfinjective algebra

Definition 6.1

$X \subseteq A$

- $l_A(X) = \{a \in A| \ \ aX = 0\}$ ***left annihilator*** of X in A
- $r_A(X) = \{a \in A| \ \ Xa = 0\}$ ***right annihilator*** of X in A

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Annihilators

I ideal of A, $B = A/I$

e idempotent of A such that $e + I$ is the identity of B

$(1_A = e_1 + \cdots + e_n + e_{n+1} + \cdots + e_r$

$e_1, \ldots, e_r$ pairwise orthogonal primitive idempotents of A,

$e = e_1 + \cdots + e_n$, $e_1, \ldots, e_n \notin I$, $e_{n+1}, \ldots, e_r \in I)$

- e is uniquely determined by I (up to an inner automorphism of A)
- e is called a ***residual identity*** of $B = A/I$
- $B \cong eAe/eIe$

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- e is uniquely determined by I (up to an inner automorphism of A)
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- $B \cong eAe/eIe$

Annihilators

I ideal of A, $B = A/I$

e idempotent of A such that $e + I$ is the identity of B

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Deforming ideals

A selfinjective algebra, I ideal of A, e residual identity of A/I

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I is said to be a ***deforming ideal*** of A if the following conditions are satisfied:

- $l_{eAe}(I) = eIe = r_{eAe}(I)$;
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Algebra A[I]

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$eAe/eIe \xrightarrow{\sim} A/I$

I is $(eAe/eIe) - (eAe/eIe)$-bimodule

$A[I]$:

- $A[I] = (eAe/eIe) \oplus I$
- $(b, x) \cdot (c, y) = (bc, by + xc + xy)$,
  for $b, c \in eAe/eIe$ and $x, y \in I$
- $1_{A[I]} = (e + eIe, 1_A - e)$

$A[I]$ is K-algebra

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Algebra A[I]

Theorem 2. (Skowroński-Yamagata)

Let A be a selfinjective algebra and I a deforming ideal of A. The following statements hold.

(i) $A[I]$ is a selfinjective algebra with the same Nakayama permutation as A and I is a deforming ideal of $A[I]$.

(ii) A and $A[I]$ are socle equivalent.

(iii) A and $A[I]$ are stably equivalent.

Algebra A[I]

Theorem 3. (Skowroński-Yamagata)

Let A be a selfinjective algebra, I an ideal of A, $B = A/I$ and e an idempotent of A. Assume that $r_A(I) = eI$ and Q_B is acyclic. The following statements hold.

(i) $A[I]$ is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.

(ii) Moreover, if $e_i \neq e_{\nu(i)}$, for any primitive summand e_i of e and ν the Nakayama permutation of A then the algebras A and $A[I]$ are isomorphic.

(i)⇒(ii)

Let

- M be a pure sectional A-module which is not the middle of a short chain
- Δ be a full valued subquiver of Γ_A^s given by the indecomposable direct summands of M
- $I = r_A(M)$, $B = A/I$
- $H = \mathrm{End}_A(M) = \mathrm{End}_B(M)$

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Then

- $\Gamma_A^s \cong \mathbb{Z}\Delta/G$ and Δ is a Dynkin quiver, $\bar{\Delta} = \Delta(A)$
- M is right B-module which is not the middle of a short chain in $\mathrm{mod}\, B \Rightarrow H$ is a hereditary algebra of Dynkin type with $Q_H = \Delta^{\mathrm{op}}$
- M is a faithful ($\Rightarrow$ sincere) right B-module which is not the middle of a short chain in $\mathrm{mod}\, B \Rightarrow B$ is a tilted algebra and M is a tilting B-module
- $T := D(M)$ is a tilting H-module, $B \cong \mathrm{End}_H(T)$, $M \cong \mathrm{Hom}_H(T, D(H)) =: M_T$
  (in particular, indecomposable direct summands of M form the canonical section $\Delta_T = \Delta$ of the connecting component $C_T = \Gamma_B$)

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Moreover

- T has no indecomposable projective direct summand
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1. A is socle equivalent to $A[I]$.
2. A is stably equivalent to $A[I]$.
3. $A[I]$ is a selfinjective algebra with the same Nakayama permutation as A and I is a deforming ideal of $A[I]$.
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- M is a pure sectional module not the middle of a short chain in $\operatorname{mod} A \Longrightarrow M = F_\lambda^\varphi(M_T)$ is a pure sectional module not the middle of a short chain in $\operatorname{mod} A[I]$
- $A[I] \cong \widehat{B}/(\varphi\nu_{\widehat{B}}) \Longrightarrow A[I] \cong \widehat{B}/(\rho\nu_{\widehat{B}}^2)$ (by Theorem 1.)
- $A[I] \cong \widehat{B}/(\rho\nu_{\widehat{B}}^2) \Longrightarrow e_i \neq e_{\nu(i)}$ for any primitive direct summand e_i of the common residual identity e of $A/I \cong A[I]/I$ (by Theorem 2.)
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