On root categories of finite-dimensional algebras

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Hall number

Let $\mathcal{A}$ be a finitary abelian category:

$$\forall X, Y \in \mathcal{A}, |\operatorname{Hom}(X, Y)| < \infty \quad \text{and} \quad |\operatorname{Ext}^1(X, Y)| < \infty.$$

For any $L, M, N \in \mathcal{A}$, the Hall number $F^N_{M,L}$ is

$$F^N_{M,L} := |\{X \subset N | X \cong L, N/X \cong M\}|.$$

Let $\operatorname{Iso}(\mathcal{A})$ be the of isomorphism classes of objects in $\mathcal{A}$ and set

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{M \in \operatorname{Iso}(\mathcal{A})} \mathbb{Z}u_M.$$

For any $L, M \in \operatorname{Iso}(\mathcal{A})$, define

$$u_L * u_M := \sum_{N \in \operatorname{Iso}(\mathcal{A})} F^N_{M,L} u_N.$$

Ringel's Theorem

Theorem (Ringel)

$(\mathcal{H}(\mathcal{A}), *)$ *is an associative algebra with unit* u_0*, where* 0 *is the zero object of* $\mathcal{A}$*.*

Applications:

- realized the positive part $U_v(b)$ of the quantized enveloping algebra $U_v(\mathfrak{g})$[Ringel, Green, etc];
- realized the positive part $\mathfrak{n}$ of the derived Kac-Moody algebra $\mathfrak{g}$.

2-periodic triangulated category

Let $\mathcal{R}$ a 2-periodic triangulated k-category with suspension functor Σ:

- $\Sigma^2 \cong \mathrm{id}$;
- for any indecomposable object $X \in \mathcal{R}$, the endomorphism algebra $\mathrm{End}(X)$ is a local k-algebra.

A triangulated k-category $\mathcal{T}$ is called finitary if

$$\forall X, Y \in \mathcal{T}, |\mathrm{Hom}(X, Y)| < \infty.$$

If k is a finite field, then this condition is equivalent to the Hom-finite condition

$$\forall X, Y \in \mathcal{T}, \dim_k \mathrm{Hom}(X, Y) < \infty.$$

Example: 2-periodic homotopy category

Example

Let A be a finite-dimensional algebra over a field k.
Let $\mathcal{P}$ be the additive category of finitely generated projective right A-modules. Let $\mathcal{C}_2(\mathcal{P})$ be the category of 2-periodic complexes of $\mathcal{P}$ and $\mathcal{H}_2(\mathcal{P})$ the associated homotopy category of $\mathcal{C}_2(\mathcal{P})$.
Then $\mathcal{H}_2(\mathcal{P})$ is a 2-periodic triangulated category.

Example: root categories of hereditary algebras

Example

A: a finite-dimensional hereditary algebra over a field k;

$\mathcal{D}^b(\operatorname{mod} A)$: bounded derived category of $\operatorname{mod} A$;

$\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$: the orbit category of $\mathcal{D}^b(\operatorname{mod} A)$ by the square of suspension functor Σ.

$\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$ is called the root category of A introduced by D. Happel in 1987.
Peng-Xiao(1997): the root category $\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$ is a 2-periodic triangulated category. In this case, $\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2 \cong \mathcal{H}_2(\mathcal{P})$ as triangulated categories.

Notations

k: finite filed with $|k| = q$;

$\mathcal{R}$: 2-periodic triangulated category over k;

$\mathrm{ind}\mathcal{R}$: set of isoclasses of indecomposable objects of $\mathcal{R}$;

h_M: image of M in the Grothendieck group $\mathrm{G}_0(\mathcal{R})$;

$\mathfrak{h}$: subgroup of $\mathrm{G}_0(\mathcal{R}) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $\frac{h_M}{d(M)}$, $M \in \mathrm{ind}\mathcal{R}$, where $d(M) = \dim_k(\mathrm{End}(X)/\operatorname{rad}\mathrm{End}(X))$;

$\mathfrak{n}$: the free abelian group with basis $\{u_X | X \in \mathrm{ind}\mathcal{R}\}$;

$\mathfrak{g}(\mathcal{R}) = \mathfrak{h} \oplus \mathfrak{n}$;

$\mathfrak{g}(\mathcal{R})_{(q-1)} = \mathfrak{g}(\mathcal{R})/(q-1)\mathfrak{g}(\mathcal{R})$.

The Lie bracket

Define a $\mathbb{Z}$-linear bracket $[-,-]$ over $\mathfrak{g}(\mathcal{R})_{(q-1)}$ as follows:

(1) $\forall X, Y \in \mathrm{ind}\mathcal{R}$,

$$[u_X, u_Y] = \sum_{L \in \mathrm{ind}\mathcal{R}} (F^L_{YX} - F^L_{XY})u_L - \delta_{X,\Sigma Y}\frac{h_X}{d(X)},$$

where $\delta_{X,\Sigma Y} = 1$ for $X \cong \Sigma Y$ and 0 else.

(2) $[\mathfrak{h}, \mathfrak{h}] = 0$.

(3) for any objects $X, Y \in \mathcal{R}$ with Y indecomposable,

$$[h_X, u_Y] = I_{\mathcal{R}}(h_X, h_Y)u_Y, \qquad [u_Y, h_X] = -[h_X, u_Y].$$

Peng-Xiao's Theorem

Theorem (Peng-Xiao2000)

Together with $[-,-]$, $\mathfrak{g}(\mathcal{R})_{(q-1)}$ is a Lie algebra over $\mathbb{Z}/(q-1)\mathbb{Z}$.

Application:

- Peng-Xiao(2000): for the root categories of finite-dimensional hereditary algebras, an integral version of $\mathfrak{g}(\mathcal{R})_{(q-1)}$ realized all the symmetrizable derived Kac-Moody algebras;
- Lin-Peng(2005): realized the elliptic Lie algebra of type $\hat{\hat{D}}_4, \hat{\hat{E}}_*, * = 6, 7, 8$ via the 2-periodic orbit categories (which is triangulated) of tubular algebras.

Realizations of Lie algebras

- To 'categorify' the elliptic Lie algebra $\mathfrak{g}$ of type $\hat{\hat{A}}_n, n \geq 1$, $\hat{\hat{D}}_m, m \geq 5$;
- To 'categorify' the Virasora algebra $\mathfrak{g}$.

At this moment, there are no suitable categorifications for $\mathfrak{g}$. It is even not clear that there is $\mathcal{R}$ such that $\mathrm{G}_0(\mathcal{R})$ realize the root lattice of $\mathfrak{g}$.

Aim

Remark

For a general finite-dimensional algebra A, its 2-periodic orbit category does not admit a canonical triangle structure: the projection functor $\pi : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{D}^b(\mathrm{mod}\, A)/\Sigma^2$ *is a triangle functor.*

To construct new 2-periodic triangulated categories which are good replacements of 2-periodic orbit categories.

Keller's construction

Let A be a finite-dimensional k-algebra of finite global dimension. Define $\mathcal{S}$ be the differential graded algebra with trivial differential whose underlying complex are

$$A \oplus \Sigma A$$

and the multiplication is given by trivial extension.

Definition

The generalized root category $\mathcal{R}_A$ of A is defined to be $\mathcal{D}^b(\mathcal{S})/\mathrm{per}(\mathcal{S})$.

The universal property

[Keller2005]The generalized root category $\mathcal{R}_A$ has the following universal property:

- There exists an algebraic triangulated functor $\pi_A : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{R}_A$;
- Let $\mathcal{B}$ be a dg category and X an object of $\mathcal{D}(A^{op} \otimes \mathcal{B})$. If there exists an isomorphism in $\mathcal{D}(A^{op} \otimes \mathcal{B})$ between $\Sigma^2 A \overset{L}{\otimes}_A X$ and X, then the triangulated algebraic functor $? \overset{L}{\otimes}_A X : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{D}(\mathcal{B})$ factorizes through π_A.

Remark

(1) we have an embedding $i : \mathcal{D}^b(\text{mod}\, A)/\Sigma^2 \hookrightarrow \mathcal{R}_A$ of categories. Once the 2-periodic orbit category admits a canonical triangle structure, then i is dense;

(2) we have an embedding $\mathcal{R}_A \hookrightarrow \mathcal{H}_2(\mathcal{P})$ of triangulated categories. If A is hereditary, then we have equivalences of triangulated categories $\mathcal{D}^b(\text{mod}\, A) \cong \mathcal{R}_A \cong \mathcal{H}_2(\mathcal{P})$.

An example of global dimension 2

Let Q be the following quiver

$$\begin{tikzcd} 2 \arrow[r, bend left, "\alpha"] & 1 \arrow[l, bend left, "\beta"] \end{tikzcd}.$$

Denote $A := kQ/\langle \beta \circ \alpha \rangle$. It is representation-finite and has global dimension 2.

Proposition

The 2-periodic orbit category $\mathcal{D}^b(\mathrm{mod}\, A)/\Sigma^2$ does not admit a canonical triangle structure.

This proposition implies particular that the notion of generalized root category makes sense.

Properties of generalized root category

Proposition

Let A be a finite-dimensional algebra over a field k. Assume that A has finite global dimension. We have

(1) *the generalized root category $\mathcal{R}_A$ is a 2-periodic triangulated category;*

(2) *the generalized root category $\mathcal{R}_A$ admits Auslander-Reiten sequence;*

(3) *the canonical functor $\pi_A : \mathcal{D}^b(\text{mod}\, A) \to \mathcal{R}_A$ induces an isometry $\mathrm{G}_0(\mathcal{D}^b(\text{mod}\, A)) \cong \mathrm{G}_0(\mathcal{R}_A)$;*

(4) *the canonical functor π_A maps AR-triangles of $\mathcal{D}^b(\text{mod}\, A)$ to AR-triangles of $\mathcal{R}_A$.*

Representation-finite hereditary algebras

By applying the part (4),

Theorem

Let A be a finite-dimensional algebra of finite global dimension. Suppose that the generalized root category $\mathcal{R}_A$ is triangle-equivalent to the root category of a representation-finite hereditary algebra kQ, then A is derived equivalent to kQ.

This theorem holds true for tame hereditary algebra of type D and E.

Conjecture

If the generalize root category $\mathcal{R}_A$ of A is triangle-equivalent to the root category of a finite-dimensional hereditary algebra kQ, then A is derived equivalent to kQ.

The construction

Let A be a finite-dimensional algebra over a finite field k. Assume that A has finite global dimension. Let $S_1, \cdots, S_n$ be all the pairwise non-isomorphic simple A-modules.

For E be a field extension of k and set $V^E = V \otimes_k E$ for any k-space V. Then A^E is an E-algebra and, for $M \in \operatorname{mod} A$, M^E has a canonical A^E-module structure. For any indecomposable A-module X, E is conservative for X, if $(\operatorname{End}(X)/\operatorname{rad}\operatorname{End}(X))^E$ is a field.

Let

$$\Omega = \{E | k \subseteq E \subseteq \overline{k} \text{ is a finite field extension}$$
$$\text{which is conservative for all simple modules}\}$$

One can show that $A^E, E \in \Omega$ have finite global dimension. Thus, one can define the generalized root category $\mathcal{R}_{A^E}$ and form the Ringel-Hall Lie algebra $\mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)}$. Consider the product

$$\prod_{E\in\Omega} \mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)},$$

let $\mathcal{LC}(\mathcal{R}_A)$ be the subalgebra generated by $u_{S_i} := (u_{S_i^E})_{E\in\Omega}$ and $u_{\Sigma S_i} := (u_{\Sigma S_i^E})$ for $1 \leq i \leq n$. We call $\mathcal{LC}(\mathcal{R}_A)$ the integral Ringel-Hall Lie algebra of A.

Peng-Xiao's realization

Theorem (Peng-Xiao2000)

Let Q be a finite acyclic quiver, kQ the path algebra over a finite field k. Then

$$\mathcal{LC}(\mathcal{R}_{kQ}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{KM}(kQ),$$

where $\mathrm{KM}(kQ)$ is the derived Kac-Moody algebra associated to the underlying diagram of Q.

Applications

- ♡ By (1) of Prop 8, one can associated a Ringel-Hall Lie algebra $\mathcal{LC}(\mathcal{R}_A)$ in the sense of Peng-Xiao to any finite dimensional algebra of finite global dimension.
- ♡ By (3) of Prop 8, one can easily construct a lot of 2-periodic triangulated categories such that its Grothendieck group characterizes the root lattice of a given elliptic Lie algebra;

A motivating example

Let Q be the following quiver

$$\begin{tikzcd} 3 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2 \arrow[r, bend left, "\beta_1"] \arrow[r, bend right, "\beta_2"'] & 1. \end{tikzcd}$$

Let I be the ideal generated by $\beta_i \circ \alpha_i$. Let $A = kQ/I$ be the quotient algebra. The global dimension of A is 2.

(a) $\mathrm{G}_0(\mathcal{R}_A)$ characterizes the root lattice of elliptic Lie algebra of $\hat{\hat{A}}_1$;

(b) $\mathcal{LC}(\mathcal{R}_A)$ is a quotient of certain GIM Lie algebra associated to C_A.

(c) Does $\mathcal{LC}(\mathcal{R}_A)$ realize the elliptic Lie algebra of $\hat{\hat{A}}_1$?

Generalized intersection matrix

A matrix $A \in M_l(\mathbb{Z})$ is called a generalized intersection matrix, if

$$\begin{aligned} A_{ii} &= 2 \\ A_{ij} &< 0 \Longleftrightarrow A_{ji} < 0 \\ A_{ij} &> 0 \Longleftrightarrow A_{ji} > 0 \end{aligned}$$

A realization of A is a triple $(H, \bigtriangledown, \bigtriangleup)$ consisting of

- a finite dimensional $\mathbb{Q}$-vector space H;
- a family $\bigtriangledown = \{\alpha_1^\vee, \cdots, \alpha_l^\vee\}$, where $\alpha_i^\vee \in H$;
- a family $\bigtriangleup = \{\alpha_1, \cdots, \alpha_l\}$, where $\alpha_i \in H^* = \mathrm{Hom}_{\mathbb{Q}}(H, \mathbb{Q})$

GIM Lie algebra of Slodowy

The GIM-Lie algebra $\mathrm{gim}(A)$ attached to the realization $(H, \nabla, \triangle)$ is given by the generators $\mathfrak{h} = H \otimes_{\mathbb{Q}} \mathbb{C}$ and $e_{\pm\alpha}, \alpha \in \triangle$ satisfying the following relations:

$$
\begin{array}{ll}
(1) & [h, h'] = 0, h, h' \in \mathfrak{h} \\
(2) & [h, e_\alpha] = \alpha(h) e_\alpha, h \in \mathfrak{h}, \alpha \in \pm\triangle \\
(3) & [e_\alpha, e_{-\alpha}] = \alpha^\vee, \alpha \in \triangle \\
(4) & ad(e_\alpha)^{max(1, 1-\beta(\alpha^\vee))} e_\beta = 0, \alpha \in \triangle, \beta \in \pm\triangle \\
(5) & ad(e_{-\alpha})^{max(1, 1-\beta(-\alpha^\vee))} e_\beta = 0, \alpha \in \triangle, \beta \in \pm\triangle \ .
\end{array}
$$

Remark

If A is a generalized Cartan matrix, then $\mathrm{gim}(A)$ *is the Kac-Moody algebra associated to A.*

Roots of GIM algebras

The adjoint action of $\mathfrak{h}$ induces a gradation of $\mathrm{gim}(A)$ as follows

$$\mathrm{gim}(A) = \bigoplus_{\gamma \in \mathfrak{h}^*} \mathrm{gim}(A)_\gamma,$$

where

$$\mathrm{gim}(A)_\gamma = \{x \in \mathrm{gim}(A) | [h, x] = \gamma(h)x \text{ for all } h \in \mathfrak{h}\}.$$

$0 \neq \gamma \in \mathfrak{h}^*$ is called a root of $\mathrm{gim}(A)$ provided $\mathrm{gim}(A)_\gamma \neq 0$.

Question

Do we have $\mathfrak{h} = \mathrm{gim}(A)_0$?

Negative answer given by Alpen using fixed point subalgebra for certain Lie algebra.

A class of algebra of global dimension 2

Let Q be the following quiver

$$
\begin{tikzcd}
 & & & 0 \arrow[d, shift right, "\delta"'] \arrow[d, shift left, "\alpha"] & & & \\
n & n-1 \arrow[l] & 2 \arrow[l, dotted] & 1 \arrow[l, "\gamma"] \arrow[r, "\beta"] & n+1 \arrow[r, dotted] & \circ \arrow[r] & n+m
\end{tikzcd}
$$

We assume $m \geq 1, n \geq 2$. Let A be the quotient algebra of path algebra kQ by the ideal generated by $\beta \circ \alpha, \gamma \circ \alpha$. It has global dimension 2.

Results

Theorem

(a) *the generalized root category $\mathcal{R}_A$ is not triangle equivalent to the root category of any hereditary algebras;*

(b) *let C_A be the Cartan matrix of A, there is a graded surjective homomorphism of Lie algebras $\phi : \mathrm{gim}(C_A)' \to \mathcal{LC}(\mathcal{R}_A) \otimes_{\mathbb{Z}} \mathbb{C}$ such that*

$$\begin{aligned} \alpha_i^\vee &\mapsto h_i, \\ e_{\alpha_i} &\mapsto u_{S_i} \\ e_{-\alpha_i} &\mapsto -u_{\Sigma S_i}, 0 \le i \le n+m. \end{aligned}$$

(c) $\dim_{\mathbb{C}} \mathrm{gim}(C_A)'_0 \geq m+n+2.$

Remarks

The above theorem gives us the following remarks about GIM Lie algebras.

Remark

- *We have* $\dim \mathrm{gim}(C_A)_0 > \dim H \otimes_{\mathbb{Z}} \mathbb{C}$. *In particular, this gives a negative answer for Slodowy's question;*
- *The idea* τ *of* $\mathrm{gim}(C_A)$ *is non-zero;*
- *GIM Lie algebras are not invariant under braid equivalence.*

Generalized root category for algebraic trian. category

The generalized root category can be defined for algebraic triangulated category $\mathcal{T}$ such that for any $X, Y \in \mathcal{T}$,

$$\dim \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(X, \Sigma^{2n} Y) < \infty.$$

In [FuYang2012], we have studied the generalized root category of algebraic triangulated category generated by a spherical object and determined the structure of the associated Ringel-Hall Lie algebra.

Thanks for your attention!