Framed Moduli Spaces, Grassmannians and tuples of operators

S. Fedotov[1]

[1] Department of Higher Algebra
Moscow State University

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A Simple Problem Of Linear Algebra

Consider the quiver L_q with one vertex and q loops.

Classifying its representations with dimension vector (m) is literally the same as classifying tuples of q operators on a m-dimensional vector space.

We are in trouble: this problem is WILD (no solution expected).

A Simple Problem Of Linear Algebra

To make things easier we should first make them worse.

Add k more arrows to a new vertex ∞ and denote the quiver by $L_{q,k}$. We will only consider its representations with dimension vector $(m, 1)$, that is tuples of q linear operators and k linear functions on an m-dimensional vector space. It turns out that on a natural Zarisky open subset of $\mathrm{Rep}(L_{q,k}, (m, 1))$ a complete classification is possible.

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Framed Representations

Definition

Let Q be a quiver, α and ζ be two dimension vectors. Consider a new quiver Q^ζ with $Q_0^\zeta = Q_0 \sqcup \{\infty\}$ and

$$Q_1^\zeta = Q_1 \sqcup \{f_{ij} : i \to \infty \mid i \in Q_0,\ j = 1, \dots, \zeta_i\}$$

Example

$$Q \quad = \quad \begin{tikzcd} 1 \arrow[r, "b"] \arrow[rr, bend right, "a"'] & 2 \arrow[r, "c"] & 3 \arrow[loop right, "d"] \end{tikzcd}$$

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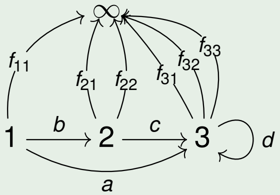
Example ($\zeta = (1, 2, 3)$)

$Q^\zeta =$



Framed Representations

Definition (Continued)

- $\mathrm{Rep}(Q, \alpha, \zeta) := \mathrm{Rep}(Q^\zeta, (\alpha, 1))$ space of framed representations of Q with dim. vectors α and ζ.
- $\mathrm{GL}(\alpha)$-action: $g \cdot (M, (f_{ij})) = (g \cdot M, (f_{ij} g_i^{-1}))$.

Example ($\zeta = (1, 2, 3)$)

$$Q^\zeta = \begin{tikzcd} & \infty & \\ 1 \arrow[ur, bend left, "f_{11}"] \arrow[r, "b"] \arrow[rr, bend right, "a"'] & 2 \arrow[u, bend left, "f_{21}"] \arrow[u, bend right, "f_{22}"'] \arrow[r, "c"] & 3 \arrow[ul, "f_{31}"] \arrow[ul, bend right, "f_{32}"'] \arrow[ul, bend right=50, "f_{33}"'] \arrow[loop right, "d"] \end{tikzcd}$$

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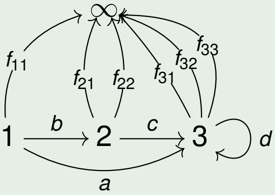

Stable Framed Representations

Definition

A pair (M, f) is `stable` is no proper nonzero subrepresentation N of M is contained in $\ker f = (\bigcap_{j=1}^{\zeta_i} \ker f_{ij})_{i \in Q_0}$.

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We denote by $\mathrm{Rep}^s(Q, \alpha, \zeta)$ the subset in $\mathrm{Rep}(Q, \alpha, \zeta)$ consisting of all stable pairs.

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Example

For $Q = L_q$, $\alpha = (m)$ and $\zeta = (k)$ framed representations are tuples

$$(a_1, \ldots, a_q, f_1, \ldots, f_k),$$

where $a_i \in L(\mathbb{k}^m)$ and $f_i \in (\mathbb{k}^m)^*$. Such a tuple is stable if no nonzero proper common invariant subspace of a_i is in $\bigcap_{i=1}^{k} \ker f_i$.

Framed Moduli Spaces

Theorem (Reineke)

The set $\mathrm{Rep}^s(Q, \alpha, \zeta)$ *admits a* `geometric quotient`, *i.e. a morphism*

$$\pi : \mathrm{Rep}^s(Q, \alpha, \zeta) \to \mathcal{M}^s(Q, \alpha, \zeta),$$

whose fibers are $\mathrm{GL}(\alpha)$*-orbits.*

Remark

The reason is that $\mathrm{Rep}^s(Q, \alpha, \zeta) \subset \mathrm{Rep}(Q^\zeta, (\alpha, 1))$ *is a subset of stable points with respect to a certain stability. So,* $\mathcal{M}^s(Q, \alpha, \zeta)$ *enjoys all the properties of a GIT quotient.*

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Grassmannians of Subrepresentations

Definition

For a representation X of a quiver Q and a dimension vector γ the `Grassmannian of` γ`-dimensional subrepresentations` of X is the closed subset $\mathrm{Gr}_\gamma^Q(X) \subseteq \prod_{i \in Q_0} \mathrm{Gr}_{\gamma_i}(X_i)$ consisting of tuples $(U_i \subseteq X_i)_{i \in Q_0}$ of subspaces satisfying $X_a(U_i) \subseteq U_j$, for all $Q_1 \ni a : i \to j$.

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Theorem (Reineke)

If Q is acyclic, we have

$$\mathcal{M}^s(Q, \alpha, \zeta) \cong \mathrm{Gr}_{(\alpha,1)}^{Q^\zeta}(I_\infty),$$

where I_∞ is the indecomposable injective representation of Q^ζ corresponding to the vertex ∞.

The projection π_s

We may also consider the standard categorical quotient

$$\mathcal{M}(Q, \alpha, \zeta) = \operatorname{Spec} \mathbb{k}[\operatorname{Rep}(Q, \alpha, \zeta)]^{\mathrm{GL}(\alpha)} = \mathcal{M}(Q, \alpha).$$

It is a general fact of GIT that there is a projection

$$\pi_s : \mathcal{M}^s(Q, \alpha, \zeta) \to \mathcal{M}(Q, \alpha).$$

So, instead of studying the whole variety $\mathcal{M}^s(Q, \alpha, \zeta)$, we may restrict our attention to fibers of π_s.

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The projection π_s

Theorem (Reineke)

Let $\Bbbk$ be algebraically closed. Then for each $y \in \mathcal{M}(Q, \alpha, \zeta)$ there is a quiver $\widetilde{Q}$ and dimension vectors $\widetilde{\alpha}$ and $\widetilde{\zeta}$ such that $\pi_s^{-1}(y) \cong \widetilde{\pi}_s^{-1}(0)$, where $\widetilde{\pi}$ is the natural projection $\mathcal{M}^s(\widetilde{Q}, \widetilde{\alpha}, \widetilde{\zeta}) \to \mathcal{M}(\widetilde{Q}, \widetilde{\alpha})$.

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Corollary (F)

Let $\mathbb{k}$ be algebraically closed. Then for each $y \in \mathcal{M}(Q, \alpha, \zeta)$ there is a finite dimensional algebra A, a dimension vector $\alpha \in \mathrm{K}_0(A)$ and an injective A-module J such that

$$\pi^{-1}(y) \cong \mathrm{Gr}_\alpha^A(J).$$

Over an arbitrary infinite field

Definition

A quiver Q is a quiver with successive cycles if whenever two oriented cycles in Q have a common vertex, they are both powers of a certain cycle.

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$$
\begin{tikzcd}
\bullet & & & \\
\bullet \arrow[u] \arrow[d] \arrow[r] & {\color{magenta}\bullet} \arrow[r] & {\color{magenta}\bullet} \arrow[r, shift left] \arrow[r, shift right] & \bullet \\
\bullet & & &
\end{tikzcd}
$$

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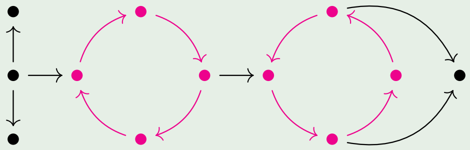

Over an arbitrary infinite field

Theorem (F)

Let Q be a quiver with successive cycles. Let also α and ζ be two dimension vectors and y be a point in $\mathcal{M}(Q, \alpha)$. There exists a quiver $Q^\spadesuit$, a dimension vector $\widetilde{\alpha} \in (\mathbb{Z}_{\geqslant 0})^{Q_0^\spadesuit}$, and a finite dimensional representation $W^\spadesuit$ of $Q^\spadesuit$ such that $\pi_s^{-1}(y) \cong \mathrm{Gr}_{\widetilde{\alpha}}^{\Bbbk Q^\spadesuit}(W^\spadesuit)$.

J-Skeleta of Representations

Definition

(1) A collection $\mathfrak{S}$ of paths in Q^ζ of nonzero length ending in ∞ is a J-`skeleton` if it whenever τa is in $\mathfrak{S}$ for a path τ and an arrow $a \in Q_1$ we have $\tau \in \mathfrak{S}$. A `dimension vector` of a skeleton $\mathfrak{S}$ is the vector $\underline{\dim}\mathfrak{S} = (|\mathfrak{S}_i|)_{i \in Q_0}$, where

$$\mathfrak{S}_i = \{\tau \in \mathfrak{S} \mid \operatorname{tail}(\tau) = i\}.$$

(2) For an α-dimensional skeleton $\mathfrak{S}$ define the following open subset in $\operatorname{Rep}(Q, \alpha, \zeta)$.

$$\operatorname{Rep}(Q, \mathfrak{S}) := \left\{ (M, f) \;\middle|\; \begin{array}{c} \text{paths from each of } \mathfrak{S}_i \\ \text{give linearly independent functions} \\ \text{on the corresponding } M_i \end{array} \right\},$$

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Example

$$
Q^\zeta = \begin{tikzcd}
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\end{tikzcd}
$$

$$\alpha = (4, 4, 5)$$

J-Skeleta of Representations

Example

$$Q^\zeta = \begin{tikzcd}[column sep=large, row sep=large] & & \infty & & \\ 1 \arrow[urr, bend left, "f_{11}"] \arrow[rr, "b"] \arrow[rrrr, bend right, "a"'] & & 2 \arrow[u, bend left, "f_{21}"] \arrow[u, bend right, "f_{22}"'] \arrow[rr, "c"] & & 3 \arrow[ull, "f_{31}"] \arrow[ull, bend right=20, "f_{32}"'] \arrow[ull, bend right=40, "f_{33}"'] \arrow[loop right, "d"] \end{tikzcd}$$

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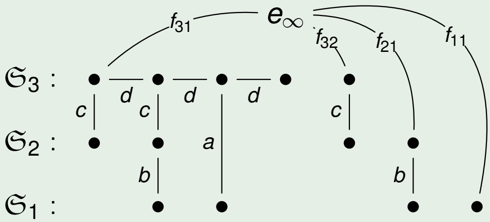

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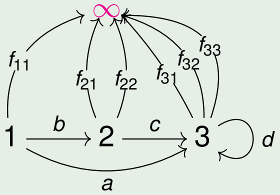

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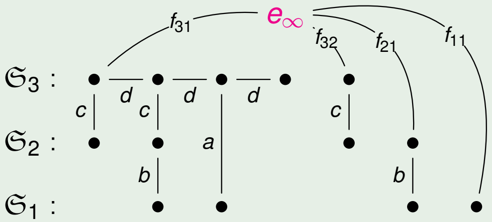

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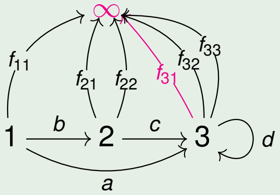
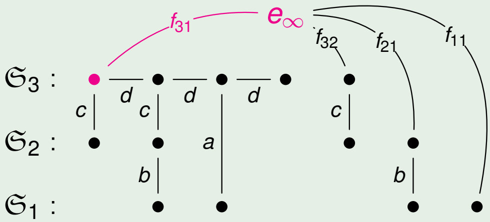


$\alpha = (4, 4, 5)$ $\tau = f_{31}$



J-Skeleta of Representations

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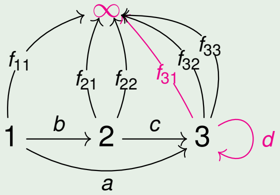

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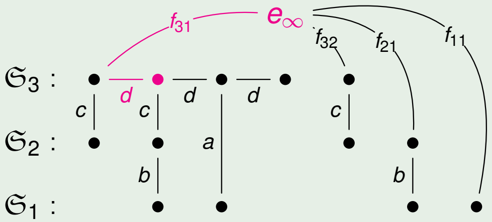

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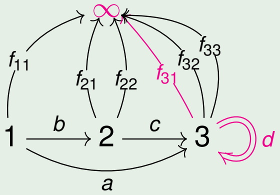
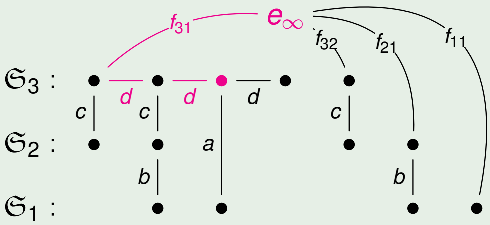


$\alpha = (4, 4, 5)$ $\tau = f_{31} d^2$



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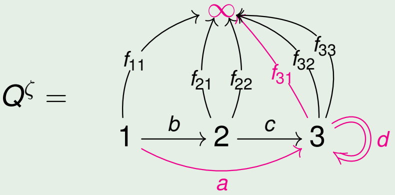
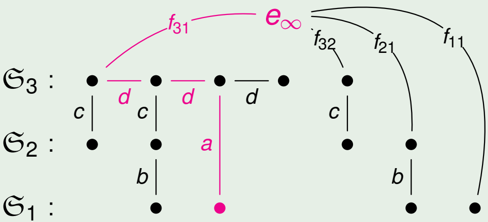


$\alpha = (4, 4, 5)$ $\tau = f_{31} d^2 a$



J-Skeleta of Representations

Example

$$D(\mathfrak{S}) = \left|\begin{matrix} f_{31}dcb \\ f_{31}d^2a \\ f_{21}b \\ f_{11} \end{matrix}\right| \cdot \left|\begin{matrix} f_{31}c \\ f_{31}dc \\ f_{32}c \\ f_{21} \end{matrix}\right| \cdot \left|\begin{matrix} f_{31} \\ f_{31}d \\ f_{31}d^2 \\ f_{31}d^3 \\ f_{32} \end{matrix}\right| .$$

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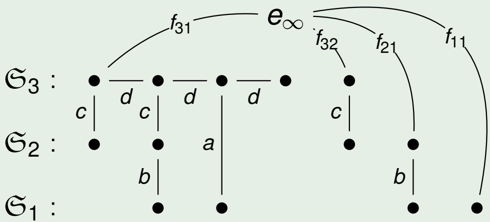

J-Skeleta of Representations

Lemma

$$\mathrm{Rep}(Q,\alpha,\zeta) = \bigcup_{\substack{\mathfrak{S} \text{ a } J\text{-skeleton,} \\ \underline{\dim}\,\mathfrak{S}=\alpha}} \mathrm{Rep}(Q,\mathfrak{S}).$$

J-Skeleta of Representations

Lemma

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Example

$$Q = L_{1,2} = \begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & \infty \end{tikzcd} \qquad \alpha = (2)$$

There are three J-skeleta with dimension vector α:

$$\mathfrak{S}_1 = (f_1, f_1 a), \qquad \mathfrak{S}_2 = (f_2, f_2 a),$$

$$\mathfrak{S}_3 = (f_1, f_2).$$

Normal Forms

Question 1: How To Parametrize Triples With A Given Skeleton?

Example (continued)

$$\mathrm{Rep}^s(Q, m, k) \ni (a, f_1, f_2)$$

Normal Forms

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Example (continued)

$$\mathrm{Rep}^s(Q,m,k) \ni (a, f_1, f_2) \rightsquigarrow \begin{pmatrix} f_1 \\ f_2 \\ f_1 a \\ f_2 a \\ f_1 a^2 \\ f_2 a^2 \\ f_1 a^3 \\ f_2 a^3 \\ \vdots \end{pmatrix}$$

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Example (continued)

$$\mathrm{Rep}^s(Q, (f_1, f_1 a)) \ni (a, f_1, f_2) \rightsquigarrow \begin{pmatrix} \boxed{\begin{matrix} f_1 \\ f_2 \\ f_1 a \\ f_2 a \end{matrix}} \\ f_1 a^2 \\ f_2 a^2 \end{pmatrix}$$

Normal Forms

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Example (continued)

$$\mathrm{Rep}^s(Q,(f_1,f_1a)) \ni (a,f_1,f_2) \rightsquigarrow \begin{pmatrix} \boxed{f_1} \\ f_2 \\ \boxed{f_1a} \\ f_2a \\ f_1a^2 \\ f_2a^2 \end{pmatrix} * \begin{pmatrix} f_1 \\ f_1a \end{pmatrix}^{-1}$$

Normal Forms

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Example (continued)

$$(a, f_1, f_2) \rightsquigarrow \begin{pmatrix} \boxed{f_1} \\ f_2 \\ \boxed{f_1 a} \\ f_2 a \\ f_1 a^2 \\ f_2 a^2 \end{pmatrix} * \begin{pmatrix} f_1 \\ f_1 a \end{pmatrix}^{-1} = \begin{pmatrix} \boxed{1\ 0} \\ \text{“}f_2\text{“} \\ \boxed{0\ 1} \\ \text{“}f_2 a\text{“} \\ \text{“}f_1 a^2\text{“} \\ \text{“}f_2 a^2\text{“} \end{pmatrix}$$

Normal Forms

Question 1: How To Parametrize Triples With A Given Skeleton?

Example (continued)

We now take rows corresponding to elements of

$$\mathfrak{S}_1 a \backslash \mathfrak{S}_1 \cup \{f_i \notin \mathfrak{S}_1\}.$$

$$(a, f_1, f_2) \rightsquigarrow \begin{pmatrix} \boxed{f_1} \\ f_2 \\ \boxed{f_1 a} \\ f_2 a \\ f_1 a^2 \\ f_2 a^2 \end{pmatrix} * \begin{pmatrix} f_1 \\ f_1 a \end{pmatrix}^{-1} = \begin{pmatrix} \boxed{1\ 0} \\ \text{``}f_2\text{``} \\ \boxed{0\ 1} \\ \text{``}f_2 a\text{``} \\ \text{``}f_1 a^2\text{``} \\ \text{``}f_2 a^2\text{``} \end{pmatrix}$$

Normal Forms

Question 1: How To Parametrize Triples With A Given Skeleton?

Example (continued)

Claim: The assignment

$$(a, f_1, f_2) \rightsquigarrow \begin{pmatrix} f_2 \\ f_1 a^2 \end{pmatrix} * \begin{pmatrix} f_1 \\ f_1 a \end{pmatrix}^{-1} = \begin{pmatrix} \text{“}f_2\text{“} \\ \text{“}f_1 a^2\text{“} \end{pmatrix}$$

$$\mathrm{Rep}(Q, \mathfrak{S}_1) \to \mathrm{Mat}_2(\Bbbk) \cong \mathbb{A}^4$$

is a geometric quotient for the action of $\mathrm{GL}_2(\Bbbk)$.

Normal Forms

Question 2: How To Get My Triple Back?

Example (continued)

We now have a 2×2-matrix. How to assign a canonical triple (a, f_1, f_2) to it?

$$? \rightsquigarrow \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Normal Forms

Question 2: How To Get My Triple Back?

Example (continued)

Recall one of the intermediate steps:

$$? \rightsquigarrow \begin{pmatrix} \boxed{1\ \ 0} \\ x_1\ \ x_2 \\ \boxed{0\ \ 1} \\ ? \\ y_1\ \ y_2 \\ ? \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Normal Forms
Question 2: How To Get My Triple Back?

Example (continued)

Trivial observation: $$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} a = \begin{pmatrix} f_1 a \\ f_2 a \end{pmatrix}$$

$$? \rightsquigarrow \begin{pmatrix} \boxed{1\ 0} \\ x_1 \ \ x_2 \\ \boxed{0\ 1} \\ ? \\ y_1 \ \ y_2 \\ ? \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Normal Forms

Question 2: How To Get My Triple Back?

Example (continued)

Trivial observation: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} a = \begin{pmatrix} 0 & 1 \\ y_1 & y_2 \end{pmatrix}$

$$? \rightsquigarrow \begin{pmatrix} \boxed{1\ 0} \\ x_1\ \ x_2 \\ \boxed{0\ 1} \\ ? \\ y_1\ \ y_2 \\ ? \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Normal Forms

Question 2: How To Get My Triple Back?

Example (continued)

Trivial observation: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} a = \begin{pmatrix} 0 & 1 \\ y_1 & y_2 \end{pmatrix}$

$$\left(\begin{pmatrix} 0 & 1 \\ y_1 & y_2 \end{pmatrix}, (1\ 0), (x_1\ \ x_2) \right) \rightsquigarrow \begin{pmatrix} \boxed{1\ 0} \\ x_1\ \ x_2 \\ \boxed{0\ 1} \\ ? \\ y_1\ \ y_2 \\ ? \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Thank you for your attention!