The crystal structure on Mirković-Vilonen polytopes and representations of preprojective algebras

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Definition

If $M_\bullet$ satisfy the edge inequalities, we can define a polytope

$$P(M_\bullet) = \{\alpha \in \mathfrak{h}_\mathbb{R} | \langle \alpha, \gamma \rangle \geq M_\gamma, \text{ for all } \gamma \in \Gamma\}$$

associated to $M_\bullet$, called a pseudo-Weyl polytope.

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- We have a map $W \twoheadrightarrow \{\text{vertices of } P(M_\bullet)\}$, $w \mapsto \mu_w$ such that

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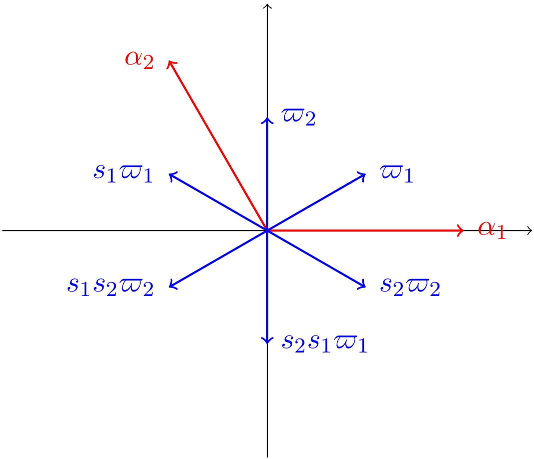

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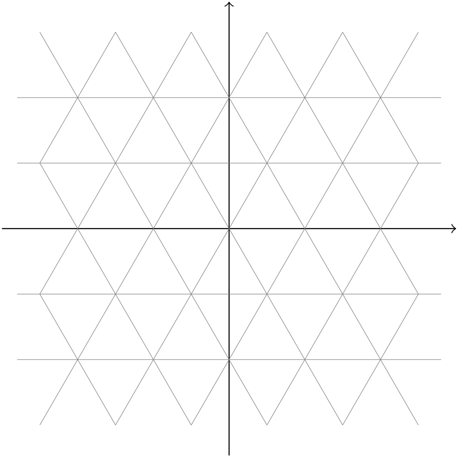

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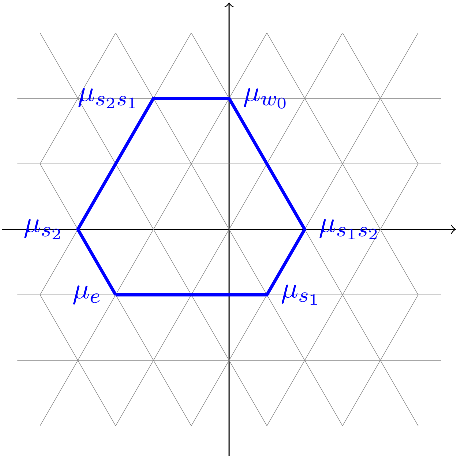

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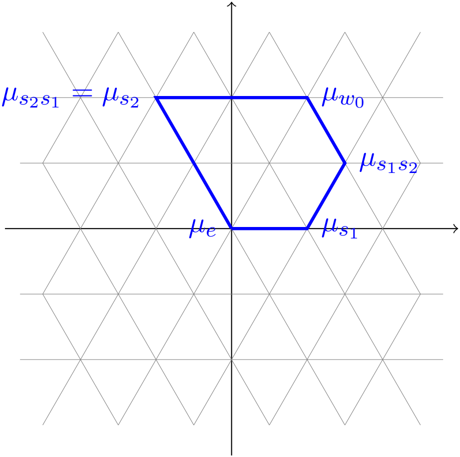

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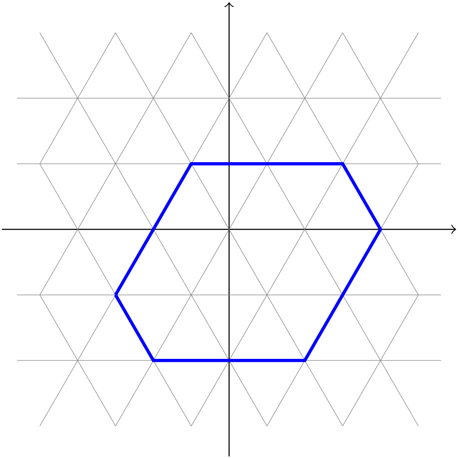

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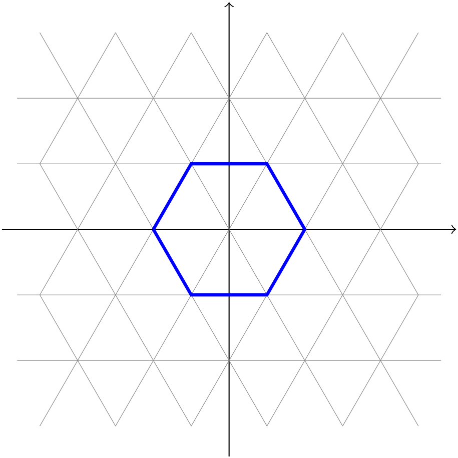

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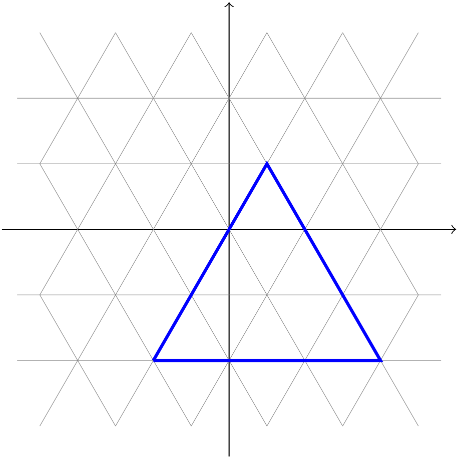

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Definition

A pseudo-Weyl polytope $P(M_\bullet)$ is called an Mirković-Vilonen (MV) polytope if $M_\bullet$ satisfy the tropical Plücker relations. And in this case $M_\bullet$ is called the Berenstein-Zelevinsky (BZ) datum of $P(M_\bullet)$.

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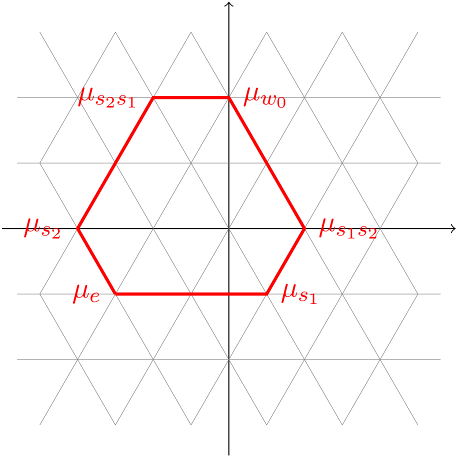

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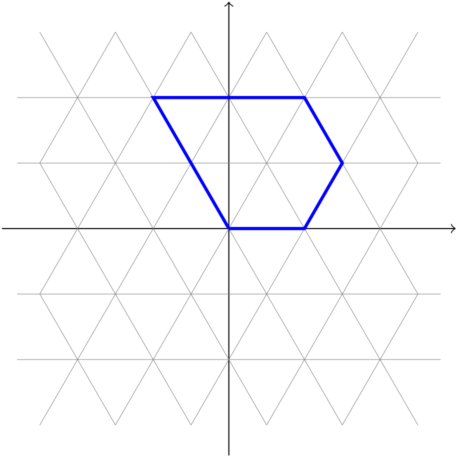

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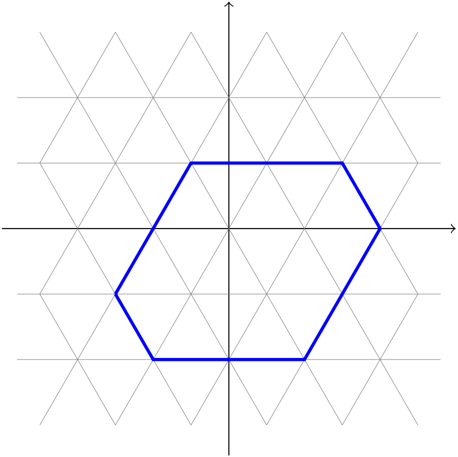

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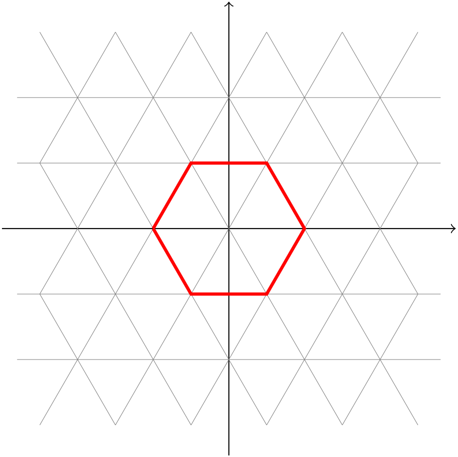

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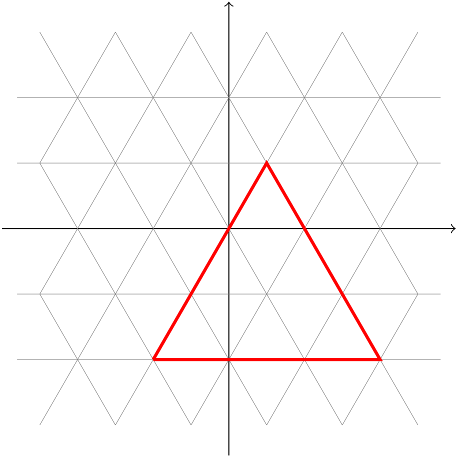

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- $\mathcal{MV}$: the set of stable MV polytopes

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Theorem (Kamnitzer, 2010)

(1). For any reduced word $\mathbf{i}$, taking $\mathbf{i}$-Lusztig datum gives a bijection $\mathcal{MV} \to \mathbb{N}^r$.
(2). For any reduced word $\mathbf{i}$, we have a coweight-preserving bijection $\mathcal{B} \leftrightarrow \mathcal{MV}$ identifying the $\mathbf{i}$-Lusztig datum, where $\mathcal{B}$ is Lusztig's canonical basis of $U_q^+(\mathfrak{g}^\vee)$.

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Thus the set $\mathcal{MV}$ inherits a crystal structure from $\mathcal{B}$.

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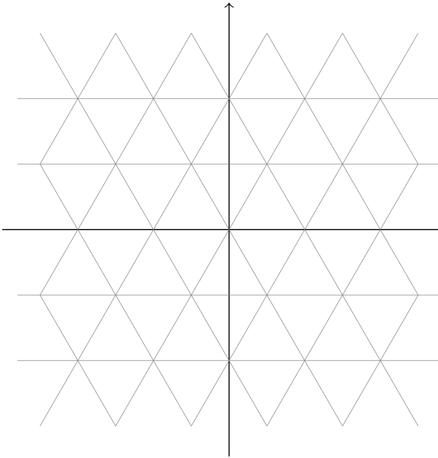

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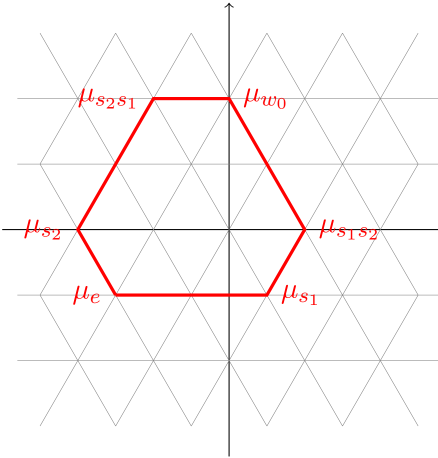

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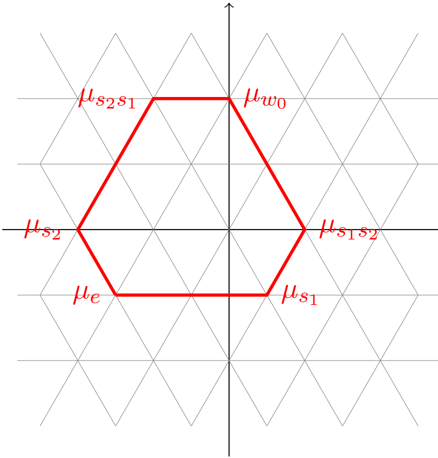

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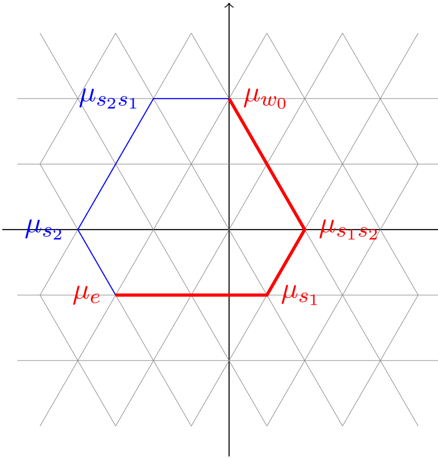

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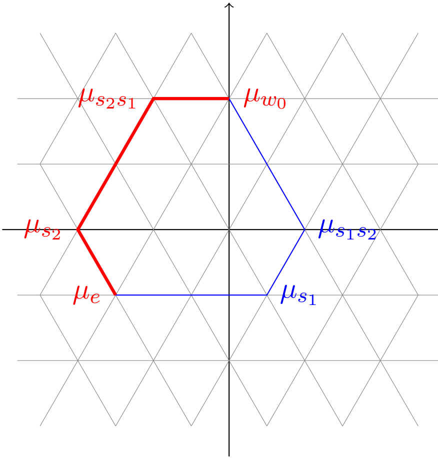

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Other M'_γs are determined by the tropical Plücker relations. Thus the description of the Kashiwara operators is non-explicit.

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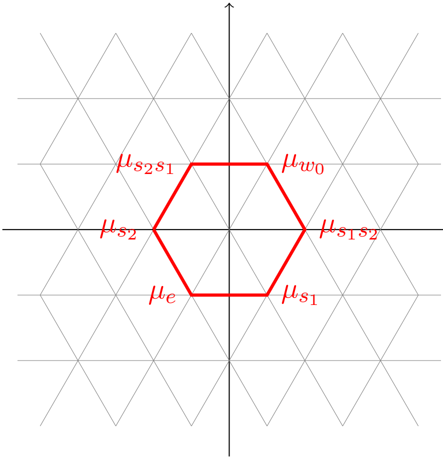

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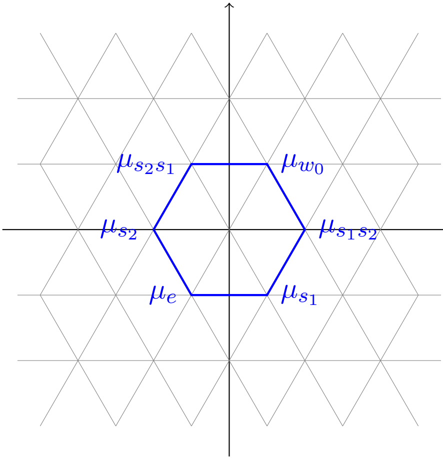

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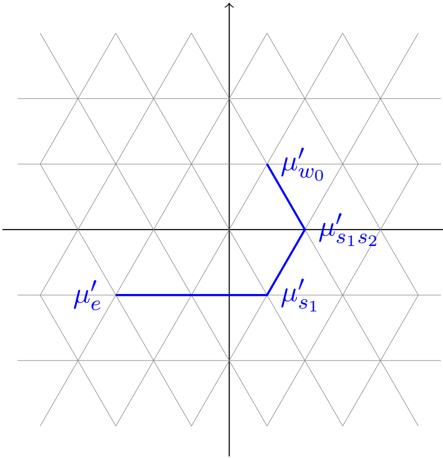

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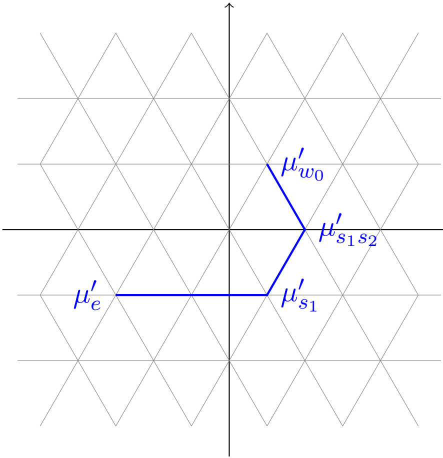

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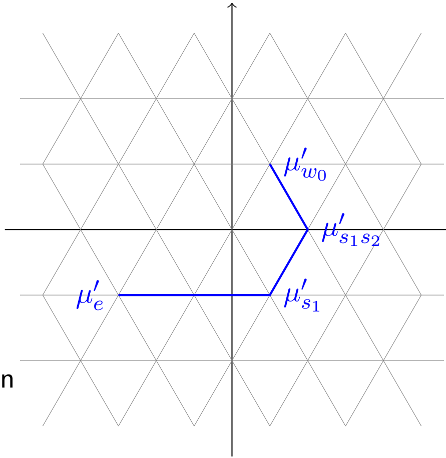

The Kashiwara operator - Examples

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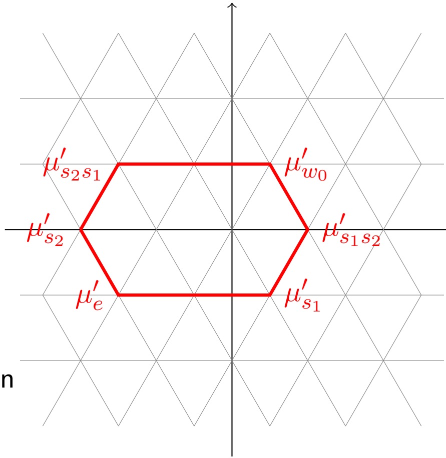

The Anderson-Mirković operators

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Definition

For each $j \in I$, define a polytope $AM_j(P) = \mathrm{conv}(\mu'_\bullet) = P'$ to be the smallest pseudo-Weyl polytope such that
(i) $\mu'_w = \mu_w$ for all $w \in W_j^-$,
(ii) $\mu'_e = \mu_e - h_j$,
(iii) P' contains μ_w for all $w \in W_j^+$,
(iv) if $w \in W_j^-$ is such that $\langle \mu_w, h_j \rangle \geq c_j$, then P' contains $r_j(\mu_w)$.

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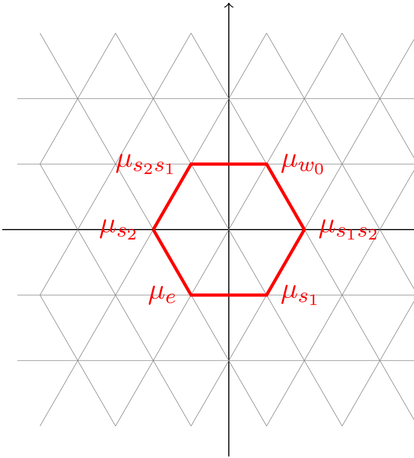

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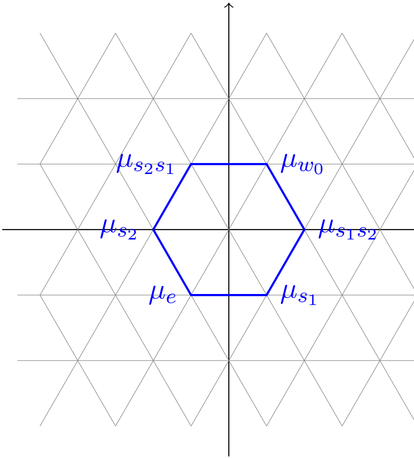

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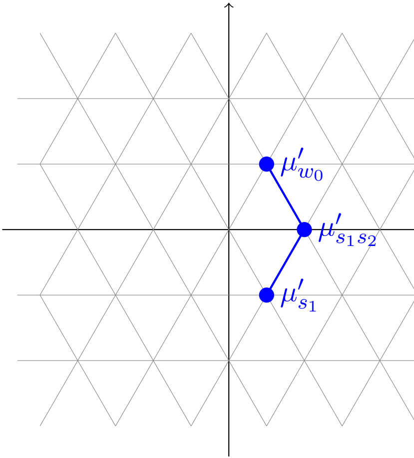

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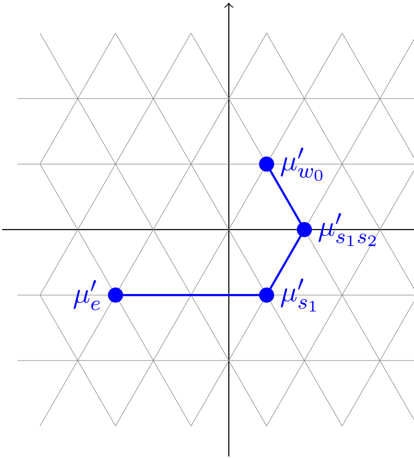

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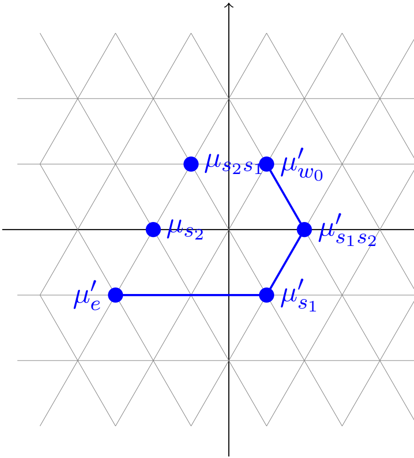

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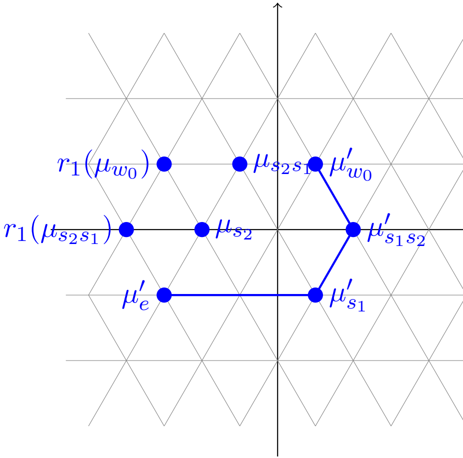

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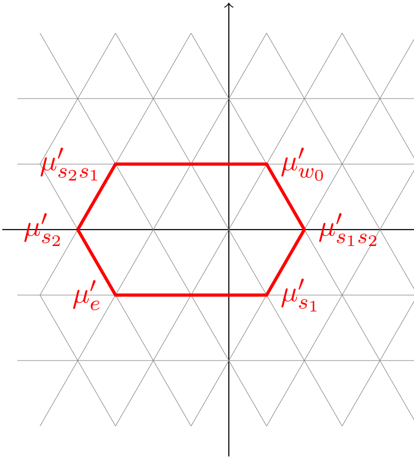

The Anderson-Mirković conjecture

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Conjecture (Anderson-Mirković)

For any MV polytope P and any $j \in I$, $AM_j(P)$ is an MV polytope and $AM_j(P) = \widetilde{f}_j(P)$.

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This answers an open question of Kamnitzer in "The crystal structure on the set of Mirković-Vilonen polytopes" (Adv. Math. 215 (2007), 66-93).

Results - continued

Theorem

(1) Let P be an MV polytope and $j \in I$, if $\varepsilon_j(P) + \varepsilon_j^*(P) - \langle \mathrm{wt}(P), \alpha_j \rangle = 0$, *then* $AM_j(P) = \widetilde{f}_j(P)$, *namely the AM conjecture holds.*
(2) In this case, assume that $P = P(M_\bullet)$ *and* $AM_j(P) = P(M'_\bullet)$, *we have*

$$M'_\gamma = \begin{cases} M_\gamma, & \text{if } \gamma \in \Gamma^j \\ M_{s_j\gamma} + c_j\langle h_j, \gamma \rangle, & \text{if } \gamma \in \Gamma \setminus \Gamma^j \end{cases}$$

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So in this case we have a very explicit description of the action of Kashiwara operators.

Results - continued

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Propostion

Let P be an MV polytope and $j \in I$ such that $\widetilde{e}_j(P) = 0$. Then there exists a positive integer $N = N(j, P)$ such that for any $n \geq N$, the assumption in the Theorem (1) holds for $\widetilde{f}_j^n P$. In particular, $AM_j(\widetilde{f}_j^n P) = \widetilde{f}_j^{n+1} P$.

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Theorem (Baumann-Kamnitzer, 2012)

(1). For any $\mathbf{v} \in \mathbb{N}^n$ and a irreducible component $Z \in \operatorname{Irr} \Lambda_{\mathbf{v}}$, $(D_\gamma(Z))_{\gamma \in \Gamma}$ is a BZ-datum.
(2). The map $\coprod_{\mathbf{v} \in \mathbb{N}^n} \operatorname{Irr} \Lambda_{\mathbf{v}} \to \mathcal{MV}$ given by $Z \mapsto P((D_\gamma(Z))_{\gamma \in \Gamma})$ is a crystal isomorphism.

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- Observation: If $M_\bullet$ satisfy the following condition

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- Use the homological properties in the module category $\mathrm{mod}(\Lambda)$.
- We also have a new proof for the AM conjecture in type A.

Thank You!