Effective construction of non finitely generated projective modules

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Traces of ideals

Proposition. (Whitehead 1980) *Let R be a ring, and let I be a two-sided ideal of R. Then the following statements are equivalent:*

(1) I is the trace ideal of a countably generated projective right R-module.

(2) There exists an ascending chain of finitely generated left ideals $\{I_i\}_{i\geq 1}$ such that $I_{i+1}I_i = I_i$ and $I = \bigcup_{i\geq 1} I_i R$.

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This result was *recovered* by Gena Puninski

Consequences

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2. If $R/J(R)$ is noetherian then I is the trace of a countably generated projective module if and only if $I = LR$ with L finitely generated left ideal such that $L^2 = L$.

Fair sized projective modules.

Let P_R be a projective module.

$$\mathcal{I}(P) = \{K \text{ ideal of } R \mid P/PK \text{ is finitely generated}\}$$

$\mathcal{I}(P)$ has a minimal element if and only if the descending sequence of right ideals $(K_n)_{n\geq 1}$ is stationary. Such a minimal element is of the form RK_n and it is an idempotent ideal.

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(Prihoda) If P and Q countably generated, fair sized projective modules over a noetherian ring with associated pair $(P/PK, K)$ and $(Q/QK', K')$ respectively, then $P \cong Q$ is and only if $P/PK \cong Q/QK'$ and $K = K'$.

FCR algebras

(Kraft-Small) An algebra R over a field k is said to be FCR if every finite dimensional representation of R is completely reducible and the intersection of kernels of finite dimensional representations is zero.

Any such kernel is an idempotent ideal so that, in the noetherian case, this gives plenty of ideals that are traces of countably generated projective modules.

Even the noetherian case can be wild

Theorem. *Assume that R is an FCR noetherian algebra with an infinite descending chain of ideals $I_0 \supsetneqq I_1 \supsetneqq \cdots$ such that*

- *For any n, R/I_n is semisimple artinian.*
- $\bigcap_{n\geq 0} I_n = \{0\}$.

Then there are, at least, uncountably many non-isomorphic countably generated, non finitely generated, projective R-modules, that are not fair-sized, such that its trace ideal is contained in I_0.

This applies to the universal enveloping algebra of a semisimple finite dimensional Lie algebra over a field of characteristic zero. In this case, non-zero, finitely generated projective modules have trace ideal equal to the ring.

Outside the noetherian case

Any countably generated projective module that is finitely generated modulo $J(R)$ is fair-sized.

Example. (Gerasimov, Sakhaev) *Let K be a field. Consider $S = K < x, y > /(yx)$ and $\varphi \colon S \to K \times K$ such that $\varphi(x) = (1, 0)$ and $\varphi(y) = (0, 1)$. Let $R = S_\Sigma$, where*

$$\Sigma = \{\text{matrices with entries in } S \text{ such that its image via } \varphi \text{ is invertible}\}.$$

So that $u = x + y$ is invertible in R. Set $a_i = u^{-(i+1)}xu^i$. Then $a_{i+1}a_i = a_i$. Then $R/J(R) \cong K \times K$ and $P = \sum a_i R$ is a ***non-finitely generated*** *projective module such that $P/PJ(R) \cong (1, 0)K$.*

This type of modules seems to be difficult to classify, even for semilocal rings, but they are very interesting. For example, using them one can produce a ring such that all its right projective modules are free but this is not true on the left.