# Spherelike Twist Functors

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This talk is about a joint work with Martin Kalck and David Ploog.
A preprint is available on-line: arXiv:1208.4046 .

# Protagonists

## Notation

Let $\mathcal{D}$ be a $\mathbf{k}$-linear algebraic triangulated category with Serre functor. All triangles are meant to be distinguished, and all functors exact.

## Definition

An object $F$ in $\mathcal{D}$ is called

| | | | |
|---|---|---|---|
| $d$-spherelike | $\Leftrightarrow$ | $\mathrm{Hom}^\bullet(F,F) = \mathbf{k} \oplus \mathbf{k}[-d]$ | $d$-Sphere |
| $d$-spherical | $\Leftrightarrow$ | $F$ is $d$-spherelike and | |
| | | $\mathrm{Hom}^\bullet(F, \cdot\,) = \mathrm{Hom}^\bullet(\,\cdot\,, F[d])^*$ | $d$-Calabi-Yau |

### Definition

For any object $F$ in $\mathcal{D}$ there is the *evaluation map*

$$\mathrm{Hom}^\bullet(F, \,\cdot\,) \otimes F \to \mathrm{id}\,.$$

We define the twist functor $\mathsf{T}_F$ as its cone. So the functor fits into the triangle

$$\mathrm{Hom}^\bullet(F, \,\cdot\,) \otimes F \to \mathrm{id} \to \mathsf{T}_F.$$

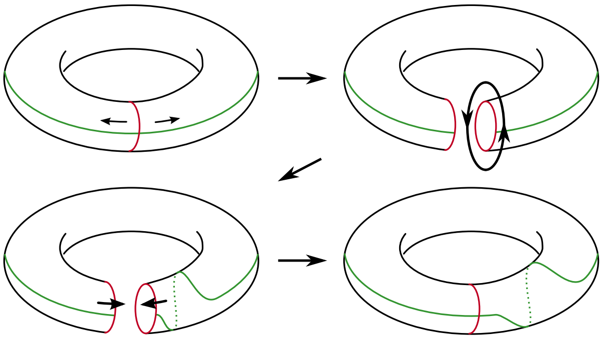
If $F$ spherelike, we call $\mathsf{T}_F$ a spherelike twist functor.
Analogously, $\mathsf{T}_F$ is called spherical twist functor, if $F$ is spherical.

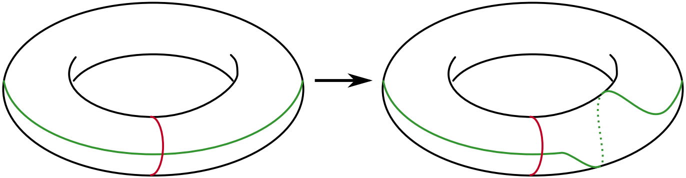
### Theorem (Paul Seidel and Richard Thomas)

Let $F$ be non-zero.
$F$ is spherical $\Leftrightarrow$ $\mathsf{T}_F$ is an auto-equivalence.

Paul Seidel and Richard Thomas were motivated by mirror symmetry. In algebraic geometry, a typical example is a $(-2)$-curve $C$ on a smooth projective surface $X$. Then $F = \mathcal{O}_C$ is 2-spherical and $\mathsf{T}_F$ is an auto-equivalence of $\mathcal{D}^b(X)$.

On the symplectic side, this twist corresponds to a Dehn twist.



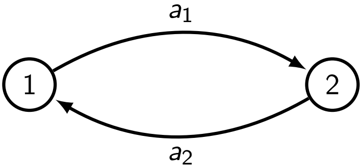


There are some immediate similarities:

| | | |
|---|---|---|
| $\mathsf{T}_F(F) = F[1-d]$ | $\leftrightarrow$ | rotating red circle |
| $\mathsf{T}_F = \text{id}$ when restricted to $F^{\perp}$ | $\leftrightarrow$ | outside red cirle<br>essentially nothing happens |

# Spherical Example: Quiver Algebra $A = \mathbf{k}Q/I$

$Q$:




$I = \langle a_2 a_1 \rangle$

The simple $A$-module $S(1)$ has the projective resolution

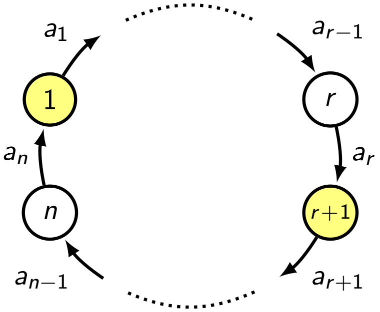
$$0 \to P(1) \to P(2) \to P(1) \to S(1) \to 0$$

Easy calculation:

$$\left.\begin{array}{l} \mathrm{Hom}^\bullet(S(1), S(1)) = \mathbf{k} \oplus \mathbf{k}[-2] \\ S(1) \text{ is 2-Calabi-Yau} \end{array}\right\} \Rightarrow S(1) \text{ is 2-spherical}$$

# Spherelike Example: Quiver Algebra $A' = \mathbf{k}Q'/I'$

$Q'$:




$I' = \langle a_n \cdots a_r \rangle$

Choosing idempotent $e = e_1 + e_{r+1}$:

$$\mathcal{D}^b(A) \cong \mathcal{D}^b(eA'e)$$

Fully faithful embedding:

$$j\colon \mathcal{D}^b(A) \hookrightarrow \mathcal{D}^b(A')$$

(induced by functor $A'e \otimes_{eA'e} \;\cdot\;$)

**Result**

$\Rightarrow$ $j(S(1))$ is still 2-spherelike,
but not 2-spherical.

$\Rightarrow$ $\mathsf{T}_{j(S(1))}$ is not an equivalence.

## The Spherical Subcategory

Let $F$ be $d$-spherelike but not $d$-spherical.
Denote the Serre functor of $\mathcal{D}$ by $\mathsf{S}$.
Then

$$\mathrm{Hom}^\bullet(F,F) = \mathrm{Hom}^\bullet(F,\mathsf{S}(F))^* \ncong \mathrm{Hom}^\bullet(F,F[d])^*$$

We can compare $\omega(F) := \mathsf{S}(F)[-d]$ and $F$. By Serre duality

$$\mathrm{Hom}^\bullet(F,\omega(F)) = \mathrm{Hom}^\bullet(F,F)^*[-d] = \mathbf{k} \oplus \mathbf{k}[-d]$$

$\Rightarrow$ canonical map

$$w\colon F \to \omega(F)$$

(even for the case $d = 0$, which needs more care)

Using the canonical map, we define the

$$F \xrightarrow{w} \omega(F) \to Q_F$$ aspherical triangle

### Properties of $Q_F$

- $F$ spherical $\Leftrightarrow$ $Q_F$ is zero
- $\mathrm{Hom}^\bullet(F, Q_F)$ vanishes

### Main Definition

Let $F$ be a spherelike object in $\mathcal{D}$.

- $\mathcal{D}_F := {}^\perp Q_F$ spherical subcategory
- $\mathcal{Q}_F := \mathcal{D}_F^\perp$ aspherical subcategory

# Main Theorem

### Theorem

Let $F$ be a $d$-spherelike object in $\mathcal{D}$.
Then $F$ is $d$-spherical in $\mathcal{D}_F$.
Moreover, $\mathsf{T}_F$ induces auto-equivalences of $\mathcal{D}_F$ and $\mathcal{Q}_F$.

### Sketch of the Proof

Easy: $\mathsf{T}_F|_{\mathcal{Q}_F} = \mathrm{id}_{\mathcal{Q}_F}$ by $F \in \mathcal{D}_F = {}^{\perp}\mathcal{Q}_F$.
$\mathsf{T}_F|_{\mathcal{D}_F}$: Apply $\mathrm{Hom}^\bullet(A, \cdot\,)$ to the aspherical triangle

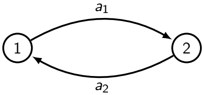
$$\begin{array}{ccccc} \mathrm{Hom}^\bullet(A, F) & \longrightarrow & \mathrm{Hom}^\bullet(A, \omega(F)) & \longrightarrow & \mathrm{Hom}^\bullet(A, Q_F) \\ & & = \mathrm{Hom}^\bullet(F, A[d])^* & & = 0 \end{array}$$

so $F$ is $d$-spherical in $\mathcal{D}_F$.
Warning: $\mathcal{D}_F$ has no Serre functor in general. Theorem of Seidel and Thomas does not apply here.

# Spherelike Example, Revisited

$$A = \mathbf{k}Q/I$$

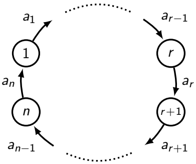
$Q$:




$I = \langle a_2 a_1 \rangle$

$$A' = \mathbf{k}Q'/I'$$

$Q'$:




$I' = \langle a_n \cdots a_r \rangle$

Special situation: $j \colon \mathcal{D}^b(A) \hookrightarrow \mathcal{D}^b(A')$ has right adjoint $i$ such that $i \circ j = \mathrm{id}$.

**Proposition**

For $F = j(S(1))$ holds

$$\mathcal{D}^b(A')_F = \langle \mathcal{D}^b(A)^{\perp} \cap {}^{\perp}F, \mathcal{D}^b(A) \rangle$$

Using this proposition, we calculate

$$\begin{aligned} \mathcal{D}^b(A')_F &\cong \langle S(k), k = 2, \ldots, r-1 \rangle \times \\ &\quad \times \langle S(k), k = r+2, \ldots, n-1 \rangle \times \mathcal{D}^b(A) \\ &\cong \mathcal{D}^b(\vec{A}_{r-2}) \times \mathcal{D}^b(\vec{A}_{n-r-2}) \times \mathcal{D}^b(A) \end{aligned}$$

with $\vec{A}_m$ the path algebra of $1 \to \cdots \to m$.