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Partial Orders on Representations of Algebras

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Outline

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New Partial Orders

Results

Examples

Notation

Throughout we use the following notation:

- k algebraically closed field or artinian ring.
- Λ finitely generated k-algebra.
- $\text{rep}_d \Lambda$ - the space of d-dimensional representations of Λ.
- $G = GL_d(k)$ - the d-dimensional general linear group over k.
- $\mathcal{M}_{n \times m}(\Lambda)$ - the set of $m \times n$-matrices with entries from Λ.

The set-up

We shall consider the space of d-dimensional representations of Λ over k, i.e.

$$\operatorname{rep}_d \Lambda = \{f \colon \Lambda \longrightarrow \mathcal{M}_{d\times d}(k) \mid f \text{ is a } k\text{-algebra homomorphism}\}$$

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Hence one can view $\operatorname{rep}_d \Lambda$ as the affine variety:

$$\{(M)_{i=1,\ldots,t} \in \mathcal{M}_{d\times d}(k)^t \mid f(M)_{i=1,\ldots,t} = 0 \; \forall f \in \operatorname{Ker} \varphi\}.$$

The geometry

G acts on $\mathrm{rep}_d \Lambda$ by conjugation, i.e. for $g \in G$ and $m \in \mathrm{rep}_d \Lambda$ $g * m := (gM_ig^{-1})_{i=1,\dots,t}$. The G-orbits correspond to the isomorphism classes of Λ-modules on k^d.

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By abuse of notation, for M, N d-dimensional Λ-modules, M degenerates to N if $n \in \overline{G * m}$ and denote this by $M \leqslant_{\mathrm{deg}} N$. Thus, $\leqslant_{\mathrm{deg}}$ is a partial order on $\mathrm{rep}_d \Lambda$.

Degenerations

Theorem (Riedtmann '86, Zwara '98)

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3. There exists $Z \in \operatorname{mod} \Lambda$ and exact sequence $0 \longrightarrow N \longrightarrow M \oplus Z \longrightarrow Z \longrightarrow 0.$

In fact Zwara showed that 2 (dually 3) is a partial order on finite length modules for an artin algebra. Hence we may take $\operatorname{rep}_d \Lambda$ to be modules of length d over an artin algebra Λ.

The Picture

$$M \leqslant_{\text{deg}} N$$

The Picture

$M \leqslant_{\text{deg}} N$ $\qquad$ $M \leqslant_{\text{vdeg}} N$

Definition (virtual degeneration)

Let $M, N \in \text{rep}_d \Lambda$, then if there is X a Λ-module such that $M \oplus X \leqslant_{\text{deg}} N \oplus X$ then M **virtually degenerates** to N.

The Picture

$M \leqslant_{\text{deg}} N$ $M \leqslant_{\text{vdeg}} N$ $M \leqslant_{\text{hom}} N$

Definition (Hom-relation)

Let $M, N \in \text{rep}_d \Lambda$, then $M \leqslant_{\text{hom}} N$ if for all $X \in \text{mod}\,\Lambda$ we have that $\ell_k(\text{Hom}_\Lambda(X, M)) \leqslant \ell_k(\text{Hom}_\Lambda(X, N))$.

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For algebras of finite representation type, tame hereditary and $k[x]$.

The Picture

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Let $\Lambda_q = k\langle X, Y\rangle / \langle X^2, Y^2, XY + qYX\rangle$ and $M_a = \Lambda / \langle X + aY\rangle$. Then $\Lambda \oplus S \leqslant_{\mathrm{deg}} M_a \oplus M_b \oplus S$ for all $a, b \in \mathbb{P}_1(k)$, however there exists a, b such that $\Lambda \nleqslant_{\mathrm{deg}} M_a \oplus M_b$.

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Open problem in general.

New orders

Definition

Let $n \in \mathbb{N}$ and $M, N \in \operatorname{rep}_d \Lambda$ then $M \leqslant_n N$ if $\ell_k(M^n/A(M^n)) \leqslant \ell_k(N^n/A(N^n))$ holds for all $A \in \mathcal{M}_{n\times n}(\Lambda)$

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Clearly this is a quasi-order for any $n \in \mathbb{N}$, i.e. reflexive and transitive. However, anti-symmetry remains to be shown.

Our Results

Proposition

For $M, N \in \text{rep}_d \Lambda$ and $r, s \in \mathbb{N}$ we have

1. $M \leqslant_r N \Rightarrow M \leqslant_s N$ whenever $r \geqslant s$, in particular $\leqslant_{n+i}$ is a partial order whenever $\leqslant_n$ is, for all $i \geqslant 0$.
2. $M \leqslant_{\text{hom}} N \Rightarrow M \leqslant_n N$ for all $n \in \mathbb{N}$.

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To show 2 we consider the sequence $\Lambda^n \xrightarrow{\varphi} \Lambda^n \longrightarrow X$ and get the following commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & (\Lambda^n/\Lambda^n\varphi, M) & \longrightarrow & (\Lambda^n, M) & \longrightarrow & (\Lambda^n, M) & & & \\
 & & & \wr\| & & \wr\| & & & \\
 & & & M^n & \longrightarrow & M^n & \longrightarrow & M^n/\varphi M^n & \longrightarrow 0
\end{array}$$

Our Results

Auslander (1982) showed that $\leqslant_{\text{hom}}$ is a partial order using AR-sequences, later Bongartz (1989) generalized this to k commutative ring and abelian k-categories. Also, in Bongartz' proof it is shown that $M \simeq N$ if and only if $\ell({}_{\Lambda}(X, M)) = \ell({}_{\Lambda}(X, N))$ for a finite test set.

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Proposition

Let $M, N \in \operatorname{rep}_d \Lambda$ and let $L_0 = M \oplus N$ and $L_{i+1} = \operatorname{rad}(\operatorname{End}_\Lambda(L_i)) \cdot L_i$ for $i = 1, \ldots, r$ with $L_{r+1} = 0$ and $L_r \neq 0$. Then

$$M \simeq N \Longleftrightarrow \ell(_\Lambda(X, M)) = \ell(_\Lambda(X, N)) \; \forall X \in \operatorname{add} \bigoplus_{i=0}^{r} L_i.$$

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$$M \simeq N \Longleftrightarrow \ell({}_\Lambda(X, M)) = \ell({}_\Lambda(X, N)) \, \forall X \in \operatorname{add} \bigoplus_{i=0}^r L_i.$$

Key ingredient: Iyama's *Finiteness of Representation Dimension*, more specefically $\operatorname{End}(\bigoplus_{i=0}^r L_i)$ has finite global dimension.

Our Results

Theorem

The relation $\leqslant_{d^3}$ is a partial order on $\operatorname{rep}_d \Lambda$.

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The proof uses the preceeding proposition and the lemma:
$M, N \in \operatorname{rep}_d \Lambda$ and $X \in \operatorname{rep}_s \Lambda$. If $M \leqslant_{d^2 s} N$ and $N \leqslant_{d^2 s} M$, then $\ell({}_\Lambda(X, M)) = \ell({}_\Lambda(X, N))$.

Examples

1. Λ - the path algebra of the quiver $\mathbb{A}_n$ with any orientation, then $\leqslant_1$ is a partial order and it coincides with $\leqslant_{\text{hom}}$.

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2. There cannot be a global n such that $\leqslant_n$ is a partial order on $\text{rep}_d\,\Lambda$ for any d. Namely, let $\delta\colon 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an almost split sequence. Then $\delta^*(X) = 0$ for all X with $\ell(X) < \ell(C)$.

Open problems

1. If $\leqslant_n$ is a partial order, is it the same as $\leqslant_{n+1}$?

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1. If $\leqslant_n$ is a partial order, is it the same as $\leqslant_{n+1}$?
2. Is $\leqslant_{d^3} \equiv \leqslant_{\text{hom}}$ on $\text{rep}_d \Lambda$, more generally is there a n such that $\leqslant_n \equiv \leqslant_{\text{hom}}$?