Large universal deformation rings

Frauke M. Bleher

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Motivation.

k = algebraically closed field, $\mathrm{char}(k) = p > 0$

$\mathcal{O}$ = complete discrete valuation ring of characteristic 0 with residue field k

G = finite group

V = finitely generated kG-module

Classical Problem:
Is there a lift of V to $\mathcal{O}$, i.e. is there an $\mathcal{O}$-free $\mathcal{O}G$-module M with $k \otimes_{\mathcal{O}} M \cong V$ as kG-modules?

Classical Answer: (J.A. Green 1959)
Yes, if $\mathrm{Ext}^2_{kG}(V, V) = 0$.

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Yes, if $\mathrm{Ext}^2_{kG}(V, V) = 0$.

Questions:

(i) How can all possible lifts of V to $\mathcal{O}$ be described?

(ii) To which complete local commutative noetherian $\mathcal{O}$-algebras R with residue field k can V be lifted?

(A lift of V to R is an R-free RG-module M together with a kG-module isomorphism $\phi : k \otimes_R M \to V$.)

Is there one particular such complete local commutative noetherian $\mathcal{O}$-algebra from which all these lifts arise?

To answer these questions, we need a more systematic way to study lifts.

This leads to Mazur's deformation rings and deformations.

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Universal deformation rings.

Let $\mathcal{C}$ be the category of all complete local commutative noetherian $\mathcal{O}$-algebras R with residue field k.

Theorem (Mazur 1980's; B-Chinburg 2000)

Suppose $\underline{\mathrm{End}}_{kG}(V) \cong k$.
Then there exists a ring $R(G,V)$ in $\mathcal{C}$ and a lift $U(G,V)$ of V over $R(G,V)$ such that for every $R \in \mathrm{Ob}(\mathcal{C})$ and every lift M of V over R there is a unique homomorphism $\alpha : R(G,V) \to R$ in $\mathcal{C}$ with

$$M \cong R \otimes_{R(G,V),\alpha} U(G,V).$$

In other words, every lift of V over a ring R in $\mathcal{C}$ arises uniquely, up to isomorphism, from $R(G,V)$ and $U(G,V)$.

$R(G,V)$ is called the universal deformation ring of V.
The isomorphism class of $U(G,V)$ is called the universal deformation of V over $R(G,V)$.

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Suppose now that V satisfies $\underline{\mathrm{End}}_{kG}(V) \cong k$.

Then $V \cong V_0 \oplus P$, where V_0 is indecomposable non-projective with $\underline{\mathrm{End}}_{kG}(V_0) \cong k$ and P is projective.

Since $R(G, V) \cong R(G, V_0)$ (B-Chinburg), we can concentrate on indecomposable V.

So V belongs to a unique block B of kG. Let D be a defect group of B.

Question (B-Chinburg 2000)

Is it true that $R(G, V)$ is always isomorphic to a subquotient ring of $\mathcal{O}D$?

In particular, is it true that $R(G, V)/pR(G, V)$ is finite dimensional over k?

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Importance of this question for deformations of Galois representations

Suppose Γ is a profinite Galois group (satisfying Mazur's finiteness condition), and

V is a finite dimensional k-vector space with continuous Γ-action and $\mathrm{End}_{k\Gamma}(V) \cong k$.

If $R(\Gamma, V)/pR(\Gamma, V)$ is finite dimensional over k, this may lead to an explicit presentation of $R(\Gamma, V)$ (see work by Böckle).

Since the Γ-action on V factors through a finite quotient group G of Γ, it follows that if $R(G, V)/pR(G, V)$ is not finite dimensional over k then $R(\Gamma, V)/pR(\Gamma, V)$ cannot be either.

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Positive results. Cyclic blocks.

Let V be an indecomposable kG-module with $\underline{\mathrm{End}}_{kG}(V) \cong k$ belonging to a block B with defect group D.

Theorem (B-Chinburg 2000)

Suppose D is cyclic of order p^d. Then $R(G,V)$ is isomorphic to

$$\text{either } \mathcal{O}, \text{ or } \mathcal{O}/p^d\mathcal{O}, \text{ or } \mathrm{Inv}_E(\mathcal{O}D)/\mathcal{O}s$$

where E is a certain group of automorphisms of D and s is either zero or the trace element in D.

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Positive results. Blocks with dihedral defect groups.

Let V be an indecomposable kG-module with $\underline{\mathrm{End}}_{kG}(V) \cong k$ belonging to a block B with defect group D.

Theorem (B; B-Llosent-Schaefer $\leq$ 2011)

Suppose $\mathrm{char}(k) = 2$ *and* D *is a dihedral group of order* $2^d \geq 4$. *If* B *is Morita equivalent to a principal block, then* $R(G, V)$ *is isomorphic*

either to a quotient ring of $\mathcal{O}$,
or to a subalgebra of $\mathcal{O}[\mathbb{Z}/2]$,
or to $\mathcal{O}[\mathbb{Z}/2 \times \mathbb{Z}/2]$,
or to $\mathcal{O}[[t]]/(t \cdot p_d(t), 2 \cdot p_d(t))$

where $p_d(t) \in \mathcal{O}[t]$ *is an explicitly given distinguished polynomial of degree* $2^{d-2} - 1$.

In all these cases, it can be shown that $R(G, V)$ *is isomorphic to a subquotient ring of* $\mathcal{O}D$.

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Negative result.

Theorem (de Smit-Rainone (2010) $p \geq 5$; B (2012) $p = 2, 3$)

For each prime p, there exists a finite group G and a finitely generated indecomposable kG-module V with $\underline{\mathrm{End}}_{kG}(V) \cong k$ *such that*

$$R(G,V)/pR(G,V) \cong k[[t]].$$

In particular, R(G, V) is not isomorphic to a subquotient ring of $\mathcal{O}D$.

Note:

- For $p = 2$, V can be chosen to belong to a tame block.
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Example for $p = 2$.

Let $\overline{G}$ be a simple group with dihedral Sylow 2-subgroups.
Let G be a central extension of $\overline{G}$ by an involution.
Then either $G = \mathrm{SL}(2, q)$ for some odd prime power q, or $G = 2.A_7$.

Case $G = \mathrm{SL}(2, q)$, $q \equiv 3 \bmod 4$.

The principal block B of kG is Morita equivalent to $\Lambda = kQ/I$ where

$Q =$

and

$$\begin{aligned} I &= \langle \delta\beta - \kappa\lambda\kappa, \gamma\eta - \lambda\kappa\lambda, \lambda\delta - \gamma\beta\gamma, \eta\kappa - \beta\gamma\beta, \\ &\quad \beta\lambda - \eta(\delta\eta)^{2^{d-1}-1}, \kappa\gamma - \delta(\eta\delta)^{2^{d-1}-1}, \delta\beta\gamma, \gamma\eta\delta, \eta\kappa\lambda\rangle. \end{aligned}$$

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$$I = \langle \delta\beta - \kappa\lambda\kappa, \gamma\eta - \lambda\kappa\lambda, \lambda\delta - \gamma\beta\gamma, \eta\kappa - \beta\gamma\beta, \beta\lambda - \eta(\delta\eta)^{2^{d-1}-1}, \kappa\gamma - \delta(\eta\delta)^{2^{d-1}-1}, \delta\beta\gamma, \gamma\eta\delta, \eta\kappa\lambda \rangle.$$

Example for $p = 2$.

Let $\overline{G}$ be a simple group with dihedral Sylow 2-subgroups.
Let G be a central extension of $\overline{G}$ by an involution.
Then either $G = \mathrm{SL}(2, q)$ for some odd prime power q, or $G = 2.A_7$.

Case $G = \mathrm{SL}(2, q)$, $q \equiv 3 \bmod 4$.

The principal block B of kG is Morita equivalent to $\Lambda = kQ/I$ where

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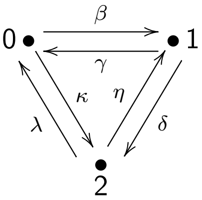

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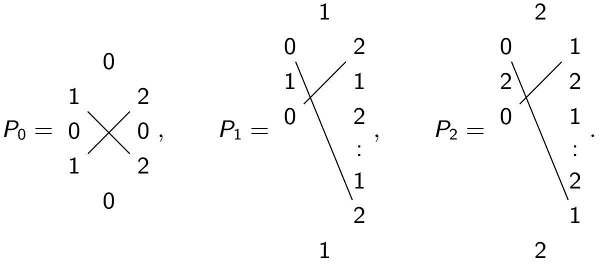

Consider the Λ-module $T = \begin{matrix} & 0 & \\ 1 & & 2 \\ & & 0 \end{matrix}$, i.e. T is a 4-dimensional k-vector space with basis b_1, b_2, b_3, b_4 such that Λ acts on this basis as follows: $\beta \mapsto E_{12}$, $\kappa \mapsto E_{32}$, $\lambda \mapsto E_{43}$ where E_{ji} sends b_i to b_j and all other basis elements to zero.

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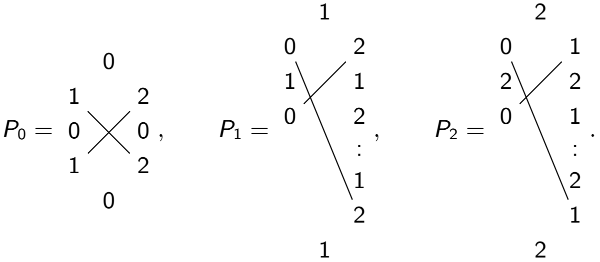

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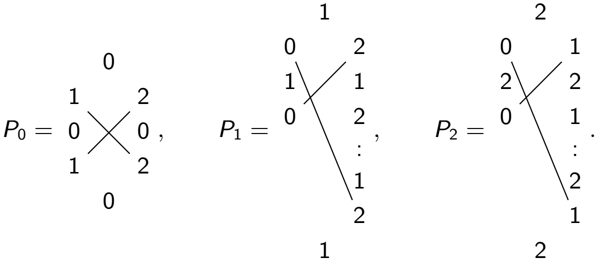

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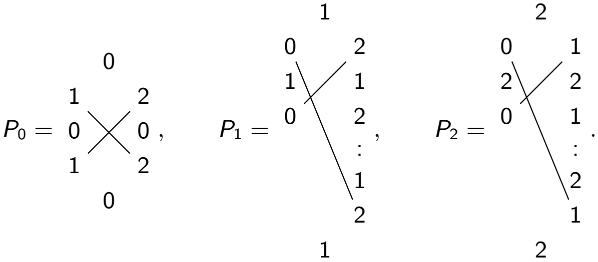

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Case $G = \mathrm{SL}(2,q)$, $q \equiv 3 \bmod 4$ (continued).

We have $\underline{\mathrm{End}}_\Lambda(T) \cong k$ and $\mathrm{Ext}^i_\Lambda(T,T) \cong k$ for $i = 1, 2$.

If V is the kG-module corresponding to T under the Morita equivalence $B \sim_M \Lambda$, then this implies that either

$$R(G,V)/2R(G,V) \cong k[[t]]$$

or

$$R(G,V)/2R(G,V) \cong k[[t]]/(t^r)$$

for some $r \geq 2$.

To show that $R(G,V)/2R(G,V) \cong k[[t]]$, it suffices to show that T has a lift L over $k[[t]]$ such that L/t^2L is not the trivial lift of T over $k[[t]]/(t^2)$, i.e. $L/t^2L \ncong k[[t]]/(t^2) \otimes_k T$.

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Let L be a free $k[[t]]$-module with basis B_1, B_2, B_3, B_4. Define a Λ-action on L as follows:

$$\beta \mapsto E_{12}, \quad \kappa \mapsto E_{32}, \quad \lambda \mapsto E_{43}, \quad \gamma \mapsto t\,E_{41}.$$

When viewed as a Λ-module, L is isomorphic to the infinite dimensional Λ-module

0 0 0 0
$\cdots$ 1 2 $\cdots$ 1 2 1 2 1 2
0 0 0 0

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$$\begin{array}{ccccc} & & 0 & & \\ \cdots & 1 & & 2 & \\ & & & & 0 \end{array} \qquad \begin{array}{ccccccccccccc} & & 0 & & & & 0 & & & & {\color{red}0} & & \\ \cdots & 1 & & 2 & & 1 & & 2 & & {\color{red}1} & & {\color{red}2} & \\ & & & & 0 & & & & 0 & & & & {\color{red}0} \end{array}$$

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