Tilted algebras and short chains of modules

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R a fixed commutative artin ring
algebra = an artin algebra over R
A an algebra
$\operatorname{mod} A$ = the category of all finitely generated right A-modules
$\operatorname{ind} A$ = the full subcategory of $\operatorname{mod} A$ formed by all indecomposable modules

$K_0(A)$ the Grothendieck group of A
$[M]$ the image of a A-module M in $K_0(A)$

$$[M] = [N] \text{ for two modules } M, N \in \operatorname{mod} A$$
$$\Leftrightarrow$$
$$M, N \text{ have the same composition factors including the multiplicities}$$

Γ_A the **Auslander-Reiten** quiver of A
$\tau_A = D\mathrm{Tr}$ the Auslander-Reiten translations in mod$A$
component of Γ_A = connected component of Γ_A

$\mathcal{C}$ a component of Γ_A
section Σ of $\mathcal{C}$= a full connected valued subquiver Σ which:

- has no oriented cycles,
- is convex in $\mathcal{C}$,
- intersects each τ_A-orbit of $\mathcal{C}$ exactly once,

Σ is **faithful** provided the direct sum of all modules lying on Σ is a faithful A-module

H a hereditary algebra
$T \in \mathrm{mod}H$
T is a **tilting** H-module:

- $\mathrm{Ext}^1_H(T,T) = 0$
- the number of pairwise nonisomorphic indecomposable direct summands = rank of $K_0(H)$

A is a **tilted algebra** if and only if $A = \mathrm{End}_H(T)$, where H is a hereditary algebra and T is a tilting module in $\mathrm{mod}H$

$(\mathcal{F}(T), \mathcal{T}(T))$ = torsion pair in mod$H$:

- torsion-free part $\mathcal{F}(T) = \{X \in \text{mod}H | \text{Hom}_H(T, X) = 0\}$
- torsion part $\mathcal{T}(T) = \{X \in \text{mod}H | \text{Ext}^1_H(T, X) = 0\}$

$(\mathcal{Y}(T), \mathcal{X}(T))$ = torsion pair in mod$B$:

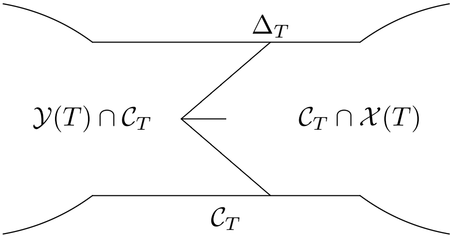
- torsion-free part $\mathcal{Y}(T) = \{Y \in \text{mod}B | \text{Tor}_1^B(Y, T) = 0\}$
- torsion part $\mathcal{X}(T) = \{Y \in \text{mod}B | Y \otimes_B T = 0\}$

Brenner-Butler Theorem.
$\text{Hom}_H(T, -) : \text{mod}H \to \text{mod}B$ induces $\mathcal{T}(T) \simeq \mathcal{Y}(T)$
$\text{Ext}^1_H(T, -) : \text{mod}H \to \text{mod}B$ induces $\mathcal{F}(T) \simeq \mathcal{X}(T)$

I indecomposable injective H-module
$M_I = \mathrm{Hom}_H(T, I) \in \mathcal{C}$ an indecomposable B-module

The modules $M_I = \mathrm{Hom}_H(T, I) \in \mathcal{C}$, with I indecomposable injective H-modules, form a faithful section Δ_T of the **connecting component** $\mathcal{C} = \mathcal{C}_T$ of Γ_B determined by T.

Δ_T connects the torsion-free part $\mathcal{Y}(T)$ with the torsion part $\mathcal{X}(T)$:
- every predecessor in ind$B$ of a module M_I from Δ_T lies in $\mathcal{Y}(T)$,
- every successor of $\tau_B^- M_I$ in ind$B$ lies in $\mathcal{X}(T)$.



$\mathrm{pd}_B Y \leqslant 1$ for any indecomposable module Y in $\mathcal{Y}(T)$

$\mathrm{id}_B X \leqslant 1$ for any indecomposable module X in $\mathcal{X}(T)$
$\mathrm{gldim} B \leqslant 2$

The Criterion of Liu and Skowroński.
Let B be an indecomposable algebra. Then B is a tilted algebra

if and only if

Γ_B admits a component $\mathcal{C}$ with a faithful section Δ such that $\mathrm{Hom}_B(X, \tau_B Y) = 0$ for all modules X and Y in Δ.

Moreover, if this is the case and T^*_Δ is the direct sum of all indecomposable modules lying on Δ, then:

- $H_\Delta = \mathrm{End}_B(T^*_\Delta)$ is an indecomposable hereditary algebra,
- $T_\Delta = D(T^*_\Delta)$ is a tilting module in $\mathrm{mod} H_\Delta$,
- the tilted algebra $B_\Delta = \mathrm{End}_{H_\Delta}(T_\Delta)$ is the basic algebra of B.

cycle in mod$A$ = a sequence $M_0 \to M_1 \to \ldots \to M_t = M_0$ of nonzero nonisomorphisms, where M_i are indecomposable, $t \geqslant 1$
short cycle in mod$A$ = a cycle with $t \leqslant 2$

Theorem (Reiten, Skowroński, Smalø).
Let M, N be indecomposable modules over artin algebra A such that $[M] = [N]$. If M does not lie on a short cycle, then $M \cong N$.

short chain of modules = a chain of nonzero homomorphisms

$$X \to M \to \tau_A X,$$

with $X \in$ ind$A$
M = the **middle** of this short chain

Proposition.
$M \in \mathrm{ind}A$:
M lies on a short cycle $\Leftrightarrow$ M is the middle of a short chain.

$\Leftarrow$ Reiten, Skowroński, Smalø
$\Rightarrow$ Happel, Liu

Theorem (Reiten, Skowroński, Smalø).
Let A be an artin algebra and assume that there is a sincere module M which is not the middle of a short chain. Then $\mathrm{gldim}A \leqslant 2$ and for any module X in $\mathrm{ind}A$ we have $\mathrm{pd}_A X \leqslant 1$ or $\mathrm{id}_A X \leqslant 1$.

Motivation:

H a hereditary algebra
T a tilting module in mod$H$
$B = \mathrm{End}_H(T)$ the associated tilted algebra

$M_T = \oplus M_I$ the direct sum of all indecomposable modules M_I forming the canonical section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_B determined by T

Then M_T is a sincere B-module which is not the middle of a short chain in mod$B$.

Reiten, Skowroński, Smalø '93

It would be interesting to know whether the existence of a sincere A-module M that is not the middle of a short chain implies that A is a tilted algebra.

Proved in the case: A is of finite representation type and M is indecomposable.

Theorem (Happel, Reiten, Smalø).
Let A be an artin algebra and X a sincere module in ind$A$ which does not lie on a short cycle. Then X does not lie on a cycle in mod$A$.

In particular, we have:
X a sincere indecomposable module in mod$A$ which is not the middle of a short chain $\Rightarrow$ X is a sincere directing module in ind$A$

$\overset{\text{Ringel}}{\Rightarrow}$ A is a tilted algebra and X lies on a section of a connecting component of Γ_A

THEOREM 1.
An artin algebra A is a tilted algebra if and only if mod$A$ admits a sincere module M which is not the middle of a short chain.

Idea of the proof:
A an artin algebra
M a sincere module in mod$A$ which is not the middle of a short chain

Then A is a quasitilted algebra (Reiten, Skowroński, Smalø).
Hence A is a tilted algebra or a quasitilted algebra of canonical type (by the theorem of Happel and Reiten).
Assume A is a quasitilted algebra which is not tilted.
Applying results on the structure of the module category of a quasitilted algebra of canonical type (established by Lenzing-Skowroński, Meltzer, Ringel, ...) we prove that mod$A$ has no sincere module which is not the middle of a short chain.

Characterization of modules not being the middle of short chains of modules:

THEOREM 2.
Let A be an algebra and M a module in mod$A$ which is not the middle of a short chain. Then there exists a hereditary algebra H, a tilting module T in mod$H$, and an injective module I in mod$H$ such that the following statements hold:

- the tilted algebra $B = \mathrm{End}_H(T)$ is a quotient algebra of A;
- M is isomorphic to the right B-module $\mathrm{Hom}_H(T, I)$.

Idea of proof:
A an artin algebra
M module in mod$A$ which is not the middle of a short chain in mod$A$
$B = A/\mathrm{ann}_A(M)$
M a sincere B-module
M is not the middle of a short chain in mod$B$ (Reiten, Skowroński, Smalø)
$B = B_1 \times ... \times B_m, \quad B_1, ..., B_m$ indecomposable algebras
$M = M_1 \oplus ... \oplus M_m, \quad M_i \in \mathrm{mod}B_i,\ B_i = A/\mathrm{ann}_A(M_i)$
M_i a sincere B_i-module which is not the middle of a short chain in modB_i
B_i a tilted algebra by **THEOREM 1**.
Hence, we may assume that B is an indecomposable algebra.
Applying results on the structure of the module category of a tilted algebra we prove that:

- $M \in \mathrm{add}(\mathcal{C}_{\overline{T}})$ for a connecting component $\mathcal{C}_{\overline{T}}$ of Γ_B of a tilting module $\overline{T}$ over a hereditary algebra $\overline{H}$ with $B = \mathrm{End}_{\overline{H}}(\overline{T})$.

Moreover, using the Auslander-Reiten theory and combinatorial arguments, we prove that:

- there is a section Δ in $\mathcal{C}_{\overline{T}}$ such that every indecomposable direct summand of M belongs to Δ.

T^*_Δ the direct sum of all indecomposable B-modules lying on Δ
$H_\Delta = \mathrm{End}_B(T^*_\Delta)$ indecomposable hereditary algebra
$T_\Delta = D(T^*_\Delta)$ tilting module in $\mathrm{mod}H_\Delta$
$B_\Delta = \mathrm{End}_{H_\Delta}(T_\Delta)$ the basic algebra of B
Take $H = H_\Delta$.
There exists a tilting module T in $\mathrm{add}(T_\Delta)$ in $\mathrm{mod}H = \mathrm{mod}H_\Delta$ such that:

- $B = \mathrm{End}_H(T)$
- $\mathcal{C}_{\overline{T}} = \mathcal{C}_T$ the connecting component of Γ_B determined by T
- $\Delta = \Delta_T$ the canonical section of $\mathcal{C}_T$
- $M \cong \mathrm{Hom}_H(T, I)$ for an injective module I in $\mathrm{mod}H$

COROLLARY.
Let A be an algebra and M a module in mod$A$ which is not the middle of a short chain. Then $\mathrm{End}_A(M)$ is a hereditary algebra.

We show that the modules not being the middle of short chains occur also in the module categories of selfinjective algebras.

Example.
K a field
H a basic indecomposable finite-dimensional hereditary K-algebra
T a multiplicity-free tilting module in $\mathrm{mod}H$
$B = \mathrm{End}_H(T)$
$T(B)^{(r)}$ r-fold trivial extension algebra, $r \geqslant 2$:

$$T(B)^{(r)} = \left\{ \begin{bmatrix} b_1 & 0 & 0 & & & \\ f_2 & b_2 & 0 & & 0 & \\ 0 & f_3 & b_3 & & & \\ & & \ddots & \ddots & & \\ & 0 & & f_{r-1} & b_{r-1} & 0 \\ & & & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \dots, b_{r-1} \in B,\ f_1, \dots, f_{r-1} \in D(B) \right\}$$

$T(B)^{(r)}$ basic indecomposable finite-dimensional **selfinjective** K-algebra
$T(B)^{(r)} \cong \widehat{B}/(\nu_{\widehat{B}}^{r})$

$A_m = T(B)^{(4(m+1))}$ for a fixed $m \geqslant 1$
$i \in \{1, ..., r\}$:
$B_{4i} = B$
$M_{4i} = \mathrm{Hom}_H(T, D(B)) \in \mathrm{mod}(B_{4i})$
$M_{4i} \in \mathcal{C}_{4i} = \mathcal{C}_T$

$$M = \bigoplus_{i=1}^{m} M_{4i}$$

- M is not the middle of a short chain in $\mathrm{mod} A_m$ (RSS)
- $A_m/\mathrm{ann}_{A_m}(M)$ is isomorphic to $\prod_{i=1}^{m} B_{4i}$

References:

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