Mutations of simple-minded systems in triangulated categories

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Notation

- k = ground field. All categories, equivalences, etc. are assumed to be k-linear.
- $(\mathcal{T}, [1])$ = Hom-finite, triangulated Krull-Schmidt category.
- $\nu : \mathcal{T} \overset{\approx}{\to} \mathcal{T}$ Serre functor; i.e., $\mathcal{T}(X, Y) \cong D\mathcal{T}(Y, \nu X)$.
- For $\mathcal{X} \subseteq \mathcal{T}$, $\mathcal{X}^{\perp} = \{Y \in \mathcal{T} \mid \mathcal{T}(X, Y) = 0 \ \forall X \in \mathcal{X}\}$, ${}^{\perp}\mathcal{X} = \{Y \in \mathcal{T} \mid \mathcal{T}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$.
- Λ, Γ = basic self-injective k-algebras (k alg. closed).
- $\underline{\text{mod}}$-Λ = stable category of f.g. right Λ-modules.
- $\mathcal{S}_{\Lambda}$ = set of (isoclasses of) simple Λ-modules.
- $D^b(\Lambda)$ = bounded derived category of f.g. right Λ-modules

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Motivation

Let $T \in D^b(\Lambda)$ be a tilting complex and $\Gamma = \mathrm{End}(T)$.
By work of Rickard, T induces equivalences:

$$\begin{tikzcd} D^b(\Gamma) \arrow[r, "F", "\approx"'] \arrow[d] & D^b(\Lambda) \arrow[d] \\ \underline{\mathrm{mod}}\text{-}\Gamma \arrow[r, "\underline{F}", "\approx"'] & \underline{\mathrm{mod}}\text{-}\Lambda \end{tikzcd}$$

Problem: How can one understand/visualize the stable equivalence $\underline{F}$?
For starters, where does it send the simple Γ-modules?

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Maximal systems of orthogonal bricks

Motivation: If $\underline{F} : \underline{\mathrm{mod}}\text{-}\Gamma \to \underline{\mathrm{mod}}\text{-}\Lambda$ is an equivalence, what properties are satisfied by the images of the simple Γ-modules under $\underline{F}$?

Definition [Pogorzały '94]

A set $\mathcal{S} = \{S_i\}_{i \in I}$ of objects of $\mathcal{T}$ is a **maximal system of orthogonal bricks** in $\mathcal{T}$ if

- $\mathcal{T}(S_i, S_j) = 0 \ \forall\ i \neq j$, and $\mathcal{T}(S_i, S_i) \cong k \ \forall\ i \in I$;
- $\forall\ X \neq 0 \in \mathcal{T}\ \exists\ i \in I$ such that $\mathcal{T}(X, S_i) \neq 0$; and
- $\nu(\mathcal{S}[1]) = \mathcal{S}$ (up to isomorphism).

Note: In $\underline{\mathrm{mod}}\text{-}\Lambda$, $\nu \circ [1]$ is the Nakayama functor, which permutes the set of simple modules, and commutes with stable equivalences.

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Simple-minded systems

For $\mathcal{C}, \mathcal{D} \subseteq \mathcal{T}$, set

$$\mathcal{C} * \mathcal{D} = \{X \in \mathcal{T} \mid \exists \text{ a triangle } C \to X \to D \to \quad C \in \mathcal{C}, D \in \mathcal{D}\}.$$

Set $(\mathcal{C})_0 := \{0\}$, and $(\mathcal{C})_n := (\mathcal{C})_{n-1} * (\mathcal{C} \cup \{0\})$ for $n \geq 1$.

$\mathcal{F}(\mathcal{C}) := \cup_{n \geq 0} (\mathcal{C})_n$ is the smallest extension-closed subcategory of $\mathcal{T}$ containing $\mathcal{C}$.

Definition [Koenig-Liu '10]

A maximal system $\mathcal{S}$ of orthogonal bricks in $\mathcal{T}$ is a **simple-minded system** if $\mathcal{F}(\mathcal{S}) = \mathcal{T}$.

Trivial Examples: $\Omega^n(\mathcal{S}_\Lambda) \subset \underline{\text{mod}}\text{-}\Lambda$ for $n \in \mathbb{Z}$.

Open Question: Is every maximal system of orthogonal bricks a simple-minded system?

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Torsion pairs

Definition

A pair $(\mathcal{C}, \mathcal{D})$ of additive subcategories of $\mathcal{T}$, closed under direct summands, form a **torsion pair** if

- $\mathcal{T}(\mathcal{C}, \mathcal{D}) = 0$; and
- $\mathcal{T} = \mathcal{C} * \mathcal{D}$.

It follows: $\forall\ X \in \mathcal{T}$, there exists a (minimal) triangle

$$C_X \xrightarrow{f_X} X \xrightarrow{g_X} D_X \rightarrow$$

with $C_X \in \mathcal{C}$ and $D_X \in \mathcal{D}$. Moreover, f_X is a (minimal) right $\mathcal{C}$-approximation and g_X is a (minimal) left $\mathcal{D}$-approximation.

Theorem

Suppose $\mathcal{X} \subseteq \mathcal{S}$ for a simple-minded system $\mathcal{S}$ in $\mathcal{T}$. Then $({}^{\perp}\mathcal{X}, \mathcal{F}(\mathcal{X}))$ and $(\mathcal{F}(\mathcal{X}), \mathcal{X}^{\perp})$ are torsion pairs in $\mathcal{T}$. In particular, $\mathcal{F}(\mathcal{X})$ is a functorially finite subcategory of $\mathcal{T}$.

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Mutation of simple-minded systems

Let $\mathcal{S}$ be a simple-minded system, and $\mathcal{X} \subset \mathcal{S}$ s.t. $\nu(\mathcal{X}[1]) = \mathcal{X}$.

For any $M \in \mathcal{T}$, we have (unique) minimal triangles

$$\underset{\in {}^{\perp}\mathcal{X}}{\mathbf{a}M} \to M \longrightarrow \underset{\in \mathcal{F}(\mathcal{X})}{\mathbf{b}M} \longrightarrow \qquad \underset{\in \mathcal{F}(\mathcal{X})}{\mathbf{c}M} \longrightarrow M \to \underset{\in \mathcal{X}^{\perp}}{\mathbf{d}M} \to$$

Definition

The **left mutation** of $\mathcal{S}$ at $\mathcal{X}$ is $\mu^+_{\mathcal{X}}(\mathcal{S}) = \{\mu^+_{\mathcal{X}}(S_i) \mid S_i \in \mathcal{S}\}$, where

$$\mu^+_{\mathcal{X}}(S_i) = \begin{cases} S_i, & \text{if } S_i \in \mathcal{X} \\ \mathbf{a}(S_i[-1]), & \text{if } S_i \notin \mathcal{X} \end{cases}$$

Right mutation of $\mathcal{S}$ at $\mathcal{X}$ is defined dually:

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For any $\mathcal{X} \subset \mathcal{S}$ satisfying $\nu(\mathcal{X}[1]) = \mathcal{X}$, $\mu^+_{\mathcal{X}}(\mathcal{S})$ and $\mu^-_{\mathcal{X}}(\mathcal{S})$ are again simple-minded systems. Moreover,

$$\mu^-_{\mathcal{X}}(\mu^+_{\mathcal{X}}(\mathcal{S})) = \mathcal{S} = \mu^+_{\mathcal{X}}(\mu^-_{\mathcal{X}}(\mathcal{S})).$$

Example. $\mu^+_1(\mathcal{S}_\Lambda), \ldots$

Let Λ have the quiver and indecomp. projectives:

$$\begin{tikzcd} & 1 \arrow[dr, shift left] \arrow[dl, shift left] & \\ 3 \arrow[ur, shift left] \arrow[rr, shift left] & & 2 \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd}$$

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$$\mu^+_1(\mathcal{S}_\Lambda) \;=\; 1, \quad \begin{array}{ccc} 3 & & \\ 1 & & 3, \\ & 2 & \end{array} \quad \begin{array}{ccc} & & 2 \\ 2 & & 1 \\ & 3 & \end{array}$$

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$$(\mu^+_1)^2(\mathcal{S}_\Lambda) \;=\; 1, \quad \begin{array}{ccccccc} 2 &&&&&& \\ & 1 && 2 &&& \\ && 3 && 1 && 2, \\ &&&&& 3 & \end{array} \quad \begin{array}{ccccccc} &&&&&& 3 \\ &&& 3 && 1 & \\ 3 && 1 && 2 && \\ & 2 &&&&& \end{array}$$

Connection to derived equivalence

- Let $\mathcal{X} \subset \mathcal{S}_\Lambda$ satisfy $\nu(\mathcal{X}[1]) = \mathcal{X}$.
- Let $P = P_{\mathcal{X}}$ be the corresponding projective module, and $Q = \Lambda/P$.
- Let $P \stackrel{f}{\to} Q'$ be a left add(Q)-approximation of P. The Okuyama tilting complex $T_{\mathcal{X}}$ associated to $\mathcal{X}$ is

$$0 \to P \stackrel{(0\ f)^T}{\to} Q \oplus Q' \to 0$$

- $\Gamma = \mathrm{End}_{D^b(\Lambda)}(T_{\mathcal{X}})$ is the "left tilting mutation of Λ at $\mathcal{X}$."
- Let $\underline{F}_{\mathcal{X}} : \underline{\mathrm{mod}}\text{-}\Gamma \to \underline{\mathrm{mod}}\text{-}\Lambda$ be the induced equivalence.

Theorem. [-, Koenig-Yang]

$$\underline{F}_{\mathcal{X}}(\mathcal{S}_\Gamma) = \mu^+_{\mathcal{X}}(\mathcal{S}_\Lambda).$$

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Application: Lifting stable equivalences to derived equivalences

Question: Determine if a stable equivalence (of Morita type) $\alpha : \underline{\text{mod}}\text{-}\Lambda' \to \underline{\text{mod}}\text{-}\Lambda$ is induced by an equivalence of derived categories $D^b(\Lambda') \to D^b(\Lambda)$.

Method of Okuyama, Linckelmann, Rickard: If $\alpha(\mathcal{S}_{\Lambda'}) = \underline{F}(\mathcal{S}_{\Gamma})$ where $\underline{F}$ is induced by an equivalence $F : D^b(\Gamma) \to D^b(\Lambda)$, then α lifts to a derived equivalence.

Corollary

If $\alpha(\mathcal{S}_{\Lambda'})$ can be obtained from $\mathcal{S}_{\Lambda}$ by a sequence of mutations and syzygies/co-syzygies, then α lifts to a derived equivalence.

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An example of Broué's conjecture (Okuyama, Rickard)

- Let $G = A_6$, $P = \mathrm{Syl}_3(G) \cong (\mathbb{Z}/3)^2$, $H = N_G(P)$.
- $\Lambda', \Lambda =$ principal blocks of kG, kH, resp., $\mathrm{char}(k) = 3$.
- Λ has quiver and indecomposable projectives

$$\begin{tikzcd} k \arrow[r, shift left] \arrow[d, shift left] & 1 \arrow[l, shift left] \arrow[d, shift left] \\ 3 \arrow[u, shift left] \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd} \qquad \begin{array}{ccccc} & & k & & \\ & 1 & & 3 & \\ 2 & & k & & 2 \\ & 1 & & 3 & \\ & & k & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 2 & & k & \\ 3 & & 1 & & 3 \\ & 2 & & k & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 3 & & 1 & \\ k & & 2 & & k \\ & 3 & & 1 & \\ & & 2 & & \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & k & & 2 & \\ 1 & & 3 & & 1 \\ & k & & 2 & \\ & & 3 & & \end{array}$$

- Restriction gives an equivalence $\underline{\mathrm{mod}}\text{-}\Lambda' \to \underline{\mathrm{mod}}\text{-}\Lambda$. The images of the simple Λ'-modules are:

$$Z_0 = k \quad Z_1 = \begin{array}{c} 1 \\ k \\ 3 \end{array} \quad Z_2 = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2 & \end{array} \quad Z_3 = \begin{array}{c} 3 \\ k \\ 1 \end{array}$$

Apply $\mu^+_{\{Z_0, Z_2\}} : Z_1 \mapsto S_1, \quad Z_3 \mapsto S_3$. Then $\mu^+_{\{k, S_1, S_3\}} : Z_2 \mapsto S_2$.

Thus restriction lifts to a derived equivalence $D^b(\Lambda') \overset{\approx}{\to} D^b(\Lambda)$.

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