

Mutations of simple-minded systems in triangulated categories

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- k = ground field. All categories, equivalences, etc. are assumed to be k -linear.
- $(\mathcal{T}, [1])$ = Hom-finite, triangulated Krull-Schmidt category.
- $\nu : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ Serre functor; i.e., $\mathcal{T}(X, Y) \cong D\mathcal{T}(Y, \nu X)$.
- For $\mathcal{X} \subseteq \mathcal{T}$, $\mathcal{X}^\perp = \{Y \in \mathcal{T} \mid \mathcal{T}(X, Y) = 0 \ \forall X \in \mathcal{X}\}$,
 ${}^\perp \mathcal{X} = \{Y \in \mathcal{T} \mid \mathcal{T}(Y, X) = 0 \ \forall X \in \mathcal{X}\}$.
- Λ, Γ = basic self-injective k -algebras (k alg. closed).
- $\underline{\text{mod}}\text{-}\Lambda$ = stable category of f.g. right Λ -modules.
- $\mathcal{S}_\Lambda$ = set of (isoclasses of) simple Λ -modules.
- $D^b(\Lambda)$ = bounded derived category of f.g. right Λ -modules

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Motivation

Let $T \in D^b(\Lambda)$ be a tilting complex and $\Gamma = \text{End}(T)$.
By work of Rickard, T induces equivalences:

$$\begin{array}{ccc} D^b(\Gamma) & \xrightarrow[\approx]{F} & D^b(\Lambda) \\ \downarrow & & \downarrow \\ \underline{\text{mod}}\text{-}\Gamma & \xrightarrow[\approx]{F} & \underline{\text{mod}}\text{-}\Lambda \end{array}$$

Problem: How can one understand/visualize the stable equivalence $\underline{F}$?

For starters, where does it send the simple Γ -modules?

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Maximal systems of orthogonal bricks

Motivation: If $\underline{F} : \underline{\text{mod}}\text{-}\Gamma \rightarrow \underline{\text{mod}}\text{-}\Lambda$ is an equivalence, what properties are satisfied by the images of the simple Γ -modules under $\underline{F}$?

Definition [Pogorzały '94]

A set $\mathcal{S} = \{S_i\}_{i \in I}$ of objects of $\mathcal{T}$ is a **maximal system of orthogonal bricks** in $\mathcal{T}$ if

- $\mathcal{T}(S_i, S_j) = 0 \ \forall \ i \neq j$, and $\mathcal{T}(S_i, S_i) \cong k \ \forall \ i \in I$;
- $\forall \ X \neq 0 \in \mathcal{T} \ \exists \ i \in I$ such that $\mathcal{T}(X, S_i) \neq 0$; and
- $\nu(\mathcal{S}[1]) = \mathcal{S}$ (up to isomorphism).

Note: In $\underline{\text{mod}}\text{-}\Lambda$, $\nu \circ [1]$ is the Nakayama functor, which permutes the set of simple modules, and commutes with stable equivalences.

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Simple-minded systems

For $\mathcal{C}, \mathcal{D} \subseteq \mathcal{T}$, set

$$\mathcal{C} * \mathcal{D} = \{X \in \mathcal{T} \mid \exists \text{ a triangle } C \rightarrow X \rightarrow D \rightarrow C \in \mathcal{C}, D \in \mathcal{D}\}.$$

Set $(\mathcal{C})_0 := \{0\}$, and $(\mathcal{C})_n := (\mathcal{C})_{n-1} * (\mathcal{C} \cup \{0\})$ for $n \geq 1$.

$\mathcal{F}(\mathcal{C}) := \cup_{n \geq 0} (\mathcal{C})_n$ is the smallest extension-closed subcategory of $\mathcal{T}$ containing $\mathcal{C}$.

Definition [Koenig-Liu '10]

A maximal system $\mathcal{S}$ of orthogonal bricks in $\mathcal{T}$ is a **simple-minded system** if $\mathcal{F}(\mathcal{S}) = \mathcal{T}$.

Trivial Examples: $\Omega^n(\mathcal{S}_\Lambda) \subset \underline{\text{mod}}\text{-}\Lambda$ for $n \in \mathbb{Z}$.

Open Question: Is every maximal system of orthogonal bricks a simple-minded system?

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Torsion pairs

Definition

A pair $(\mathcal{C}, \mathcal{D})$ of additive subcategories of $\mathcal{T}$, closed under direct summands, form a **torsion pair** if

- $\mathcal{T}(\mathcal{C}, \mathcal{D}) = 0$; and
- $\mathcal{T} = \mathcal{C} * \mathcal{D}$.

It follows: $\forall X \in \mathcal{T}$, there exists a (minimal) triangle

$$C_X \xrightarrow{f_X} X \xrightarrow{g_X} D_X \rightarrow$$

with $C_X \in \mathcal{C}$ and $D_X \in \mathcal{D}$. Moreover, f_X is a (minimal) right $\mathcal{C}$ -approximation and g_X is a (minimal) left $\mathcal{D}$ -approximation.

Theorem

Suppose $\mathcal{X} \subseteq \mathcal{S}$ for a simple-minded system $\mathcal{S}$ in $\mathcal{T}$. Then $({}^\perp \mathcal{X}, \mathcal{F}(\mathcal{X}))$ and $(\mathcal{F}(\mathcal{X}), \mathcal{X}^\perp)$ are torsion pairs in $\mathcal{T}$. In particular, $\mathcal{F}(\mathcal{X})$ is a functorially finite subcategory of $\mathcal{T}$.

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Mutation of simple-minded systems

Let $\mathcal{S}$ be a simple-minded system, and $\mathcal{X} \subset \mathcal{S}$ s.t. $\nu(\mathcal{X}[1]) = \mathcal{X}$.

For any $M \in \mathcal{T}$, we have (unique) minimal triangles

$$\begin{array}{ccccccc} \mathbf{a}M \rightarrow M \rightarrow \mathbf{b}M \rightarrow & & \mathbf{c}M \rightarrow M \rightarrow \mathbf{d}M \rightarrow \\ \in {}^\perp\mathcal{X} & \in \mathcal{F}(\mathcal{X}) & \in \mathcal{F}(\mathcal{X}) & \in \mathcal{X}^\perp \end{array}$$

Definition

The **left mutation** of $\mathcal{S}$ at $\mathcal{X}$ is $\mu_{\mathcal{X}}^+(\mathcal{S}) = \{\mu_{\mathcal{X}}^+(S_i) \mid S_i \in \mathcal{S}\}$, where

$$\mu_{\mathcal{X}}^+(S_i) = \begin{cases} S_i, & \text{if } S_i \in \mathcal{X} \\ \mathbf{a}(S_i[-1]), & \text{if } S_i \notin \mathcal{X} \end{cases}$$

Right mutation of $\mathcal{S}$ at $\mathcal{X}$ is defined dually:

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Mutation of simple-minded systems (cont.)

Theorem

For any $\mathcal{X} \subset \mathcal{S}$ satisfying $\nu(\mathcal{X}[1]) = \mathcal{X}$, $\mu_{\mathcal{X}}^+(\mathcal{S})$ and $\mu_{\mathcal{X}}^-(\mathcal{S})$ are again simple-minded systems. Moreover,

$$\mu_{\mathcal{X}}^-(\mu_{\mathcal{X}}^+(\mathcal{S})) = \mathcal{S} = \mu_{\mathcal{X}}^+(\mu_{\mathcal{X}}^-(\mathcal{S})).$$

Example. $\mu_1^+(\mathcal{S}_{\Lambda}), \dots$

Let Λ have the quiver and indecomp. projectives:



Mutation of simple-minded systems (cont.)

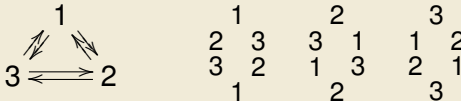
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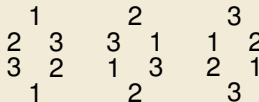
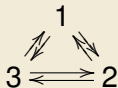
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Connection to derived equivalence

- Let $\mathcal{X} \subset \mathcal{S}_\Lambda$ satisfy $\nu(\mathcal{X}[1]) = \mathcal{X}$.
- Let $P = P_{\mathcal{X}}$ be the corresponding projective module, and $Q = \Lambda/P$.
- Let $P \xrightarrow{f} Q'$ be a left $\text{add}(Q)$ -approximation of P . The Okuyama tilting complex $T_{\mathcal{X}}$ associated to $\mathcal{X}$ is

$$0 \rightarrow P \xrightarrow{(0 \ f)^T} Q \oplus Q' \rightarrow 0$$

- $\Gamma = \text{End}_{D^b(\Lambda)}(T_{\mathcal{X}})$ is the “left tilting mutation of Λ at $\mathcal{X}$.”
- Let $\underline{F}_{\mathcal{X}} : \underline{\text{mod}}\text{-}\Gamma \rightarrow \underline{\text{mod}}\text{-}\Lambda$ be the induced equivalence.

Theorem. [–, Koenig-Yang]

$$\underline{F}_{\mathcal{X}}(\mathcal{S}_\Gamma) = \mu_{\mathcal{X}}^+(\mathcal{S}_\Lambda).$$

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Application: Lifting stable equivalences to derived equivalences

Question: Determine if a stable equivalence (of Morita type) $\alpha : \underline{\text{mod}}\text{-}\Lambda' \rightarrow \underline{\text{mod}}\text{-}\Lambda$ is induced by an equivalence of derived categories $D^b(\Lambda') \rightarrow D^b(\Lambda)$.

Method of Okuyama, Linckelmann, Rickard: If $\alpha(\mathcal{S}_{\Lambda'}) = \underline{F}(\mathcal{S}_{\Gamma})$ where $\underline{F}$ is induced by an equivalence $F : D^b(\Gamma) \rightarrow D^b(\Lambda)$, then α lifts to a derived equivalence.

Corollary

If $\alpha(\mathcal{S}_{\Lambda'})$ can be obtained from $\mathcal{S}_{\Lambda}$ by a sequence of mutations and syzygies/co-syzygies, then α lifts to a derived equivalence.

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Method of Okuyama, Linckelmann, Rickard: If $\alpha(\mathcal{S}_{\Lambda'}) = \underline{F}(\mathcal{S}_{\Gamma})$ where $\underline{F}$ is induced by an equivalence $F : D^b(\Gamma) \rightarrow D^b(\Lambda)$, then α lifts to a derived equivalence.

Corollary

If $\alpha(\mathcal{S}_{\Lambda'})$ can be obtained from $\mathcal{S}_{\Lambda}$ by a sequence of mutations and syzygies/co-syzygies, then α lifts to a derived equivalence.

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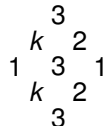
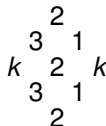
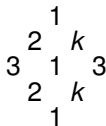
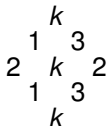
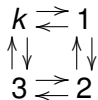
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- Let $G = A_6$, $P = \text{Syl}_3(G) \cong (\mathbb{Z}/3)^2$, $H = N_G(P)$.
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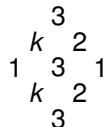
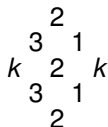
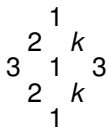
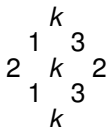
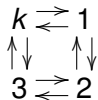
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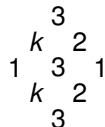
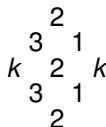
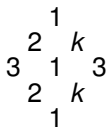
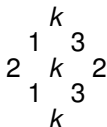
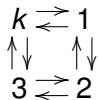
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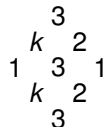
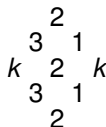
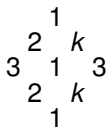
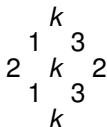
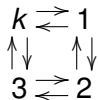
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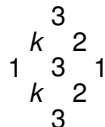
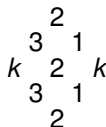
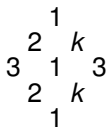
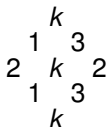
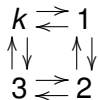
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