Partial tilting complexes and beyond

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About PLACES

BIELEFELD Universität &

the University of PADOVA =

the most important places where I studied, worked long time ago.
Results of this talk are based of the hope and feeling that INDECOMPOSABLE complexes may play a big role [bigger than that of indecomp. modules]. Very often, the same holds for the complexes whose INDECOMPOSABLE summands have INDECOMPOSABLE non-zero components ( = ”well - behaved” for short ).
The trick to delete many complicated properties and to look only some of them, if possible of "combinatorial" type (concerning the underlying vector spaces of algebras, modules, complexes) = the trick I used several times from the very beginning in quite different situations.
More or less old situations:

Abelian Groups & finiteness conditions showed up in my “tesi di laurea” on “Abelian Groups whose endomorphism ring is locally compact in the finite topology” (under the direction of A. ORSATTI)
Adalberto ORSATTI =
- supervisor of the master (master + PhD) thesis of many Italian colleagues who studied or worked in Padova for some time.
- organizer of algebra meetings in Italy.
Endomorphism rings of abelian groups - equipped with the **finite topology** - and Corner’s type realization theorems = subject of my first talk in BIELEFELD ……
and of my first conversation with

Claus Michael RINGEL

during a coffee break...
Notation / Conventions

\( K \) = algebraically closed field

MODULE = left module over a \( K \) - algebra

COMPLEX = right bounded complex with projective components

MORPHISM of complexes = morphism / homotopy

\( M^\circ \) = projective resolution of the module \( M \)

T partial n - tilting module = . . . . . .
With some hypothesis on the **Hom spaces of NON surjective morphisms** between ..... 

A **right bounded** string of integers $> 0$ 
$[ \ldots \ldots , m(2), m(1)]$ stands for the **indecomposable right bounded** complex $\mathbf{C}^\circ$ s.t. (proceeding from right to left) the indecomp. projective modules $\ldots \ldots, P(m(2)), P(m(1))$ as the non-zero components of $\mathbf{C}^\circ$. 
The meaning of **PARTIAL TILTING** (or \( n \)-tilting) module in this talk:

**T PARTIAL TILTING:**

- \( \text{proj dim } T \) at most \( n \)

- \( \text{Ext} \left( T, \bigoplus T \right) = 0 \) for all \( i > 0 \)
The meaning of **TILTING** (or **n-tilting**) module **T** in this talk:

- the **projective** dimension of **T** is at most **n**.
- \( \text{Ext} (T, \Sigma) = 0 \) where \( \Sigma \) is any direct sum of copies of **T** and \( i = 1, \ldots, n \).
- There is a **long** exact sequence of the form
  \[
  0 \longrightarrow R \longrightarrow * \longrightarrow \cdots \cdots \longrightarrow * \longrightarrow 0
  \]
  where the **n + 1** symbols * stand for direct summands of **direct sums** of copies of **T**.
The meaning of **LARGE** partial tilting module in this talk

\( T \) partial tilting module \( \text{s.t.} \)

\[
\text{Hom} ( T, X ) = \text{Ext}^* ( T, X ) = 0
\]

implies \( X = 0 \).
FOR SHORT (in this talk)

$C \circ$ is orthogonal to $T \circ$:

any morphism from $T \circ$ to any shift $C \circ [i]$ of $C \circ$ is homotopic to zero.
What is used to deal with complexes:

a characterization of tilting complexes given by Y. MIYACHI (in “Extensions of rings and tilting complexes “)

which replaces a condition on triangulated categories by a condition on morphisms homotopic to zero.
Starting points:

- **BAZZONI** ‘s question on the relationship between tilting modules and large partial tilting modules (i.e. with the functorial property described in the abstract).

- **MANTESE & TONOLO** ‘s question on the relationship between bounded and right bounded “real” complexes “orthogonal” to the projective resolutions of ...........
Strategies used to deal with ..... 

RIGHT BOUNDED complexes of PROJECTIVE modules and their morphisms / homotopy:

Use as many as possible NEW modules (with “dual” properties)

NEW directions (if possible)
(A) Use as many as possible

- **INJECTIVE** modules
- **indecomposable** modules $P, Q$ with a **rigid** structure [i.e. $\text{Hom}(P,Q)$ is a vector space of **dimension** $< 3$, and
$< 2$ if $P$ and $Q$ are not isomorphic].
FEW morphisms between ...

... indecomposable projective modules

= reason why non-zero morphisms of this form (which are not isomorphisms) are uniquely determined up to scalar, so that strings \([ \ldots m(2), m(1) ]\) denote many useful complexes.
( B ) Use as many as possible " directions "

to investigate morphisms between

**BOUNDED** complexes (in the category of

**right bounded** ones) :

from **RIGHT** to **LEFT** ( = **THE** obvious
direction in the **WHOLE** category)

from **LEFT** to **RIGHT** ( = **THE** **NEW**
possible & less natural direction )
A few words on different points of view:

- A. De Saint Exupery
A. De Saint Exupery’s assertion:

“To see clearly it is often enough to change our viewing direction.”

sums up the strategy used to deal with complexes, and - more generally - to simplify complicated objects.
Part 1 ( on modules )

A result on **CANCELLATIONS** of the **OBVIOUS** direct summands of tilting modules, used to obtain **LARGE** partial tilting modules.
The meaning of \textbf{LARGE PARTIAL TILTING} module in this talk

\( T \) partial tilting module \textbf{s.t.}\n
\[ \text{Hom} ( T, X ) = \text{Ext}^* ( T, X ) = 0 \]

implies \( X = 0 \)
COLPI's result (the "classical" case)

LARGE partial TILTING modules of projective dimension at most 1

= TILTING modules of ..............
BAZZONI's result (the "general" case)

For any $n$, any TILTING module of projective dimension $n$ is a LARGE PARTIAL TILTING module.
What I proved:

For any $n > 1$ (i.e. in all possible cases) there are LARGE partial tilting modules of projective dimension $n$ which are NOT tilting modules.
Some properties of LARGE partial tilting modules $T$ (of finite length)

These modules are SINCERE but NOT always faithful. They may be rather small, i.e. indecomposable injective, and their dimension / $K$ may be equal to the $\# \text{ of simples modules}$ (= least dimension for a sincere module).
No restriction on $\#$:

$\#$ runs over all $n > 1$ even under the additional hypotheses that

- $T$ is INJECTIVE & uniserial
- The class of all modules $X$ s.t. $\text{Ext}^*(T, X) = 0$ for all $* > 0$ is the class of INJECTIVE modules.
Consequence:

LARGE partial tilting modules & proper direct summands of tilting modules may NOT be ALMOST COMPLETE tilting modules.
My answer to the following question:

- **WHY LARGE** partial tilting modules which are **NOT** tilting modules?

is that
Among many other things
(classes of modules, functors, ... )

**AUSLANDER - REITEN** quivers
make these modules **visible** & give
the idea to find the “minimal” ones.
Idea suggested by:

**AUSLANDER - REITEN** quivers:

**SOMETIMES** every **SINCERE** summand $T$ of a **LARGE** partial tilting module $M = T \oplus P$ with $P$ projective inherits from $M$ the property of being a **LARGE** partial tilting module.
The following property:

"The class of all modules $X$ s.t. $\text{Ext}^* (T, X) = 0$ for all $* > 0$ is the class of INJECTIVE modules" is satisfied by many LARGE partial tilting modules (and "explains" why may be rather small).
THEOREM (possible choice of SOMETIMES)

(a) $M$ large partial $n$-tilting module such that the orthogonal class

$$M_{\infty} = \bigcap_{i \geq 1} \text{Ker Ext}^i(\ M \ , \ - )$$

is the class of all injective modules

(b) $T$ SINCERE summand of $M$ with a PROJECTIVE complement

(a) & (b) IMPLY $T$ LARGE partial $n$-tilting
Example

If \( n > 1 \), there is an \( A \)-module \( T \) s.t. \( T \) (unique indecomp. injective module which is NOT projective) is a NON faithful large partial tilting module obtained from \( D(A) \), the \( K \)-dual of \( A \), after CANCELLATION of all its indecomposable projective summands \& \[ 2(n - 1) = \text{projective dimension of } T = \text{global dimension of } A \]
A $K$-algebra given by the following quiver with $n \geq 2$ vertices

with relation

$d_n \cdot d_{n-1} \cdots d_1 = 0$
\[ n \geq 2, \ m = 2n - 2 \]

The module \( T \):

(unique indecomposable injective module which is NOT projective)

is a NON FAITHFUL large partial \( m \)-tilting module
Part 2 ( on complexes )
By RICKARD + MIYACHI `s results:

the projective resolution $T^\circ$ of a LARGE partial tilting module $T$ (which is NOT a tilting module) is a partial tilting complex $T^\circ$ s.t. for every non-zero module $M$ there is a morphism from $T^\circ$ to shift $M^\circ[i]$ of $M^\circ$ which is NOT homotopic to zero, BUT
...... BUT

there is a non-zero complex $C^\circ$ s.t.

any morphism from $T^\circ$ to $C^\circ$ is homotopic to zero for any integer $i$ [i.e. "$C^\circ$ orthogonal to $T^\circ"$].
Reasonable “conjectures” (more or LESS correct):

- The indecomposable right bounded complexes $C^\circ$ (of projective modules) orthogonal to $T^\circ$ are as different as possible from “concealed” complexes, that is projective resolutions of indecomposable modules.
Natural question:

How many choices, up to shifts, for a well-behaved indecomp. complex $C^\circ$ orthogonal to $T^\circ$?

[well-behaved complex : the non-zero components of its indecomposable summands are indecomposable].
ANSWER to the natural question:

With the special hypothesis that $T^\circ$ is a well-behaved complex, the answer to the above question may be ........................
finitely many but > 1

( and only 1 left unbounded )

\[ \mathbb{N}_0 \] ( and only \( \mathbb{N}_0 \) bounded )
3 possible constructions used to find $C^\circ$ orthogonal to $T^\circ$:

**CANCELLATIONS** [2 or 3 different types]

**ADDITIONS** [2 different types]

**LEGO - TYPE CONSTRUCTIONS** [oo - many types] to get more complicated (even **LEFT unbounded**) complexes from the minimal ones.
FOR ME

CANCELATION = the best & easy construction
LEGO - TYPE construction = the best & more complicated construction
RIGHT ADDITION = the less natural & oldest construction (sometimes the unique possible one)
2 (quite different) examples

where the choice of $C^\circ$ is unique
and $T^\circ$ has at most 2 indecomp. summands:
Ex. A = example with only 1 choice for $C^\circ$, obtained by means of **RIGHT addition** from the indecomposable complex $T^\circ$:

$C^\circ : \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \end{bmatrix}$

$T^\circ : \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$
Ex. B = example with only 1 choice for $C^\circ$, obtained by means of **LEFT cancellation & addition** from the unique **indecomposable** non stalk summand $X^\circ$ of $T^\circ$:

$C^\circ : [ \ldots \ldots \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 1 \ 3 \ ]$

$X^\circ : [ \ 1 \ 2 \ 1 \ 3 \ ]$
Ex. $C = $ example with $\infty$ - many (but countably many)

choices for $C^\circ$, where $T^\circ$ has 2 indecomposable summands, i.e.
- the stalk complex $[2]\]

and

- the complex $X^\circ = [3121]$ and .....
..... $X^\circ = [3\ 1\ 2\ 1]$ 

and the following strings describe all the possible choices of $C^\circ$:

$[1\ 2\ 1]$,
$[1\ 2\ 2\ 1]$,
$[1\ 2\ 2\ 2\ 1]$,

.............................................

.............................................

$[\ ..................\ 2\ 2\ 2\ 2\ 2\ 1\ ]$
Remarks

- Only in one case (= Lego - type case) one proceeds in the most obvious direction (from RIGHT to LEFT), but the ingredients (building blocks) are complexes and NOT “isolated” modules.
- The less obvious construction (= RIGHT addition) may be the unique possible one (Example A).
Ex. D = CANCELLATIONS

For any $m > 1$, there is a large partial tilting module $T$ s.t. $\text{pdim } T = 2m > 2$ and $T$ is injective & uniserial.

If $P$ & $Q$ are indec. projective and $C^\circ$ is an indecomp. complex of the form

$$0 \longrightarrow P \longrightarrow Q \longrightarrow 0,$$

then TFAE:
TFAE:

1) $C^\circ$ is orthogonal to $T^\circ$.

2) $P$ & $Q$ injective, not isomorphic and we obtain $C^\circ$ from $T^\circ$ by means of cancellations (LEFT, RIGHT,...).

3) $P$ & $Q$ injective, not isomorphic and the morphism from $P$ to $Q$ is a composition of IRREDUCIBLE maps $X \rightarrow Y$ with either $X$ or $Y$ injective.
Continuation of TFAE:

(4) (reduction to an easy case): for any morphism of complexes from $T^\circ$ to a shift of $C^\circ$ of the form $(f, 0)$ where $f : X \rightarrow P$ and $X$ is the last non-zero components of $T^\circ$, we have $f = 0$. 
The complex $T^\circ$ in Ex. D is the projective resolution of the uniserial module $T$, considered in the first Example of this talk, of the form:

\[
\begin{array}{cccccc}
& & & & & \\
1 & 2 & 3 & \cdots & n & \vdots \\
\end{array}
\]
Ex. E = example with uncountably many choices for $C^\circ$:

$T = \text{direct sum of 2 indecomp. modules:}$

- $P$ uniserial & projective
- $S$ simple and pdim $S = 5$

$4 = \text{number of simple modules}$

$5 = \text{dimension of } T \text{ over } K$

$\text{Hom}(P, S) = 0 = \text{Hom}(S, P)$ .....
The shape of the proj. resol. $S^\circ$ of $S$ is

$\begin{bmatrix} 1 & 3 & 1 & 2 & 1 & 4 \end{bmatrix}$

and the indecomp. projectives look like as follows:

$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
P(1) = 2 & P(2) = 3 & P(3) = 1 & P(4) = 1 \\
3 & 1 & 2 & \\
\end{array}$
Two types of indecomp. complexes orthogonal to the proj. resol. of \( T \)

(1) the "simple ones" (no indecomposable subcomplexes has the same property).

(2) the "non simple" indecomp. complexes, obtained from complexes of type (1) by means of a Lego-type construction.
The shape of some complexes of type \((1)\)

\[
\begin{align*}
[3 & 2 2 3], & [3 2 2 2 3], & [3 2 2 2 2 3], & \ldots & \text{and their “limit”} & \[ \ldots \ldots \ 2 2 2 2 \ldots \ldots \ 2 3 \].
\end{align*}
\]

\[
[1 2 1 4] \text{ subcomplex of } S^\circ \text{ obtained after left cancellation of 2 components.}
\]
Remark

All possible first (resp. last) non-zero components, that is \( P(1) \ & \ P(3) \) (resp. \( P(3) \ & \ P(4) \)) show up.
The shape of some complexes of type \((2)\)

- complexes with \textbf{injective} components

\[
[3 \ 2 \ ... \ 2 \ 3 \ 3 \ 2 \ .... \ 2 \ 3], \ \\
[3 \ 2 \ .... \ 2 \ 3 \ 3 \ 2 \ .... \ 2 \ 3 \ 3 \ 2 \ .... \ 2 \ 3], \ \\
\ldots
\]

- complexes with \textbf{NON injective} components

\[
[3 \ 2 \ .... \ 2 \ 3 \ 1 \ 2 \ 1 \ 4], \ \\
[3 \ 2 \ .... \ 2 \ 3 \ 3 \ 2 \ .... \ 2 \ 3 \ 1 \ 2 \ 1 \ 4], \ \\
[ \ldots \ldots \ldots \ 2 \ .... \ 2 \ 3 \ 1 \ 2 \ 1 \ 4]
\]
THANKS & picture(s)
Thank you very much to ALL the ORGANIZERS for the great hospitality & the informal atmosphere during:
- this big & international Conference
- many other meetings
- less official
Possible meanings for me of less official … :

- Seminar Darstellungstheorie

- Vorlesung
Picture taken by
ANNETTE HOEWELMANN
and
MONIKA HAENSCH

somewhere in this building in the last millennium
2 (unofficial) pictures

Niagara Falls (ICRA X)

On the left: BAROT, BRENNER, BUTLER, KRAUSE, SCHROER

On the right: young people attending WYD2002 (World Youth Day 2002)