A Bracket for Monoidal Categories

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- **Well-known fact:** $\mathrm{HH}^{\bullet}(A)$ is a Gerstenhaber k-algebra (G-algebra), i.e. it also has a graded Lie-bracket

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  satisfying the graded Poisson identity.
- If A is projective over k, then $\mathrm{HH}^\bullet(A) \cong \mathrm{Ext}^\bullet_{A^{\mathrm{ev}}}(A, A)$. $\rightsquigarrow$ Categorical interpretation of $\{-,-\}$ (Schwede, 1998).

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$$\mathrm{H}^\bullet(\mathcal{E}, \mathbb{1}) := \mathrm{Ext}^\bullet_{\mathcal{E}}(\mathbb{1}, \mathbb{1}),$$

where $(\mathcal{E}, \otimes, \mathbb{1})$ is a 'nice' exact monoidal category.

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Abbreviation: $\mathrm{s}(X, Y) \equiv 0 \to Y \to \mathbb{E} \to X \to 0.$

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- **Objects:** a.e.s. $\mathsf{s}(X, Y) \equiv 0 \to Y \to \mathbb{E} \to X \to 0$, where $\mathbb{E}$ is a complex in $\mathcal{E}$ concentrated in degrees 0 up to $n-1$

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- **Morphisms** $\mathrm{s}(X, Y) \to \mathrm{s}'(X, Y)$**:** morphisms $f : \mathbb{E} \to \mathbb{E}'$ of complexes s.t. the following commutes:

$$\begin{array}{ccccccccccc} \mathrm{s}(X,Y) & \equiv & 0 & \longrightarrow & Y & \longrightarrow & \mathbb{E} & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & & & \| & & \downarrow f & & \| & & \\ \mathrm{s}'(X,Y) & \equiv & 0 & \longrightarrow & Y & \longrightarrow & \mathbb{E}' & \longrightarrow & X & \longrightarrow & 0\,. \end{array}$$

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More concretely: $\mathcal{L}(\mathcal{C}, X) = \{\text{loops at } X\}$, i.e. the set of zig-zags

$$X \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_n \to X.$$

in $\mathcal{C}$.

Then: $\pi_1(\mathcal{C}, X) = \mathcal{L}(\mathcal{C}, X)/\text{homotopy relations}.$

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  $\rightsquigarrow$ $\pi_1(\mathcal{E}xt^{n+1}_{\mathcal{E}}(X, Y), \mathsf{s}(X, Y))$ is an abelian group and independent of the base point.

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- every object in $\mathcal{E}$ is *flat*, i.e. $- \otimes X$ is an exact functor for every $X \in \mathrm{Ob}\mathcal{E}$,
- $\mathcal{E}$ is closed under kernels of epimorphisms.

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1. $\mathcal{P} := {}_A\mathrm{Proj}_A \subseteq \mathrm{Mod}(A^{\mathrm{ev}})$ full subcategory of A^{ev}-modules which are projective on each side. $\mathcal{P}$ is extension closed and monoidal with exact tensor functor $\otimes_A$ and tensor unit A.

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1. $\mathrm{H}^\bullet(\mathcal{P}, A) = \mathrm{Ext}^\bullet_{\mathcal{P}}(A, A) \cong \mathrm{Ext}^\bullet_{A^{ev}}(A, A)$ (Hochschild cohomology)
2. $\mathrm{H}^\bullet(\mathcal{M}, k) = \mathrm{Ext}^\bullet_{\mathcal{M}}(k, k) \cong \mathrm{Ext}^\bullet_{H}(k, k)$ (cohomology of Hopf algebras)
3. $\mathrm{H}^\bullet(\mathcal{F}, \mathsf{Id}_{\mathcal{A}}) = \mathrm{Ext}^\bullet_{\mathcal{F}}(\mathsf{Id}_{\mathcal{A}}, \mathsf{Id}_{\mathcal{A}}) =: \mathrm{HH}^\bullet(\mathcal{A})$ (Hochschild cohomology of abelian categories)

The Bracket and its Properties

- **Eckmann-Hilton argument:** $\mathrm{H}^\bullet(\mathcal{E}, \mathbb{1}) := \mathrm{Ext}^\bullet_{\mathcal{E}}(\mathbb{1}, \mathbb{1})$ is a graded commutative *k*-algebra.

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The tensor functor on $\mathcal{E}$ yields

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$\rightsquigarrow$ Loop in $\pi_1(\mathcal{E}xt^{m+n}_{\mathcal{E}}(\mathbb{1}, \mathbb{1}), \mathsf{s} \boxtimes \mathsf{t}) \cong \mathrm{Ext}^{m+n-1}(\mathbb{1}, \mathbb{1})$ for all sequences $\mathsf{s} := \mathsf{s}(\mathbb{1}, \mathbb{1})$, $\mathsf{t} := \mathsf{t}(\mathbb{1}, \mathbb{1})$ with $|\mathsf{s}| = m$, $|\mathsf{t}| = n$.

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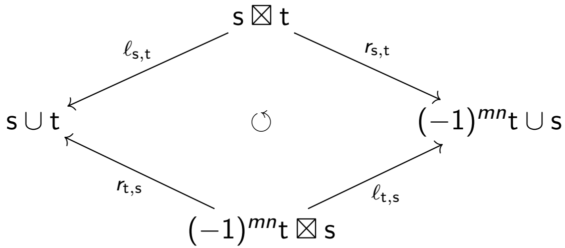

Upper hemisphere of the loop $\rightsquigarrow \mathrm{H}^\bullet(\mathcal{E},\mathbb{1})$ is graded commutative.

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Lemma

Let $\mathcal{E}'$ be another k-linear exact and monoidal category having the same properties as $\mathcal{E}$. Let $\mathcal{F} : \mathcal{E} \to \mathcal{E}'$ be an exact and monoidal functor. Then $\mathcal{F}[-,-] = [\mathcal{F}(-), \mathcal{F}(-)]'$.

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Proposition

Suppose that the monoidal structure on $\mathcal{E}$ admits a braiding, that is 'nice' natural morphisms $\gamma_{X,Y} : X \otimes Y \to Y \otimes X$, $X, Y \in Ob\mathcal{E}$. Then $[-,-]$ is the zero map.

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The functor $\mathcal{F}$ gives rise to an exact and monoidal functor

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Moreover, it induces a split monomorphism of graded k-algebras:

$$\mathrm{H}^\bullet(H, k) \cong \mathrm{Ext}^\bullet_{\mathcal{M}}(k, k) \to \mathrm{Ext}^\bullet_{\mathcal{P}}(H, H) \cong \mathrm{HH}^\bullet(H).$$

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Theorem

Let H be a cocommutative Hopf k-algebra being projective over k. Then the restriction of $\{-,-\}$ on $\mathrm{HH}^\bullet(H)$ to the subring $\mathrm{H}^\bullet(H,k) \subseteq \mathrm{HH}^\bullet(H)$ is zero:

$$\{\mathrm{H}^\bullet(H,k), \mathrm{H}^\bullet(H,k)\} = 0.$$

Perspectives

- Schwede's Theorem $\rightsquigarrow \mathrm{H}^\bullet(\mathcal{E}, \mathbb{1})$ is a Gerstenhaber k-algebra for $\mathcal{E} \in \{{}_A\mathsf{Proj}_A,\ \mathrm{End}_k(\mathsf{Mod}(A))\}$, if A is k-projective.

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- **Ultimative dream:** Show that $\mathrm{H}^\bullet(\mathcal{E}, \mathbb{1})$ is a Gerstenhaber k-algebra for general $\mathcal{E}$!

The End

Thank you for your attention!