On a definition of multi-Koszul algebras

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Estanislao Herscovich (joint with Andrea Rey)
Université Grenoble, Grenoble, France

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Plan of the talk

1. History
2. Definition of multi-Koszul algebras
3. Several properties

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Short summary on (part of) the history of the subject

(i) Inspired on work by J.-L. Koszul in the 50's, S. Priddy introduced in 1970 the **Koszul** property for (graded) algebras, which has been intensively studied later on. They are necessarily **quadratic** (*i.e.* $TV/\langle R\rangle$ for $R \subseteq V^{\otimes 2}$).

(ii) In 2001, R. Berger defined the **generalized Koszul** property for **homogeneous algebras** (*i.e.* $TV/\langle R\rangle$ for $R \subseteq V^{\otimes N}$ and $N \in \mathbb{N}_{\geq 2}$). They were independently (defined and) studied by E. Green, E. Marcos, R. Martínez-Villa and P. Zhang (2004).

(iii) There were other *Koszul-like* definitions, *e.g.* **almost Koszul algebras** introduced by S. Brenner, M. Butler and A. King (2002), δ-**Koszul algebras** defined by E. Green and E. Marcos (2005), **piecewise-Koszul algebras** by J.-F. Lü, J.-W. He and D.-M. Lu (2007), $\mathcal{K}_2$ **algebras** by B. Cassidy and T. Shelton (2008), 2-p-**Koszul algebras** defined by E. Green and E. Marcos (2011), etc.

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Setting

Let k be a field, and A be a finitely generated nonnegatively graded connected algebra which is generated in degree 1, *i.e.* $A \simeq TV/I$, where V is concentrated in degree 1. We assume that $V \simeq A_{>0}/(A_{>0}.A_{>0})$ and $I \subseteq TV_{\geq 2}$ to avoid redundancy.

Let $R = \oplus_{s \in S} R_s \subseteq I$ be a **space of relations** of A, *i.e.* $R \simeq I/(TV_{>0}I + ITV_{>0})$ and suppose $\dim(R) < +\infty$.

For $s \in \mathbb{N}_{\geq 2}$, consider $n_s : \mathbb{N}_0 \to \mathbb{N}_0$ given by $n_s(2m) = sm$ and $n_s(2m+1) = sm + 1$, for $m \in \mathbb{N}_0$.

If $s \in S$, we will denote

$$J_i^s = \bigcap_{j=0}^{n_s(i)-s} V^{\otimes j} \otimes R_s \otimes V^{\otimes (n_s(i)-s-j)},$$

for $i \geq 2$,

$$J_i = \bigoplus_{s \in S} J_i^s,$$

if $i \geq 2$, and $J_i = V^{\otimes i}$, if $i = 0, 1$.

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Definition

The **(left) multi-Koszul complex** $(K(A)_\bullet, \delta_\bullet)$ of A is defined by $K(A)_0 = A$, $K(A)_1 = A \otimes V$ and $K(A)_i = A \otimes J_i$ for $i \geq 2$, with differential $\delta_\bullet$, where δ_1 is induced by the multiplication on A, and, for $i \geq 2$,

$$\delta_i : A \otimes J_i \to A \otimes J_{i-1}$$

is given by the restriction of the map $\hat{\delta}_i : A \otimes (\oplus_{s \in S} V^{\otimes n_s(i)}) \to A \otimes (\oplus_{s \in S} V^{\otimes n_s(i-1)})$, where

$$\hat{\delta}_i(\alpha \otimes v_{j_1} \cdots v_{j_{n_s(i)}}) = \begin{cases} \alpha v_{j_1} \cdots v_{j_{s-1}} \otimes v_{j_s} \cdots v_{j_{n_s(i)}}, & \text{if } i \text{ is even,} \\ \alpha v_{j_1} \otimes v_{j_2} \cdots v_{j_{n_s(i)}}, & \text{if } i \text{ is odd,} \end{cases}$$

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Examples and comments

Proposition

Let $\{B^s : s \in S\}$, where $S \subseteq \mathbb{N}_{\geq 2}$, be a finite collection of homogeneous algebras such that B^s is s-Koszul, for each $s \in S$. Then, the free product $A = \coprod_{s \in S} B^s$ of the collection $\{B^s : s \in S\}$ is a multi-Koszul algebra.

Remark

The (left) multi-Koszul property is not equivalent to the trivial A-module k having a minimal projective resolution $(P_\bullet, d_\bullet)$ whose i-th projective P_i is generated in degrees $\{n_s(i) : s \in S\}$, for all $i \in \mathbb{N}_0$. For instance, for $A = k\langle x, y, z\rangle/\langle x^2y, z^2x\rangle *_k k\langle u\rangle/\langle u^4\rangle$, the i-th projective module of the minimal projective resolution of k is pure in degrees $n_3(i)$ and $n_4(i)$, for all $i \in \mathbb{N}_0$, but it is not multi-Koszul.

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An equivalent description

Proposition

The following are equivalent

(i) *A is left (resp., right) multi-Koszul algebra*

(ii) *There is an isomorphism of graded vector spaces* $\mathrm{Tor}_i^A(k,k) \simeq J_i = \bigoplus_{s \in S} J_i^s$, *for all* $i \in \mathbb{N}_0$.

(iii) *There is an isomorphism of graded vector spaces* $\mathcal{E}xt_A^i(k,k) \simeq J_i^{\#}$, *for all* $i \in \mathbb{N}_0$.

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Relation to (generalized) Koszul components

Given $A = TV/\langle R \rangle$, where $R = \oplus_{s \in S} R_s$, we define the **associated** s**-homogeneous component** A^s of A given by $TV/\langle R_s \rangle$.

Theorem

The following conditions are equivalent:

(i) *A is (left) multi-Koszul.*

(ii) *For each $s \in S$, A^s is s-Koszul, $r\text{-pdim}_{A^s}(A) \leq 1$, and $\mathrm{Ker}(\delta_2) = \oplus_{s \in S}(\mathrm{Ker}(\delta_2) \cap (A \otimes R_s))$, where $\delta_2 : A \otimes R \to A \otimes V$ is the second differential of the left multi-Koszul complex of A.*

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The Yoneda algebra

By the functoriality of the Yoneda algebra construction, the morphisms of algebras $TV \to A^s$ and $A^s \to A$, for $s \in S$, induce morphisms $E(A) \to E(A^s)$ and $E(A^s) \to E(TV)$, for $s \in S$.

Proposition

The algebra $E(A)$ of a multi-Koszul algebra A is the inverse limit in the category of graded algebras of the system given by $\{E(A^s) \to E(TV),\ for\ s \in S\}$.

Corollary

The graded algebra $E(A)$ of a multi-Koszul algebra A is generated by $E^1(A) = \mathrm{Ext}^1_A(k,k)$ and $E^2(A) = \mathrm{Ext}^2_A(k,k)$, i.e. it is $\mathcal{K}_2$ (in the sense of T. Cassidy and B. Shelton). Moreover, the A_∞-algebra structure of $E(A)$ can be easily computed computed in an explicit manner.

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The Yoneda algebra (cont.)

(i) There is a quasi-isomorphism of A_∞-algebras from $E(A)$ to the A_∞-algebra given by the cohomology of the inverse limit in the category of differential graded algebras of the system $\{F_s : B(A^s)^* \to B(TV)^*, \text{ for } s \in S\}$.

(ii) On the other hand, using the Merkulov's procedure to obtain a canonical A_∞-algebra structure on the homology of a differential graded algebra, it is trivial to see that we may choose the higher multiplications of the inverse limit such that $m_n(a_1 \otimes \cdots \otimes a_n) = 0$ if there are indices $1 \leq i \neq j \leq n$ satisfying that $a_i \in E^{d_i}(A^s)$ and $a_j \in E^{d_j}(A^{s'})$ for $s \neq s'$, and $d_i, d_j > 1$.

(iii) Finally, the restriction of the higher multiplication m_n of $E(A)$ to each $E(A^s)^{\otimes n}$ is given by the corresponding Merkulov's construction of the higher multiplication of $E(A^s)$, for each $s \in S$, which were completely described by J.-W. He and D.-M. Lu.

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