Combinatorial formulas for cluster algebras from surfaces

Ilke Canakci

University of Connecticut, 196 Auditorium Road, Unit 3009 Storrs, CT 06269, USA

ilke.canakci@uconn.edu

Abstract

Recall from [MSW] that there is a positive combinatorial formula for the Laurent expansion of any cluster variable in a cluster algebra arising from a surface [FST] given by the perfect matchings of snake graphs associated to arcs in the surface, that is $x_\gamma = \frac{1}{x(\mathcal{G})} \sum_{P \in \text{Match}\,\mathcal{G}_\gamma} x(P)y(P)$. In this work, we introduce the notion of abstract snake graphs and develop a graphical calculus for surface cluster algebras. Moreover, we give a new proof of Skein relations.

Notation

- Let the triple (S, M, T) be a bordered surface S with marked points $M \in \partial S$ together with an associated triangulation T.
- $\mathcal{A} = \mathcal{A}(S, M, T)$ be the cluster algebra associated to the surface (S, M, T) with principal coefficients.
- Smoothing of two arcs γ_1 and γ_2 at a point is defined to be the pair of curves obtained by the local change of crossing point with the pair of segments and . The pair of arcs obtained by smoothing is denoted by (γ_3, γ_4) and (γ_5, γ_6), respectively.
- Skein relations is given by $x_{\gamma_1}x_{\gamma_2} = y_- x_{\gamma_3}x_{\gamma_4} + y_+ x_{\gamma_5}x_{\gamma_6}$.

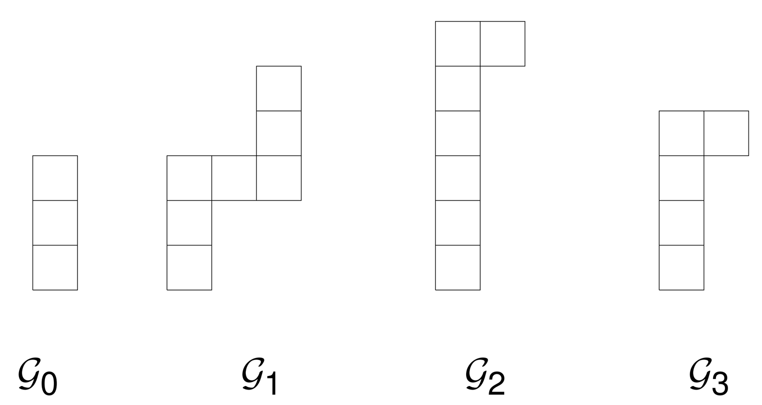
Snake graphs

Definition 1. A **snake graph** $\mathcal{G}$ is a connected graph consisting of finite ordered sequence of tiles $G_1, G_2, \cdots, G_d$ such that

1. G_i and G_{i+1} share exactly one edge e_i for each i, and
2. G_i and G_j are disjoint whenever $i - j \geq 2$.

Remark 2. A snake graph can be viewed as a finite path in the **Z**-grid where we can only allowed to go to the north or to the east.

Example Snake Graph



Local overlap

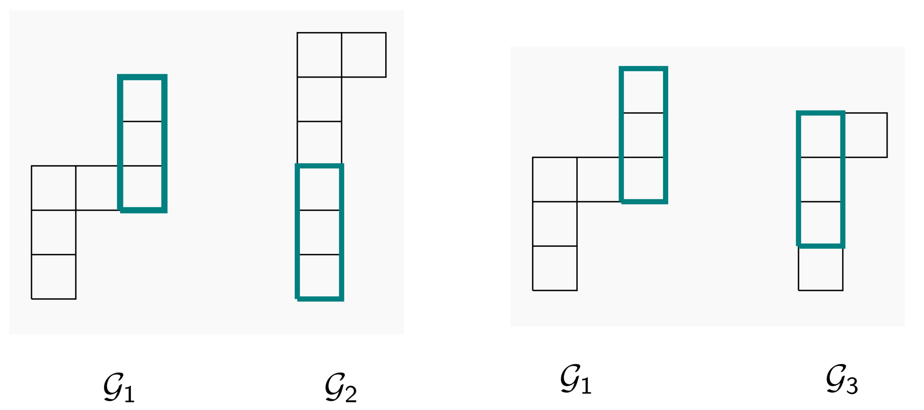
Definition 3. Let $\mathcal{G}_1 = (G_1, G_2, \cdots, G_d)$ and $\mathcal{G}_2 = (G'_1, G'_2, \cdots, G'_{d'})$ be two snake graphs. We say that $\mathcal{G}_1$ and $\mathcal{G}_2$ have a **local overlap** $\mathcal{G}$ if $\mathcal{G}$ is a snake graph and there exist two embeddings $i_1 : \mathcal{G} \to \mathcal{G}_1$ and $i_2 : \mathcal{G} \to \mathcal{G}_2$ such that

- (*Maximality*) If $\mathcal{G}$ has at least two tiles, and there exists a snake graph $\mathcal{G}'$ and two embeddings $i'_1 : \mathcal{G}' \to \mathcal{G}_1$ and $i'_2 : \mathcal{G}' \to \mathcal{G}_2$ such that $i_j(\mathcal{G}) \subset i'_j(\mathcal{G}')$ then $i_j(\mathcal{G}) = i'_j(\mathcal{G}')$.
- If $\mathcal{G}$ is a single tile and $i_1(\mathcal{G}) = G_k$ and $i_2(\mathcal{G}) = G'_{k'}$ then either $k \in \{1, d\}$ or $k' \in \{1, d'\}$ or $1 < k < d$ and the subgraphs G_{k-1}, G_k, G_{k11} and $G'_{k'-1}, G'_{k'}, G'_{k'+1}$ are isomorphic.

Remark 4. Two snake graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ may have several overlaps.

Example Local overlap

Local overlap of the pairs $(\mathcal{G}_1, \mathcal{G}_2)$ and $(\mathcal{G}_1, \mathcal{G}_3)$ of snake graphs are both given by the green snake graph below.

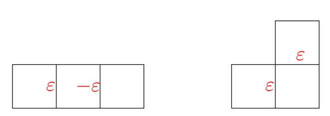


Sign function on snake graphs

Definition 5. A **sign function** on $\mathcal{G} = (G_1, G_2, \cdots, G_d)$ is a map f from the edges e_i of $\mathcal{G}$ to $\{\pm\}$ where e_i is the interior edge shared by the tiles G_i and G_{i+1}, for each $i \in \{1, \cdots, d-1\}$, such that

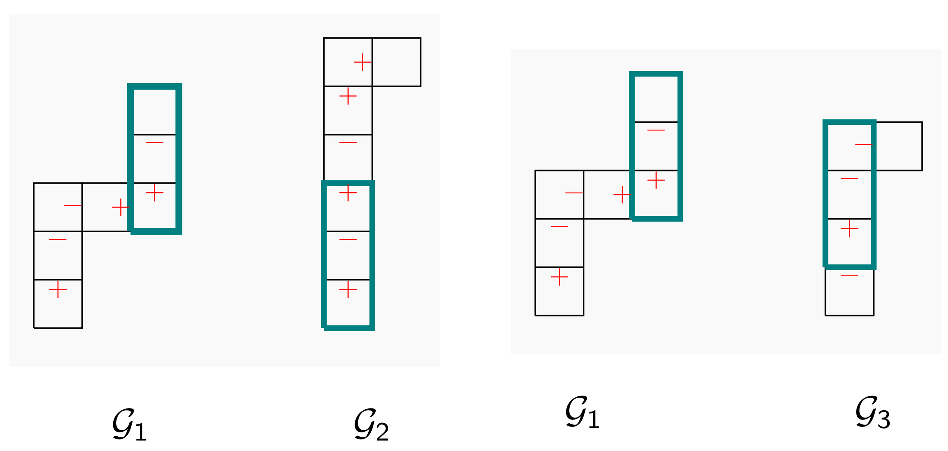
- $f(e_i) = -f(e_{i+1})$ if G_i, G_{i+1}, G_{i+2} form a straight subgraph,
- $f(e_i) = f(e_{i+1})$ if G_i, G_{i+1}, G_{i+2} form a zig-zag subgraph.

We indeed have the following local configurations



where $\varepsilon = \pm$.

Example Sign Function

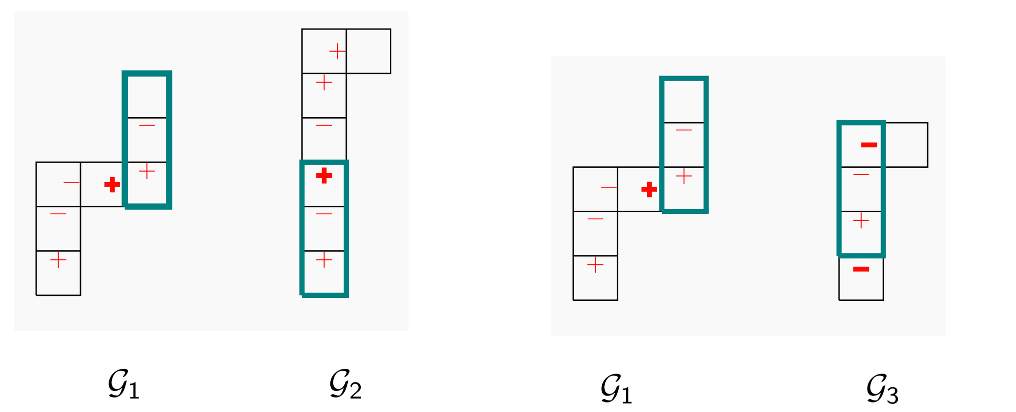


Crossing of snake graphs

Definition 6. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two snake graphs with a non-empty local overlap $\mathcal{G}$, f be a sign function on $\mathcal{G}$ and f_1, f_2 be sign functions on $\mathcal{G}_1, \mathcal{G}_2$, respectively, induced by f. We say $\mathcal{G}_1$ and $\mathcal{G}_2$ **cross** if one of the following holds.

- $f_1(e_{s-1}) = -f_1(e_t)$ where $s > 1, t < d$ or $f_2(e'_{s'-1}) = -f_2(e'_{t'})$ where $s' > 1, t' < d'$,
- $f_1(e_t) = f_2(e'_{s'-1})$ where $s = 1, t < d, s' > 1, t' = d'$ or $f_1(e_{s-1}) = f_2(e'_{t'})$ where $s > 1, t = d, s' = 1, t' < d'$.

Example Crossing



In the figure above, $\mathcal{G}_1$ and $\mathcal{G}_2$ cross. However, $\mathcal{G}_1$ and $\mathcal{G}_3$ do not cross.

Resolution of snake graphs

Let $\mathcal{G}[i, j] := (G_i, \cdots, G_j)$. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ cross at a local overlap $\mathcal{G}$ where $s > 1, s' = 1, d = t, d' > t'$. Define four connected snake graphs as follows.

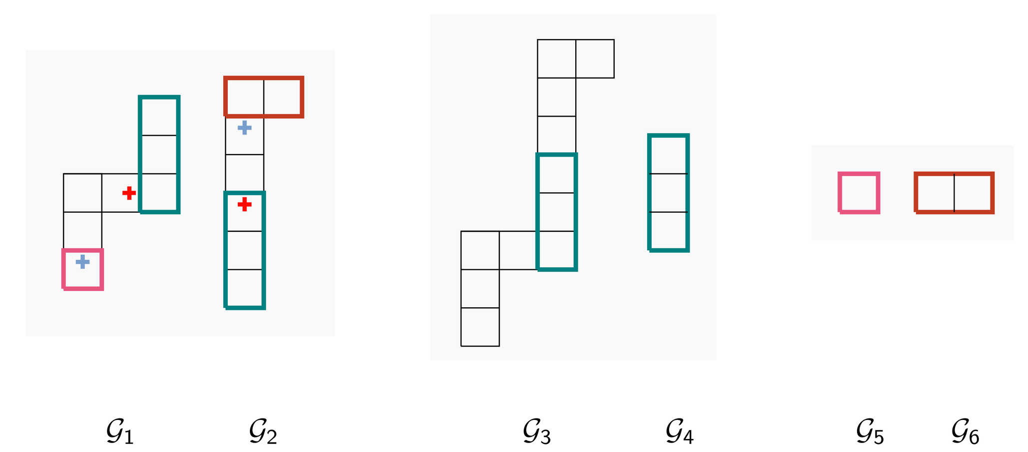
- $\mathcal{G}_3 = \mathcal{G}_1[1, t] \cup \mathcal{G}_2[t' + 1, d']$,
- $\mathcal{G}_4 = \mathcal{G}_2[1, t'] \cup \mathcal{G}_1[t + 1, d]$,
- $\mathcal{G}_5 = \mathcal{G}_1[1, k]$ where $k < s - 1$ is the largest integer such that $f_1(e_k) = f_1(e_{s-1})$,
- $\mathcal{G}_6 = \mathcal{G}_2[k', d']$ where $k' > t' + 1$ is the least integer such that $f_2(e'_{t'}) = f_2(e'_{k'-1})$.

The **resolution of the crossing** of $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathcal{G}$ is defined to be $(\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$ and denoted by $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$.

Theorem 7. *There is a bijection*

$$\text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \to \text{Match}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2))$$

Example Resolution



$\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2) = (\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$.

Relation to cluster algebras

Definition 8. Define the **Laurent polynomlal** of the resolution by

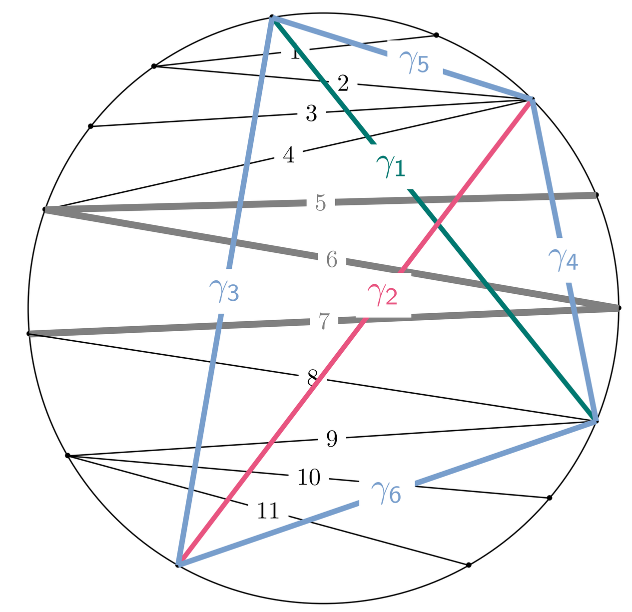
$$\mathcal{L}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) = \mathcal{L}(\mathcal{G}_3 \sqcup \mathcal{G}_4) + y(\bar{\mathcal{G}})\mathcal{L}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$$

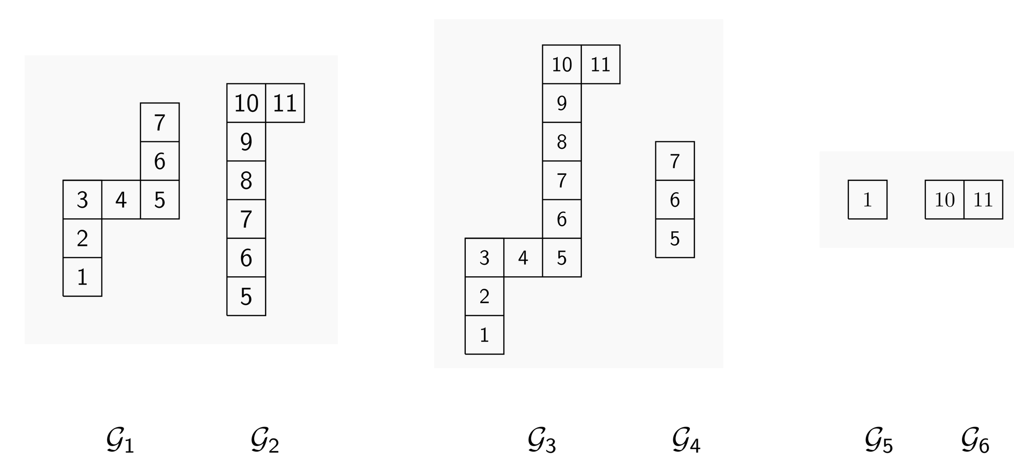
where $\mathcal{L}(\mathcal{G}_k \sqcup \mathcal{G}_l) = \frac{1}{x(\mathcal{G}_k)x(\mathcal{G}_l)} \sum_{P \in \text{Match}(\mathcal{G}_k \sqcup \mathcal{G}_l)} x(P)y(P)$.

Theorem 9. *Let (S, M, T) be a surface with triangulation T. Let γ_1 and γ_2 be two arcs in (S, M) which cross with a nonempty local overlap $\mathcal{G}$, and let $\mathcal{G}_1, \mathcal{G}_2$ be the corresponding snake graphs. Then*

- $\mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) = \mathcal{L}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2))$
- *The snake graphs of the arcs obtained by smoothing γ_1 and γ_2 are given precisely by $\text{Res}_{\mathcal{G}}(\mathcal{G}_{\gamma_1}, \mathcal{G}_{\gamma_2})$ where $\mathcal{G} = \mathcal{G}_{\gamma_1} \cap \mathcal{G}_{\gamma_2}$.*

Example





References

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[FT] S. Fomin and D. Thurston, Cluster algebras and triangulated surfaces. Part II: Lambda Lengths, preprint (2008), http://www.math.lsa.umich.edu/ fomin/Papers/cats2.ps

[MSW] G. Musiker, R. Schiffler, and L. Williams, Positivity for cluster algebras from surfaces, *Adv. Math.* **227**, (2011), 2241-2308.

* Joint work with Ralf Schiffler