Normality of maximal orbit closures for Euclidean quivers

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Representation space

Assumption

Throughout k is a fixed algebraically closed field.

Definition

For a quiver Q and a dimension vector $\mathbf{d}$ we define

$$\mathrm{rep}_Q(\mathbf{d}) := \{M \in \mathrm{rep}\, Q : M(x) = k^{\mathbf{d}(x)} \text{ for each } x \in Q_0\}.$$

Remarks

1. $\mathrm{rep}_Q(\mathbf{d}) = \prod_{\alpha \in Q_1} \mathbb{M}_{\mathbf{d}(t\alpha)\times\mathbf{d}(s\alpha)}(k)$.
2. For each $M \in \mathrm{rep}\, Q$ with $\mathbf{dim}\, M = \mathbf{d}$ there exists $N \in \mathrm{rep}_Q(\mathbf{d})$ with $M \simeq N$.
3. $\mathrm{GL}(\mathbf{d}) := \prod_{x \in Q_0} \mathrm{GL}(\mathbf{d}(x))$ acts such that

$$\mathrm{GL}(\mathbf{d})M = \mathrm{GL}(\mathbf{d})N \iff M \simeq N.$$

Guiding problem

Problem

Is $\overline{\mathrm{GL}(\mathbf{d})M}$ a normal variety?

Facts

1. *If Q is of Dynkin type $\mathbb{A}$ or $\mathbb{D}$, then $\overline{\mathrm{GL}(\mathbf{d})M}$ is always normal. [B/Zwara]*
   (The type $\mathbb{E}$ is open.)
2. *If Q is of infinite type, then there exists M such that $\overline{\mathrm{GL}(\mathbf{d})M}$ is not normal. [Zwara, Chindris]*

Remark

The orbit $\mathrm{GL}(\mathbf{d})M$ in 2 is not maximal, i.e., there exists N such that $\mathrm{GL}(\mathbf{d})M \subseteq \overline{\mathrm{GL}(\mathbf{d})N}$ and $M \not\simeq N$.

Main result

Problem

Is $\overline{\mathrm{GL}(\mathbf{d})M}$ a normal variety if $\overline{\mathrm{GL}(\mathbf{d})M}$ is maximal?

Remark

Yes, if Q is Dynkin ($\overline{\mathrm{GL}(\mathbf{d})M} = \mathrm{rep}_Q(\mathbf{d})$).

Theorem

Yes, if Q is Euclidean.

Proof.

1. $\mathrm{Ext}^1_Q(M, M) = 0 \implies \overline{\mathrm{GL}(\mathbf{d})M} = \mathrm{rep}_Q(\mathbf{d})$.
2. $\mathrm{Ext}^1_Q(M, M) \neq 0 \implies M$ is regular.

Semi-invariants

Conclusion

We may assume that M is regular.

Assumption

For simplicity of the presentation we assume that Q is the Kronecker quiver.

Notation (Schofield's semi-invariants)

For $\lambda \in \mathbb{P}^1(k)$:

1. we denote by R_λ the corresponding regular quasi-simple representation,
2. we construct $c_\lambda : \mathrm{rep}_Q(\mathbf{d}) \to k$ such that

$$c_\lambda(N) = 0 \iff \mathrm{Hom}_Q(R_\lambda, N) \neq 0.$$

Equations

Proposition

Assume $\mathbf{d} = (p, p)$. *There exist* $\lambda_0, \ldots, \lambda_p \in \mathbb{P}^1(k)$ *and* $\mu_1, \ldots, \mu_p \in k$ *such that*

$$\overline{\mathrm{GL}(\mathbf{d})M} = \{N \in \mathrm{rep}_Q(\mathbf{d}) : c_{\lambda_i}(N) = \mu_i \cdot c_{\lambda_0}(N) \text{ for each } i = 1, \ldots, p\}.$$

Corollary

$\overline{\mathrm{GL}(\mathbf{d})M}$ *is a (set-theoretic) complete intersection.*

Conclusion

In order to prove normality we need to calculate $\dim \mathrm{Sing}\, \overline{\mathrm{GL}(\mathbf{d})M}$ (more precisely, of the scheme given by the above equations).

Tangent space

Facts

Let $N \in \operatorname{rep}_Q(\mathbf{d})$.

1. *[Voigt] There is a (canonical) epimorphism*

$$\pi_N : T_N \operatorname{rep}_Q(\mathbf{d}) \twoheadrightarrow \operatorname{Ext}^1_Q(N, N).$$

2. *[Riedtmann/Zwara] If* $\dim_k \operatorname{Hom}_Q(R_\lambda, N) = 1$, *then*

$$\operatorname{Ker} \partial c_\lambda = \pi_N^{-1}(R_\lambda\text{-}\textit{exact sequences}),$$

where a sequence $0 \to N \to L \to N \to 0$ *is* R_λ*-exact, if the induced sequence*

$$0 \to \operatorname{Hom}_Q(R_\lambda, N) \to \operatorname{Hom}_Q(R_\lambda, L) \to \operatorname{Hom}_Q(R_\lambda, N) \to 0$$

is exact.

Nonsingular points

Proposition

In each irreducible component of $\mathcal{V}(c_{\lambda_0}, \dots, c_{\lambda_p})$ *there exists* N *such that*

$$\dim \bigcap_{i=0}^{p} \operatorname{Ker} \partial_{c_{\lambda_i}} = \dim \operatorname{rep}_Q(\mathbf{d}) - (p+1).$$

Immediate Consequence

$\mathcal{I}(\overline{\mathrm{GL}(\mathbf{d})M}) = (c_{\lambda_1} - \mu_1 \cdot c_{\lambda_0}, \dots, c_{\lambda_p} - \mu_p \cdot c_{\lambda_0})$.

Proof of normality.

1. $\overline{\mathrm{GL}(\mathbf{d})M} = \mathcal{V}(c_{\lambda_0}, \dots, c_{\lambda_p}) \cup (\overline{\mathrm{GL}(\mathbf{d})M} \cap \{\text{regular representation}\})$.
2. $\overline{\mathrm{GL}(\mathbf{d})M} \cap \{\text{regular representation}\} =$ union finitely many orbits.
3. Degenerations of codimension one are regular [Zwara].

Generalization

Theorem

Let M be a τ_Λ-periodic Λ-module of dimension vector $\mathbf{d}$ over a tame concealed canonical algebra Λ.
If $\overline{\mathrm{GL}(\mathbf{d})M}$ is maximal in $\mathrm{mod}_\Lambda(\mathbf{d})$ then $\overline{\mathrm{GL}(\mathbf{d})M}$ is normal iff at least one of the following conditions is satisfied:

1. $\tau M \not\simeq M$,
2. *there do not exist dimension vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ such that*
   - $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$,
   - $\langle \mathbf{d}_1, \mathbf{d}_1 \rangle = 0 = \langle \mathbf{d}_2, \mathbf{d}_2 \rangle$,
   - $\langle \mathbf{d}_1, \mathbf{d}_2 \rangle = 2$ *(consequently,* $\langle \mathbf{d}_2, \mathbf{d}_1 \rangle = -2$*).*

Remark

Non-normal maximal orbit closure appear only in the case $(2, 2, 2, 2)$.
If $\overline{\mathrm{GL}(\mathbf{d})M}$ is not normal, then M is quasi-simple indecomposable from a homogeneous tube.