A quantum analogue of a dihedral group action on Grassmannians

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Properties of Grassmannians

Recall that $\mathrm{Gr}(k,n)$ is the projective variety of k-dimensional subspaces of an n-dimensional space. Its coordinate ring $\mathscr{O}(\mathrm{Gr}(k,n))$ is generated by the Plücker coordinates $\{\Delta^I \mid I \subseteq \{1,\dots,n\}, |I| = k\}$, subject to the Plücker relations.

One may specify a point in the Grassmannian $\mathrm{Gr}(k,n)$ as a $k \times n$ matrix of rank k and the symmetric group S_n acts on the Grassmannian via permutation of columns. Thus the dihedral subgroup D_{2n} of S_n generated by the n-cycle $c = (1\,2\,\cdots\,n)$ and the longest element w_0 acts on $\mathrm{Gr}(k,n)$ and hence on $\mathscr{O}(\mathrm{Gr}(k,n))$.

The Grassmannian also admits an action of a torus $T_n \cong (\mathbb{K}^*)^n$. The complex Grassmannian is a Poisson homogeneous space for the Poisson–Lie group $SL_n(\mathbb{C})$ and Yakimov has shown that $D_{2n} \ltimes T_n$ acts by Poisson automorphisms and anti-automorphisms on $\mathscr{O}(\mathrm{Gr}(k,n))$.

Our goal is to describe a quantum analogue of this action.

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Quantum matrices

We let $\mathbb{K}$ be a field and let $q \in \mathbb{K}^*$.

The quantum matrix algebra $\mathscr{O}_q(\mathrm{M}(k,n))$ is the $\mathbb{K}$-algebra generated by the set $\{X_{ij} \mid 1 \leq i \leq k,\ 1 \leq j \leq n\}$ subject to the quantum 2×2 matrix relations on each 2×2 submatrix of

$$\begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kn} \end{pmatrix}$$

where the quantum 2×2 matrix relations on $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are

$$\begin{array}{lll} ab = qba & ac = qca & bc = cb \\ bd = qdb & cd = qdc & ad - da = (q - q^{-1})bc. \end{array}$$

These relations arise naturally from considering coactions on quantum affine n-space, $\mathscr{O}_q(\mathbb{K}^n) \stackrel{\text{def}}{=} \mathbb{K}\langle\, x_1, \ldots, x_n \,\rangle / (x_i x_j = q x_j x_i \ \forall\ i < j)$.

Quantum Grassmannians

The quantum Grassmannian $\mathscr{O}_q(\mathrm{Gr}(k,n))$ is the subalgebra of $\mathscr{O}_q(\mathrm{M}(k,n))$ generated by the quantum Plücker coordinates

$$\mathscr{P}_q(k,n) = \{\Delta_q^I \mid I \subseteq \{1,\dots,n\}, |I| = k\}.$$

Here if $I = \{i_1 < i_2 < \cdots < i_k\}$ is a k-subset of $\{1,\dots,n\}$ then we define

$$\Delta_q^I \stackrel{\text{def}}{=} \sum_{\sigma \in S_k} (-q)^{l(\sigma)} X_{1i_{\sigma(1)}} \cdots X_{ki_{\sigma(k)}}$$

where S_k is the symmetric group of degree k and l is the usual length function on this, e.g. $\Delta_q^{ij} = X_{1i}X_{2j} - qX_{1j}X_{2i} \in \mathscr{O}_q(\mathrm{Gr}(2,n))$.

The quantum Grassmannian $\mathscr{O}_q(\mathrm{Gr}(k,n))$ is a finitely generated $\mathbb{Z}^n$-graded noncommutative $\mathbb{K}$-algebra of GK-dimension $k(n-k)+1$, is a noetherian domain and has semi-classical limit $\mathscr{O}(\mathrm{Gr}(k,n))$.

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The n-cycle and quantum Grassmannians

As we noted before, the Grassmannian admits an action of the symmetric group S_n and hence so does $\mathscr{O}(\mathrm{Gr}(k,n))$, with permutations acting on Plücker coordinates by permuting their indexing sets. However Launois and Lenagan have shown that the n-cycle $c = (1\,2 \cdots n)$ does not induce a corresponding automorphism of the quantum Grassmannian if $q^2 \neq 1$, in contrast to the classical situation.

To see this, one simply calculates in $\mathscr{O}_q(\mathrm{Gr}(2,4))$. We attempt to define an automorphism by

$$\theta(\Delta_q^{ij}) = \Delta_q^{\widetilde{(i+1)}\widetilde{(j+1)}}$$

on generating minors, extended linearly and multiplicatively. Here, $\widetilde{a}$ indicates that the index should be taken modulo $n = 4$ and from the set $\{1,2,3,4\}$. Applying this map to a short quantum Plücker relation and using the relations in $\mathscr{O}_q(\mathrm{Gr}(2,4))$ one finds a contradiction unless $q^2 = 1$.

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Twisting

Launois and Lenagan fixed this by twisting the quantum Grassmannian by a 2-cocycle. Their main theorem is that the twisted algebra $T(\mathscr{O}_q(\mathrm{Gr}(k,n)))$ is naturally isomorphic to the untwisted quantum Grassmannian $\mathscr{O}_q(\mathrm{Gr}(k,n))$ by an isomorphism

$$\Theta_0 \colon T(\mathscr{O}_q(\mathrm{Gr}(k,n))) \to \mathscr{O}_q(\mathrm{Gr}(k,n))$$

with

$$\Theta_0(T(\Delta_q^I)) = \lambda_I \Delta_q^{\widetilde{I+1}} = \lambda_I \Delta_q^{c(I)},$$

for $\lambda_I = 1$ if $n \notin I$, $\lambda_I = q^{-2}$ if $n \in I$. Here $\widetilde{I+1} = \{\widetilde{i+1} \mid i \in I\}$.

For $\mathscr{O}_q(\mathrm{Gr}(2,4))$, $\Theta_0(T(\Delta_q^{12})) = \Delta_q^{23}$ but $\Theta_0(T(\Delta_q^{14})) = q^{-2}\Delta_q^{12}$ for example.

That is, although c does not induce an automorphism of $\mathscr{O}_q(\mathrm{Gr}(k,n))$, there is an isomorphism of $\mathscr{O}_q(\mathrm{Gr}(k,n))$ with a twist of itself that behaves in the way we would like, up to some powers of q.

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A quantum dihedral action

Allman and myself have obtained a quantum analogue of the action of D_{2n}, by using powers of the Launois–Lenagan cocycle and another similar cocycle to cycle by powers of c.

Twisting by powers of these cocycles, for each positive integer l we obtain algebras $T^l(\mathscr{O}_q(\mathrm{Gr}(k,n)))$ and $\tau^l(\mathscr{O}_q(\mathrm{Gr}(k,n)))$ and isomorphisms

$$\Theta_{-l+1} \colon T^l(\mathscr{O}_q(\mathrm{Gr}(k,n))) \to T^{l-1}(\mathscr{O}_q(\mathrm{Gr}(k,n))), \quad T^l(\Delta_q^I) \mapsto \lambda_I T^{l-1}(\Delta_q^{c(I)})$$

$$\Theta_l \colon \tau^{l-1}(\mathscr{O}_q(\mathrm{Gr}(k,n))) \to \tau^l(\mathscr{O}_q(\mathrm{Gr}(k,n))), \quad \tau^{l-1}(\Delta_q^I) \mapsto \lambda_I \tau^l(\Delta_q^{c(I)})$$

This gives us our quantum rotations, by repeated composition.

A quantum dihedral action

For the analogue of the relation $c^n = 1$ we look at the composition of n consecutive isomorphisms: there is an isomorphism

$$\begin{aligned} \mathscr{O}_q(\mathrm{Gr}(k,n)) &\to \tau^n(\mathscr{O}_q(\mathrm{Gr}(k,n))) \\ \Delta_q^I &\mapsto \Lambda_I(n)\tau^n(\Delta_q^{c^n(I)}) = \Lambda_I(n)\tau^n(\Delta_q^I) \end{aligned}$$

given by the composition $\Theta_n\Theta_{n-1}\cdots\Theta_1$. Here $\Lambda_I(n) = \prod_{s=1}^{n} \lambda_{c^{s-1}(I)}$ is fact independent of I: explicitly, $\Lambda_I(n) = q^{-2k}$ for any I.

For generic q, this map is not the identity, nor even a scalar multiple of it, but its classical limit is exactly the identity map on $\mathscr{O}(\mathrm{Gr}(k,n))$. Indeed, it is easy to see that any composition of the form $\Theta_{n+r-1}\Theta_{n+r-2}\cdots\Theta_{r+1}\Theta_r$, that is n consecutive Θ_i's, also has this property.

A quantum dihedral action

The analogue of the action of w_0 arises from the observation that one may combine the maps

$$f\colon \mathscr{O}_q(\mathrm{Gr}(k,n)) \to \mathscr{O}_{q^{-1}}(\mathrm{Gr}(k,n)), \qquad f(\Delta_q^I) = \Delta_{q^{-1}}^I$$
$$g\colon \mathscr{O}_q(\mathrm{Gr}(k,n)) \to \mathscr{O}_{q^{-1}}(\mathrm{Gr}(k,n)), \qquad g(\Delta_q^I) = \Delta_{q^{-1}}^{w_0(I)}$$

to obtain

$$\Omega_0 \stackrel{\mathrm{def}}{=} g^{-1} \circ f\colon \mathscr{O}_q(\mathrm{Gr}(k,n)) \to \mathscr{O}_q(\mathrm{Gr}(k,n)), \qquad \Omega_0(\Delta_q^I) = \Delta_q^{w_0(I)},$$

an involutive anti-automorphism of $\mathscr{O}_q(\mathrm{Gr}(k,n))$. Setting

$$\Omega_l = (\Theta_l \Theta_{l-1} \cdots \Theta_1)\Omega_0(\Theta_0 \Theta_{-1} \cdots \Theta_{-l+1})$$

gives a family of anti-automorphisms, our quantum reflections.

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A quantum dihedral groupoid

We can interpret these results as saying that the dihedral subgroup $\langle\, c, w_0 \,\rangle \cong D_{2n}$ of the automorphism group S_n of $\mathrm{Gr}(k,n)$ has been quantized (and/or categorified) by an automorphism groupoid

$$
\begin{tikzcd}
\ddots \arrow[dr] & & & & & & \iddots \\
& T^2(\mathscr{O}_q(\mathrm{Gr}(k,n))) \arrow[rrrr, "\Omega_2"] \arrow[dr, "\Theta_{-1}"'] & & & & \tau^2(\mathscr{O}_q(\mathrm{Gr}(k,n))) \arrow[ur] & \\
& & T(\mathscr{O}_q(\mathrm{Gr}(k,n))) \arrow[rr, "\Omega_1"] \arrow[dr, "\Theta_0"'] & & \tau(\mathscr{O}_q(\mathrm{Gr}(k,n))) \arrow[ur, "\Theta_2"'] & & \\
& & & \mathscr{O}_q(\mathrm{Gr}(k,n)) \arrow[loop above, "\Omega_0"] \arrow[ur, "\Theta_1"'] & & &
\end{tikzcd}
$$

Orbits on the set of torus-invariant prime ideals

The quantum Grassmannian also admits an action of the torus $T_n \cong (\mathbb{K}^*)^n$. The above isomorphisms induce a dihedral group action on the set of torus-invariant prime ideals of $\mathcal{O}_q(\mathrm{Gr}(k,n))$.

For given a torus-invariant prime ideal P of $\mathcal{O}_q(\mathrm{Gr}(k,n))$, we can apply the vector space isomorphism T to obtain $T(P) \subseteq T(\mathcal{O}_q(\mathrm{Gr}(k,n)))$. It is easy to see that $T(P)$ is a torus-invariant prime ideal of $T(\mathcal{O}_q(\mathrm{Gr}(k,n)))$. Then $\Theta_0(T(P)) \subseteq \mathcal{O}_q(\mathrm{Gr}(k,n))$ is a torus-invariant prime ideal, since Θ_0 is an algebra isomorphism.

Similarly, since Ω_0 is an anti-automorphism of $\mathcal{O}_q(\mathrm{Gr}(k,n))$, it also sends torus-invariant prime ideals to torus-invariant prime ideals.

We illustrate this by showing the orbits for this action in the case of $\mathcal{O}_q(\mathrm{Gr}(2,4))$.

Orbits on torus-invariant prime ideals for $\mathscr{O}_q(\mathrm{Gr}(2,4))$

$12, 13, 14, 23, 24, 34$

$12, 13, 23, 24, 34$ | $13, 14, 23, 24, 34$ | $12, 13, 14, 24, 34$ | $12, 13, 14, 23, 24$ | $12, 13, 14, 23, 34$ | $12, 14, 23, 24, 34$

$12, 13, 23, 24$ | $13, 23, 24, 34$ | $12, 13, 23, 34$ | $14, 23, 24, 34$ | $12, 23, 24, 34$ | $13, 14, 23, 34$

$12, 13, 14, 24$ | $13, 14, 24, 34$ | $12, 14, 23, 24$ | $12, 13, 14, 34$ | $12, 13, 14, 23$ | $12, 14, 24, 34$

$12, 13, 23$ | $23, 24, 34$ | $13, 14, 34$ | $12, 14, 24$

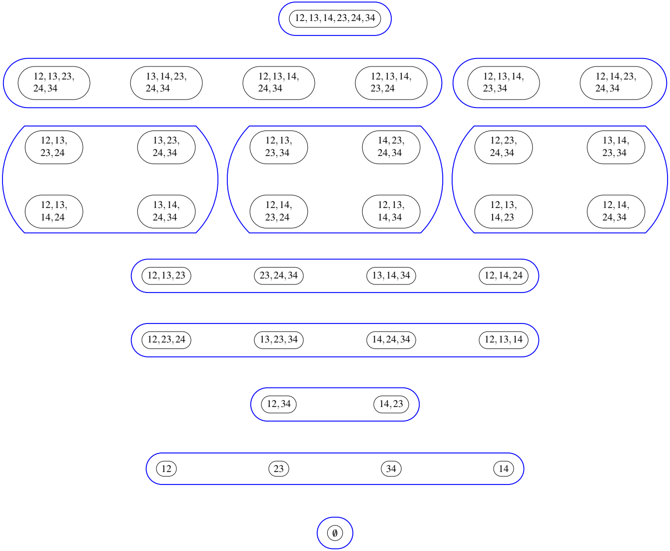
$12, 23, 24$ | $13, 23, 34$ | $14, 24, 34$ | $12, 13, 14$

$12, 34$ | $14, 23$

12 | 23 | 34 | 14

$\emptyset$

Orbits on torus-invariant prime ideals for $\mathcal{O}_q(\mathrm{Gr}(2,4))$



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