Characterization of representation types of quivers, using the Gabriel-Roiter measures

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Two years ago (2010)...... Tokyo

History of GR measure

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- Gabriel (1973), introduced "Roiter measure"
- Ringel (2004), algebras of infinite representation type.

Definition

Let Λ be an artin algebra.

- The *Gabriel-Roiter measure* $\mu(M)$:

$$\mu(M) = \max_{N \subset M}\{\mu(N)\} + \begin{cases} 0 & \text{if } M \text{ is decomposable,} \\ \frac{1}{2^{|M|}} & \text{if } M \text{ is indecomposable} \end{cases}.$$

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- A rational number μ is called a GR measure for Λ if $\mu(M) = \mu$ for some indecomposable Λ-module M.
- If X is cogenerated by Y, then $\mu(X) \leq \mu(Y)$.

An example, finite representation type

Path algebra: $1 \longrightarrow 2 \longrightarrow 3$,

$$
\begin{tikzcd}
 & & P_1 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
S_3 \arrow[ur] & & S_2 \arrow[ur] & & S_1
\end{tikzcd}
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\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & \frac{7}{8} \arrow[dr] & & \\
 & \frac{3}{4} \arrow[ur] \arrow[dr] & & \frac{3}{4} \arrow[dr] & \\
\frac{1}{2} \arrow[ur] & & \frac{1}{2} \arrow[ur] & & \frac{1}{2}
\end{tikzcd}$$

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\begin{tikzcd}
& P_2 \arrow[dr, shift left] \arrow[dr, shift right] & & P_4 \cdots \\
P_1 \arrow[ur, shift left] \arrow[ur, shift right] & & P_3 \arrow[ur, shift left] \arrow[ur, shift right] & \cdots ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \\
R_3 \arrow[d, shift left] \\
R_2 \arrow[u, shift left] \arrow[d, shift left] \\
R_1 , \arrow[u, shift left]
\end{tikzcd}
\qquad
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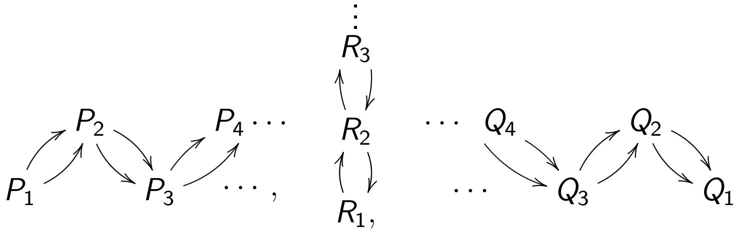

GR measures: $\frac{1}{2}, \frac{5}{8}, \frac{21}{32}, \cdots \,|\, \frac{3}{4}, \frac{13}{16}, \frac{213}{256}, \cdots \,|\, \cdots, \frac{107}{128}, \frac{27}{32}, \frac{7}{8}.$

Ringel's Partition

Theorem (Ringel)

Let Λ be a representation-infinite artin algebra. Then there are GR measures I_i and I^i

$$I_1 < I_2 < I_3 < \ldots \quad \ldots < I^3 < I^2 < I^1$$

such that any other GR measure I satisfies $I_i < I < I^i$ for all i.

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- **Successor Lemma (Ringel)** Any GR measure, not maximal, has a direct successor.
- 'Predecessor Lemma' does **Not** hold.
- Fix a GR measure μ_0. We obtain a sequence of GR measures by taking direct predecessors and direct successors:

$$\ldots < \mu_{-2} < \mu_{-1} < \mu_0 < \mu_1 < \mu_2 < \ldots$$

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Let Λ be an artin algebra. Then T.F.A.E.:

- *Λ is of finite representation type.*
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- *There is a finite GR segment.*

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- *There is only one GR segment.*
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If Λ is of infinite representation type: Then a GR segment is of type $\mathbb{N}$, type $-\mathbb{N}$ or type $\mathbb{Z}$.

Known result and question

Theorem

Let Q be an Euclidean quiver of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$.

1. *The number of GR segments is bounded by b.*

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Question

Let Q be a finite connected acyclic quiver. Are the following statements equivalent?

1. *Q is of wild representation type.*
2. *There are infinitely many GR segments.*
3. *There are infinitely many GR segments of type $\mathbb{N}$.*

Wake Up!

Two years later (2012)...... Bielefeld

Main theorem

Theorem

Let Q be a finite connected acyclic quiver. Then T.F.A.E.

1. *Q is of wild representation type.*
2. *There are infinitely many GR segments.*

Proof I, part I: two vertices

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha_1"] \arrow[r, phantom, "\vdots"] \arrow[r] \arrow[r, bend right, "\alpha_n"'] & 2
\end{tikzcd}$$

Let $I^m = \{1, 2, 4, \ldots, 2m, 2m+1\}$ and $\mu^m = \sum_{i \in I^m} \frac{1}{2^i}$. Then these μ^m's do NOT admit direct predecessors and are contained in different GR segments of type $\mathbb{N}$.

Proof I, part II: at least three vertices

Theorem

Let Q be a wild quive with $n \geq 3$ vertices. For each indecomposable regular module X, there is an indecomposable regular module Y such that the GR measures $\mu(X)$ and $\mu(Y)$ are in different GR segments.

Proof I, part II: Continue

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3. $\tau^i T$ is again a tilting module, and $\mathrm{Hom}(\tau^i T, X[k]) = 0$ for all $k \geq 1$. (X not necessary quasi-simple).

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5. All $X[k]$ are cogenerated by $\tau^{i+1} T$ and take Y a summand of $\tau^{i+1} T$ with maximal GR measure.

$\square$

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3. Let $Y \cong \tau^m M$ such that M is cogenerated by Y and $\mathrm{Ext}^1(M, Y) = 0$.
4. Ringel's Simplification: $\mathcal{F}(M)$ the full subcategory consisting of modules N which have filtrations $0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_r = N$ such that $N_i/N_{i-1} \cong M$. Then $\mathcal{F}(M)$ contains infinitely many indecomposable objects.

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5. If $N \in \mathcal{F}(M)$, then N is cogenerated by Y. In particular, $\mu(X)$ and $\mu(Y)$ are in different GR segments.

Interested?

Let Λ be an artin aglebra. The following are equivalent (?):

1. Λ is of finite representation type.
2. There is a finite GR segment.
3. There is a finite AR component.

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Let Λ be an artin aglebra. The following are equivalent (?):

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2. There is a finite GR segment.
3. There is a finite AR component.
4. There is only one GR segment.
5. There is only one AR component?

(Survive 2012)
I will talk ...... in 2014, Beijing

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Thank you!