Identification of simple representations of affine q-Schur algebras

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(Joint with Jie Du and Qiang Fu)

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Main aim

In [D–Du–Fu] (*A double Ringel–Hall algebra approach to affine quantum Schur–Weyl theory*), two approaches of classifying simple representations of affine q-Schur algebras have been given:

- the upward approach: applying the classification of simple modules over affine Hecke algebras of type A due to Zelevinsky and Rogawski (indexed by *multisegments*).

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- The quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ is the $\mathbb{C}$-algebra generated by $\mathtt{x}^{\pm}_{i,s}$ $(1 \le i < n,\ s \in \mathbb{Z})$, $\mathtt{k}_i^{\pm 1}$ and $\mathtt{g}_{i,t}$ $(1 \le i \le n,\ t \in \mathbb{Z}\backslash\{0\})$ with relations:

(1) $\mathtt{k}_i\mathtt{k}_i^{-1} = 1 = \mathtt{k}_i^{-1}\mathtt{k}_i,\ [\mathtt{k}_i, \mathtt{k}_j] = 0,\ [\mathtt{k}_i, \mathtt{g}_{j,s}] = 0,\ [\mathtt{g}_{i,s}, \mathtt{g}_{j,t}] = 0$;

(2) $\mathtt{k}_i\mathtt{x}^{\pm}_{j,s} = q^{\pm(\delta_{i,j}-\delta_{i,j+1})}\mathtt{x}^{\pm}_{j,s}\mathtt{k}_i$;

(3) $$[\mathtt{g}_{i,s}, \mathtt{x}^{\pm}_{j,t}] = \begin{cases} 0 & \text{if } i \ne j,\ j+1, \\ \pm q^{-is}\frac{[s]}{s}\mathtt{x}^{\pm}_{i,s+t}, & \text{if } i = j, \\ \mp q^{(1-i)s}\frac{[s]}{s}\mathtt{x}^{\pm}_{i-1,s+t}, & \text{if } i = j+1; \end{cases}$$

(4) $[\mathtt{x}^{+}_{i,s}, \mathtt{x}^{-}_{j,t}] = \delta_{i,j}\frac{\phi^{+}_{i,s+t} - \phi^{-}_{i,s+t}}{q-q^{-1}}$;

(5) $\mathtt{x}^{\pm}_{i,s}\mathtt{x}^{\pm}_{j,t} = \mathtt{x}^{\pm}_{j,t}\mathtt{x}^{\pm}_{i,s}$, if $|i-j| > 1$;

(6) $[\mathtt{x}^{\pm}_{i,s}, [\mathtt{x}^{\pm}_{j,t}, \mathtt{x}^{\pm}_{i,p}]_q]_q = -[\mathtt{x}^{\pm}_{i,p}, [\mathtt{x}^{\pm}_{j,t}, \mathtt{x}^{\pm}_{i,s}]_q]_q$, if $|i-j| = 1$;

(7) $[\mathtt{x}^{\pm}_{i,s+1}, \mathtt{x}^{\pm}_{j,t}]_{q^{\pm c_{ij}}} = -[\mathtt{x}^{\pm}_{j,t+1}, \mathtt{x}^{\pm}_{i,s}]_{q^{\pm c_{ij}}}$.

where $[x, y]_a = xy - ayx$.

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- Here $\phi^{\pm}_{i,s}$ are defined by the generating function:

$$\Phi_i^{\pm}(u) := \widetilde{\mathrm{k}}_i^{\pm 1} \exp(\pm(q-q^{-1})\sum_{m\geq 1} \mathrm{h}_{i,\pm m} u^{\pm m}) = \sum_{s\geq 0} \phi^{\pm}_{i,\pm s} u^{\pm s},$$

where $\widetilde{\mathrm{k}}_i = \mathrm{k}_i/\mathrm{k}_{i+1}$ and

$$\mathrm{h}_{i,m} = q^{(i-1)m}\mathrm{g}_{i,m} - q^{(i+1)m}\mathrm{g}_{i+1,m}\ (1 \leq i < n).$$

- The $\mathbb{C}$-subalgebra generated by $\mathrm{x}^{\pm}_{i,s}$, $\widetilde{\mathrm{k}}_i^{\pm 1}$, and $\mathrm{h}_{i,m}$ $(1 \leq i < n,$ $s, m \in \mathbb{Z}$, $m \neq 0)$ is the quantum enveloping algebra $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$ (This is the so-called Drinfeld new presentation of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$).

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Double Ringel–Hall algebras of cyclic quivers

- Let $\mathfrak{D}(\triangle)$ denote the double Ringel–Hall algebra of the cyclic quiver $\triangle = \triangle_n$ with n vertices. (Ringel, J.A. Green, and Xiao)
- Following Schiffmann and Hubery, $\mathfrak{D}(\triangle)$ can be generated by $\mathrm{k}_i^{\pm 1}, E_i, F_i$ $(1 \le i \le n)$ and $\mathrm{z}_m^{\pm}$ $(m \ge 1)$ with $\mathrm{z}_m^{\pm}$ being *primitive* and *central*. The subalgebra generated by $\widetilde{\mathrm{k}}_i^{\pm 1}$, E_i, and F_i $(1 \le i \le n)$ gives the Jimbo–Drinfeld presentation of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$. Moreover, there is a Hopf algebra isomorphism

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- Ω: $\mathbb{C}$-vector space with basis $\{\omega_s \mid s \in \mathbb{Z}\}$
- There is a natural $\mathfrak{D}(\triangle)$-module structure on Ω defined by

$$E_i \cdot \omega_s = \delta_{\bar{s},\overline{i+1}}\omega_{s-1}, \quad F_i \cdot \omega_s = \delta_{\bar{s},\bar{i}}\omega_{s+1},$$

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Thus, for each $r \geq 1$, the above action induces an action of $\mathfrak{D}(\triangle)$ on $\Omega^{\otimes r}$ via the Hopf algebra structure on $\mathfrak{D}(\triangle)$, that is, $\Omega^{\otimes r}$ is a left $\mathfrak{D}(\triangle)$-module.

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- Following Varagnolo–Vasserot, there is a right $\mathcal{H}_\vartriangle(r)$-module structure on $\Omega^{\otimes r}$, where $\mathcal{H}_\vartriangle(r)$ is the *affine Hecke akgebra of type A*, i.e., $\mathbb{C}$-algebra generated by $T_i, X_j^{\pm 1}$ $(i = 1, \ldots, r-1,\ j = 1, \ldots, r)$ with relations

$$\begin{aligned}
&(T_i + 1)(T_i - q^2) = 0,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ \ T_iT_j = T_jT_i\ (|i-j| > 1),\\
&X_iX_i^{-1} = 1 = X_i^{-1}X_i,\ \ X_iX_j = X_jX_i,\\
&T_iX_iT_i = q^2X_{i+1},\ \ X_jT_i = T_iX_j\ (j \neq i, i+1).
\end{aligned}$$

- The two actions on $\Omega^{\otimes r}$ commute, i.e., $\Omega^{\otimes r}$ becomes a $\mathfrak{D}(\triangle)$-$\mathcal{H}_\vartriangle(r)$-bimodule. Moreover, it induces a surjective algebra homomorphism:

$$\xi_r : \mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \cong \mathfrak{D}(\triangle) \twoheadrightarrow \mathrm{End}_{\mathcal{H}_\vartriangle(r)}(\Omega^{\otimes r}) =: \mathcal{S}_\vartriangle(n, r).$$

The algebra $\mathcal{S}_\vartriangle(n, r)$ is called the *affine q-Schur algebra*.

Affine Hecke algebras of type A

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- Suppose $n \geq r$. Then the functor

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  is an equivalence. It also induces a category equivalence between $\mathcal{H}_\vartriangle(r)$-mod and $\mathcal{S}_\vartriangle(n,r)$-mod.

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- A segment s of length $k = |\mathsf{s}|$ with center $a \in \mathbb{C}^\times$ is by definition a sequence

$$\mathsf{s} = (aq^{-k+1}, aq^{-k+3}, \ldots, aq^{k-1}) \in (\mathbb{C}^\times)^k.$$

- Let $\mathscr{S}_r$ be the set of multisegments $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_p)$ with $|\mathsf{s}_1| + \cdots + |\mathsf{s}_p| = r$.
- [Zelevinsky, Rogawski] For each $\mathbf{s} \in \mathscr{S}_r$, there is an associated simple $\mathcal{H}_\Delta(r)$-module $V_\mathbf{s}$. Moreover, the set

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The set
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The downward approach

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Case $n > r$ [D–Du–Fu]:

- a category equivalence of Chari–Pressley from the category of finite dimensional $\mathcal{H}_\vartriangle(r)$-modules to the category of finite dimensional $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$-modules of type 1 which are of level r as $\mathrm{U}_q(\mathfrak{sl}_n)$-modules.
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