Characterization of representation types of quivers, using the Gabriel-Roiter measures

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Two years ago (2010)...... Tokyo
Roiter’s proof of BTh I (1968) marks the beginning of the new representation theory (of finite dimensional algebras).
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History of GR measure

- Roiter’s proof of BTh I (1968) marks the beginning of the new representation theory (of finite dimensional algebras)
- Gabriel (1973), introduced ”Roiter measure”
- Ringel (2004), algebras of infinite representation type.
Let $\Lambda$ be an artin algebra.

- The Gabriel-Roiter measure $\mu(M)$:

$$
\mu(M) = \max_{N \subseteq M} \{ \mu(N) \} + \begin{cases} 
0 & \text{if } M \text{ is decomposable}, \\
\frac{1}{2|M|} & \text{if } M \text{ is indecomposable}
\end{cases}
$$
Definition

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- A rational number $\mu$ is called a GR measure for $\Lambda$ if $\mu(M) = \mu$ for some indecomposable $\Lambda$-module $M$. 

Bo Chen  IAZ, Uni. Stuttgart Representation types of quivers, using GR measure
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- A rational number $\mu$ is called a GR measure for $\Lambda$ if $\mu(M) = \mu$ for some indecomposable $\Lambda$-module $M$.

- If $X$ is cogenerated by $Y$, then $\mu(X) \leq \mu(Y)$. 

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An example, finite representation type

Path algebra: $1 \rightarrow 2 \rightarrow 3$, 

Diagram: 

$$
\begin{array}{c}
P_1 \\
| \downarrow \\
P_2 \\
| \downarrow \\
S_3 \\
| \downarrow \\
S_2 \\
| \downarrow \\
S_1
\end{array}
$$
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Affine quiver $\widetilde{A}_1$: $\bullet \xrightarrow{\sim} \bullet$
An example, infinite representation type

Affine quiver $\tilde{A}_1$: $\bullet \xrightarrow{\ldots} \bullet$

$P_1 \xrightarrow{P_2} P_3 \xrightarrow{P_4} \ldots$, $R_1 \xleftarrow{R_2} R_3 \xleftarrow{\ldots}$

$Q_1 \xleftarrow{Q_2} Q_3 \xleftarrow{Q_4} \ldots$

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Representation types of quivers, using GR measure
An example, infinite representation type

Affine quiver $\tilde{A}_1$: $\bullet \xrightarrow{\sim} \bullet$

$$
\begin{align*}
&\quad P_1 \xrightarrow{} P_2 \xrightarrow{} P_3 \xrightarrow{} \cdots, \\
&\quad R_1, R_2, \ldots \xleftarrow{}
\end{align*}
$$

$$
\begin{align*}
&\quad Q_1 \xrightarrow{} Q_2 \xrightarrow{} Q_3 \xrightarrow{} \cdots, \\
&\quad \cdots \xrightarrow{R_3}
\end{align*}
$$

GR measures: $\frac{1}{2}, \frac{5}{8}, \frac{21}{32}, \cdots | \frac{3}{4}, \frac{13}{16}, \frac{213}{256}, \cdots | \cdots, \frac{107}{128}, \frac{27}{32}, \frac{7}{8}$. 
Theorem (Ringel)

Let \( \Lambda \) be a representation-infinite artin algebra. Then there are GR measures \( I_i \) and \( I^i \)

\[
l_1 < l_2 < l_3 < \ldots \quad \ldots < l^3 < l^2 < l^1
\]

such that any other GR measure \( I \) satisfies \( I_i < I < I^i \) for all \( i \).
Direct successors and direct predecessors

A GR measure $J$ is called a **direct successor** of $I$ if $I < J$ and $I \leq I' \leq J$ implies $I' = I$ or $I' = J$. 

Successor Lemma (Ringel)

Any GR measure, not maximal, has a direct successor.

'Predecessor Lemma' does not hold.

Fix a GR measure $\mu_0$. We obtain a sequence of GR measures by taking direct predecessors and direct successors:

$\ldots < \mu_{-2} < \mu_{-1} < \mu_0 < \mu_1 < \mu_2 < \ldots$
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Definition

The set of GR measures is a totally ordered set. A connected component of the Hasse diagram is called a GR segments.
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Theorem
Let $\Lambda$ be an artin algebra. Then T.F.A.E.:

- $\Lambda$ is of finite representation type.
- There is only one GR segment.
- There is a finite GR segment.
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- There is only one GR segment.
- There is a finite GR segment.

If $\Lambda$ is of infinite representation type: Then a GR segment is of type $\mathbb{N}$, type $-\mathbb{N}$ or type $\mathbb{Z}$. 
Theorem

Let $Q$ be an Euclidean quiver of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

1. The number of GR segments is bounded by $b$.

2. The number of GR segments of type $Z$ is bounded by $a$. Here $a$ and $b$ are numbers relating to the ranks of exceptional tubes.

Question

Let $Q$ be a finite connected acyclic quiver. Are the following statements equivalent?

1. $Q$ is of wild representation type.
2. There are infinitely many GR segments.
3. There are infinitely many GR segments of type $N$. 

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Known result and question

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1. The number of GR segments is bounded by $b$.
2. The number of GR segments of type $\mathbb{Z}$ is bounded by $a$.

Here $a$ and $b$ are numbers relating to the ranks of exceptional tubes.

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Let $Q$ be a finite connected acyclic quiver. Are the following statements equivalent?

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Bo Chen IAZ, Uni. Stuttgart Representation types of quivers, using GR measure
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Let $Q$ be a finite connected acyclic quiver. Are the following statements equivalent?

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2. There are infinitely many GR segments.
3. There are infinitely many GR segments of type $\mathbb{N}$.
Wake Up!
Two years later (2012)...... Bielefeld
Theorem

Let $Q$ be a finite connected acyclic quiver. Then T.F.A.E.

1. $Q$ is of wild representation type.
2. There are infinitely many GR segments.
Proof I, part I: two vertices

Let \( I^m = \{1, 2, 4, \ldots, 2m, 2m + 1\} \) and \( \mu^m = \sum_{i \in I^m} \frac{1}{2^i} \). Then these \( \mu^m \)'s do NOT admit direct predecessors and are contained in different GR segments of type \( \mathbb{N} \).
Theorem
Let $Q$ be a wild quive with $n \geq 3$ vertices. For each indecomposable regular module $X$, there is an indecomposable regular module $Y$ such that the GR measures $\mu(X)$ and $\mu(Y)$ are in different GR segments.
Proof.

1. Let $T = \bigoplus T_i$ be a basic regular tilting module.
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1. Let $T = \bigoplus T_i$ be a basic regular tilting module.
2. $\tau^iT$, $i \geq 0$, such that $\text{Hom}(\tau^iT, \tau^{-j}X) = 0$ for all $j \geq 0$.

3. $\tau^iT$ is again a tilting module, and $\text{Hom}(\tau^iT, X[k]) = 0$ for all $k \geq 1$. ($X$ not necessary quasi-simple).

4. Property of tilting module: $\text{Hom}(T, M) = 0$ iff $M$ is cogenerated by $\tau^iT$.

5. All $X[k]$ are cogenerated by $\tau^iT+1$ and take $Y$ a summand of $\tau^iT+1$ with maximal GR measure.
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Proof I, part II: Continue

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5. All $X[k]$ are cogenerated by $\tau^{i+1} T$ and take $Y$ a summand of $\tau^{i+1} T$ with maximal GR measure.
1. Let $M$ be an indecomposable module with $\text{End}(M) = k$ and $\text{Ext}^1(M, M) \neq 0$. 
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2. (If necessary, shift $\tau^i$, $i \geq 0$) there is a monomorphism $X \rightarrow M$.  

3. Let $Y \sim = \tau^m M$ such that $M$ is cogenerated by $Y$ and $\text{Ext}^1(M, Y) = 0$.

4. Ringel's Simplification: $F(M)$ the full subcategory consisting of modules $N$ which have filtrations $0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_r = N$ such that $N_i / N_{i-1} \sim M$. Then $F(M)$ contains infinitely many indecomposable objects.

5. If $N \in F(M)$, then $N$ is cogenerated by $Y$. In particular, $\mu(X)$ and $\mu(Y)$ are in different GR segments.
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3. Let $Y \cong \tau^m M$ such that $M$ is cogenerated by $Y$ and $\text{Ext}^1(M, Y) = 0$. 

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Proof II, anonymous

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5. If $N \in \mathcal{F}(M)$, then $N$ is cogenerated by $Y$. In particular, $\mu(X)$ and $\mu(Y)$ are in different GR segments.
Let \( \Lambda \) be an artin algebra. The following are equivalent (\( ? \)):

1. \( \Lambda \) is of finite representation type.
2. There is a finite GR segment.
3. There is a finite AR component.
Let $\Lambda$ be an artin algebra. The following are equivalent (?):

1. $\Lambda$ is of finite representation type.
2. There is a finite GR segment.
3. There is a finite AR component.
4. There is only one GR segment.
5. There is only one AR component?
I will talk ...... in 2014, Beijing
(Survive 2012)
I will talk …… in 2014, Beijing

Thank you!