# Loewy lengths of tensor products of $kD_{2^l}$-modules

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# The representation ring

Let $G$ be a group and $k$ a field. A tensor product is defined on $kG \bmod$ by $M \otimes N = M \otimes_k N$, with $G$-action defined by $g \cdot (m \otimes n) = gm \otimes gn$.

The representation ring (or Green ring) $A(kG)$ of $kG$ is

- the abelian group generated by isoclasses of $kG$-modules, subject to the relation $[M \oplus N] = [M] + [N]$,
- a commutative, unital ring with multiplication defined by $[M] \cdot [N] = [M \otimes N]$.

# Dihedral 2-groups

Let $k$ be a field of characteristic 2.

The dihedral group of order $4q$, where $q = 2^l$:

$$D_{4q} = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^{2q} = 1 \rangle \,,$$

$$\begin{aligned} kD_{4q} &\simeq k\langle X, Y\rangle / (\, X^2,\ Y^2,\ (XY)^q - (YX)^q \,)\,, \\ \sigma &\mapsto 1 + X, \\ \tau &\mapsto 1 + Y. \end{aligned}$$

# Modules

The algebra $kD_{4q}$ is special biserial. Its indecomposable modules were first classified by Ringel in 1975.

1. strings,
2. bands,
3. projectives.

# Binary expansions and stupid addition

1. For $n \in \mathbb{N}$, write $n = \sum_{i \geqslant 0} [n]_i 2^i$.
2. For $r, s \in \mathbb{N}$, define

$$r \# s = r' + s' + 2^a - 1,$$

where $a \in \mathbb{N}$ is the smallest number such that $[r]_i + [s]_i \leqslant 1$ for all $i \geqslant a$, and

$$r' = \sum_{i \geqslant a} [r]_i 2^i \,, \quad s' = \sum_{i \geqslant a} [s]_i 2^i \,.$$

3. $r, s \leqslant r \# s$, and
$r \# s \leqslant r + s$, with "$=$" if and only if $[r]_i + [s]_i \leqslant 1$ for all $i \in \mathbb{N}$.

# Loewy length: The uniserial case

Set

$$M_r \quad = \quad u_0 \xrightarrow{X} u_1 \xrightarrow{Y} u_2 \xrightarrow{X} \cdots\cdots\cdots \longrightarrow u_r$$

$$N_r \quad = \quad v_0 \xrightarrow{Y} v_1 \xrightarrow{X} v_2 \xrightarrow{Y} \cdots\cdots\cdots \longrightarrow v_r$$

**Proposition**

$$\ell\ell(M_r \otimes N_s) = \begin{cases} 2 + r\#s & \textit{if } a \geqslant 1, \\ 1 + r\#s = 1 + r + s & \textit{if } a = 0, \end{cases}$$

$$\ell\ell(M_r \otimes M_s) = \begin{cases} 2 + r\#s & \textit{if } \exists t < a-1 \ : \ [r]_t = 1 \textit{ or } [s]_t = 1, \\ 1 + r\#s & \textit{if } [r]_t = [s]_t = 0 \textit{ for all } t < a-1\,. \end{cases}$$

# Loewy length: The general case

For arbitrary indecomposable modules $M, N$,

$$\ell\ell(M \otimes N) = \max \left\{ \ell\ell(U \otimes V) \;\middle|\; \begin{array}{l} U \text{ is a subquotient of } M \\ V \text{ is a subquotient of } N \\ \operatorname{top} U, \operatorname{top} V, \operatorname{soc} U, \operatorname{soc} V \text{ are simple} \end{array} \right\}$$