CATEGORIES OF REPRESENTATIONS OF CYCLIC POSETS

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1.1. **Introduction.** K is any field. $R = K[[t]]$.

Given a "recurrent" cyclic poset X and "admissible automorphism" ϕ, we construct a Frobenius category $\mathcal{F}_\phi(X)$ which is R-linear. By [Happel] the stable category $\mathcal{C}_\phi(X)$ will be a triangulated category over K. In each of our examples, $\mathcal{C}_\phi(X)$ will be a cluster category:

cyclic poset X	automorphism ϕ	cluster category $\mathcal{C}_\phi(X)$	comments
Z_n $1 < 2 < \cdots < n < \sigma 1$	$\phi(i) = i + 1$	$\mathcal{C}(A_{n-3})$	2-CY
$\mathbb{Z}$ (with cyclic order)	$\phi(i) = i + 1$	$\mathcal{C}(A_\infty)$ infinity-gon	2-CY
S^1	id	$\mathcal{C}$ continuous cluster category	not 2-CY but has clusters $Y[1] \cong Y$
$S^1 * \mathbb{Z}$	id $\phi(x, i) = (x, i + 1)$	$\tilde{\mathcal{C}}$ $\tilde{\mathcal{C}}'$	not 2-CY ($Y[1] \cong Y$) 2-CY
$Z_m * \mathbb{Z}$	$\phi(i, j) = (i + 1, j)$ $\phi(m, j) = (1, j + 1)$	contains m-cluster category of type A_∞	$(m + 1)$-CY
$\mathcal{P}(1)/3\mathbb{Z} * \mathbb{Z}$	$\phi^3(x, i) = (x, i + 1)$	$\begin{pmatrix}\text{3-cluster category} \\ \text{of type } A_\infty\end{pmatrix}^3$	4-CY

1.2. **Cyclic poset.** is same as periodic poset $\tilde{X}$. i.e. $\exists$ poset automorphism $\sigma : \tilde{X} \to \tilde{X}$ so that $x < \sigma x$ for all x. Also:

- $(\forall x, y \in \tilde{X})\ x \leq \sigma^j y$ for some $j \in \mathbb{Z}$.

(1) Z_n: $\tilde{X} = \mathbb{Z}$, $\sigma(x) = x + n$ (n fixed).
(2) $\tilde{X} = \mathcal{P}(1)$ (from Schmidmeier's lecture), σ : go up three steps.
(3) $\tilde{X} * \mathbb{Z}$ means $\tilde{X} \times \mathbb{Z}$ with lexicographic order (from van Roosmalen).

Let X = set of σ orbits. How to describe cyclic poset structure just in terms of X?

(1) Choose representative $\tilde{x} \in \tilde{X}$ for each orbit $x \in X$.
(2) $(\forall x, y \in X)$ let $b = b(x, y)$ be minimal so that $\tilde{x} \leq \sigma^b \tilde{y}$.
(3) Let $c = \delta b$:

$$c(xyz) := b(xy) + b(yz) - b(xz)$$

Then $c : X^3 \to \mathbb{N}$ is independent of the choice of representatives $\tilde{x}$.

- X is in *cyclic order* iff $c(xyz) \leq 1$. In that case:

$$c(xyz) = \begin{cases} 0 & \text{if xyz in cyclic order} \\ 1 & \text{otherwise} \end{cases}$$

Proposition 1.2.1. *The cyclic poset structure on a set X is uniquely determined by the function $c : X^3 \to \mathbb{N}$ which is an arbitrary reduced cocycle (*reduced *means $c(xxy) = 0 = c(xyy)$.* cocycle *means $\delta c = 0$.)*

1.3. **Representations.**

Definition 1.3.1. A *representation* M of (X, c) over R is

(1) An R-module M_x for each $x \in \tilde{X}$ so that $M_x = M_{\sigma x}$.
(2) An R-linear map $M_y \to M_x$ for $x < y$ so that all diagrams commute and
(3) $M_{\sigma x} \to M_x$ is $\cdot t$ (multiplication by t).

Definition 1.3.2. Let $\mathcal{P}(X)$ be the category of f.g. projective representations of X over $R = K[[t]]$.

Let P_x = indec. projective rep. generated at $x \in X$.

1.4. **Frobenius category.**

Definition 1.4.1. Let $\mathcal{F}(X)$ denote the category of all pairs (P, d) where $P \in \mathcal{P}(X)$ and $d : P \to P$ so that $d^2 = \cdot t$ (mult by t). Morphism $f : (P, d) \to (Q, d)$ are maps $f : P \to Q$ so that $df = fd$.

Theorem 1.4.2. *In all examples on page 1, $\mathcal{F}(X)$ is Krull-Schmidt with indecomposable objects:*

$$M(x, y) := \left(P_x \oplus P_y, \begin{bmatrix} 0 & \beta \\ \alpha & 0 \end{bmatrix} \right) : \qquad P_x \underset{\alpha}{\overset{\beta}{\leftrightarrows}} P_y$$

with $\alpha\beta = \cdot t$, $\beta\alpha = \cdot t$.

Lemma 1.4.3. *The functor $G : \mathcal{P}(X) \to \mathcal{F}(X)$ given by*

$$GP := \left(P \oplus P, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right) : \qquad P \underset{id}{\overset{\cdot t}{\leftrightarrows}} P$$

is both left and right adjoint to the forgetful functor $F : \mathcal{F}(X) \to \mathcal{P}(X)$.

Theorem 1.4.4. *For any cyclic poset X, $\mathcal{F}(X)$ is a Frobenius category where a sequence*

$$(A, d) \to (B, d) \to (C, d)$$

is defined to be exact in $\mathcal{F}(X)$ if $A \to B \to C$ is (split) exact in $\mathcal{P}(X)$. GP are the projective injective objects. $\underline{f} = \underline{g}$ iff $f - g = ds + sd$ for some $s : P \to Q$.

1.5. **Twisted version.** An automorphism ϕ of X is *admissible* if:

$$x \le \phi(x) \le \phi^2(x) \le \sigma x$$

for all $x \in \tilde{X}$. Then we get

$$P_x \xrightarrow{\eta_x} \phi P_x = P_{\phi(x)} \xrightarrow{\xi_x} P_x$$

giving natural transformations

$$P \xrightarrow{\eta_P} \phi P \xrightarrow{\xi_P} P$$

Definition 1.5.1. Let $\mathcal{F}_\phi(X)$ be the full subcategory of $\mathcal{F}(X)$ of all (P, d) where d factors through $\eta_P : P \to \phi P$.

Theorem 1.5.2. *$\mathcal{F}_\phi(X)$ is a Frobenius category with projective-injective objects*

$$G_\phi P := \left(P \oplus \phi P, \begin{bmatrix} 0 & \xi_P \\ \eta_P & 0 \end{bmatrix} \right) : \qquad P \underset{\eta_P}{\overset{\xi_P}{\rightleftarrows}} \phi P$$

1.6. m**-cluster categories.**

Theorem 1.6.1. *Let $m \geq 3$, let X be a cyclically ordered set (equivalently, $c(xyz) \leq 1$), and ϕ is an admissible automorphism of $X * \mathbb{Z}$ so that $\phi^m(x, i) = (x, i + 1)$ then $\mathcal{F}_\phi(X * \mathbb{Z})$ is $(m + 1)$-Calabi-Yau.*

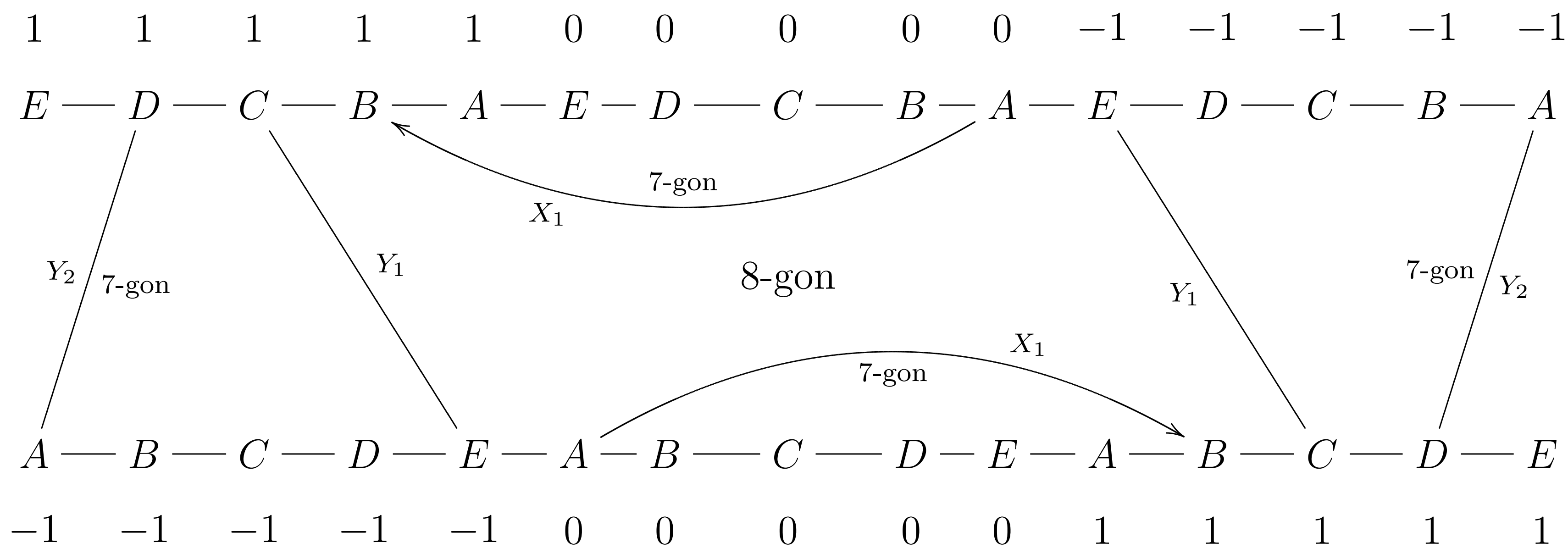
- On $Z_m * \mathbb{Z}$ let $\phi(i, j) = (i + 1, j)$ for $i < m$ and $\phi(m, j) = (1, j + 1)$.
- All objects of $\mathcal{C}_\phi(Z_m * \mathbb{Z})$ are "standard" if $m = 3$.
- All objects are $m + 1$ rigid iff $m \leq 4$.

Theorem 1.6.2. *The "standard objects" form a thick subcategory $\mathcal{C}^m_\infty$ of $\mathcal{C}_\phi(Z_m * \mathbb{Z})$. This subcategory is a true m-cluster category in the following sense.*

- *All standard objects X are $m + 1$ rigid in the sense that* $\mathrm{Hom}(X, X[i]) = 0$ *for $1 \leq i \leq m$.*
- *Maximal compatible sets of standard objects form m-clusters (usual sense).*
- *Isomorphism classes of standard m-clusters are in 1-1 correspondence with the partitions of the ∞-gon into $m + 2$-gons.*

Theorem 1.6.3. *Maximal compatible sets of $m+1$ rigid objects (including nonstandard objects) correspond to 2-periodic partitions of the doubled ∞-gon into $m+2$-gons (except for the one in the middle).*

Example 1.6.4. ($m = 5$). Example of a maximal compatible set of 6-rigid objects in $\mathcal{C}_\phi(Z_5 * \mathbb{Z})$. $M(x, y)$ is arc from x to y (horizontal if standard, vertical if nonstandard). Compatible arcs do not cross. There is 8-gon in center. Other regions have 7 sides.



Standard: $X_1 = M(A_0, B_1)$ (horizontal).
$Y_1 = M(C_1, E_{-1})$, $Y_2 = M(A_{-1}, D_1)$ are nonstandard but $(m + 1)$-rigid (vertical).
Notation: $(1, j) = A_j$, $(2, j) = B_j$, etc.

Thank you for you attention!