On the quasi-coherent sheaves over a weighted projective line

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Weighted projective lines $\mathbb{X}$

$k = \overline{k}$;

$p = (p_1, \cdots, p_n)$: an n-tuple of integers with $p_i > 1$;

$\mathrm{L}(p)$: rank 1 abelian group on generators $\vec{x_1}, \cdots, \vec{x_n}$ with relations

$$p_1\vec{x_1} = \cdots = p_n\vec{x_n} =: \vec{c};$$

$\lambda = (\lambda_1, \cdots, \lambda_n)$: pairwise distinct elements of $\mathbb{P}^1_k$;

$S(p, \lambda) = k[X_1, \cdots, X_n]/\langle X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1},\ i = 3, \cdots, n\rangle$.

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Let $\mathbb{X}$ be the curve corresponding to $S(p, \lambda)$. Then $\mathbb{X}$ is called the weighted projective line of type $p = (p_1, \cdots, p_n)$.

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For the detail structures of the category of coherent sheaves on a weighted projective line, we can see [L1].

[L1] H. Lenzing, Hereditary categories, Handbook of tilting theory, London Mathematical Society Lecture Note Series **332**(2007), 105-146.

Tilting objects in vect$\mathbb{X}(2,2,2,2)$

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Let vect$\mathbb{X}$ be the subcategory of vector bundles on the weighted projective lines. [GL] show that,

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Recently, [KLM] gives an explicit tilting object in $\underline{\mathrm{vect}}\mathbb{X}$ of domestic and tubular cases except type (2,2,2,2) .

[GL] W. Geigle, H. Lenzing, A class of weighted projective curves arising in representation theory of finite dimensional algebras. Singularities, representations of algebras, and Vector bundles, Springer Lect. Notes Math. **1273** (1987), 265-297.

[KLM] Kussin, Lenzing and Meltzer. Triangle singularities, ADE-chains and weighted projective lines. arXiv:1203.5505

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(iii) Not all rank two bundles are exceptional.

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Solution:

(i) Describe projective cover and injective hull of all the rank two bundles.

(ii) Give a formula to compute $\mu(X[1])$

(iii) Show all the exceptional objects.

Tilting objects in $\underline{\mathrm{vect}}\mathbb{X}(2,2,2,2)$

Let E, E_1, E_2, E_3, E_4 and F be the middle term of the following non-split exact sequences in $\mathrm{coh}\mathbb{X}$, respectively.

$$\eta : 0 \longrightarrow O(\vec{\omega}) \longrightarrow E \longrightarrow O \longrightarrow 0,$$

$$\xi_i : 0 \longrightarrow O(\vec{\omega}) \longrightarrow E_i \longrightarrow O(\vec{x_i}) \longrightarrow 0, (1 \leq i \leq 4),$$

$$\zeta : 0 \longrightarrow O(\vec{\omega}) \longrightarrow F \longrightarrow E(\vec{\omega} + \vec{x_i}) \longrightarrow 0.$$

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Theorem

$T = E \bigoplus (\bigoplus_{i=1}^{4} E_i) \bigoplus F$ is a tilting object in $\underline{\text{vect}}\mathbb{X}$ such that End$T$ is a canonical algebra of type $(2,2,2,2)$.

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Remark: T consists of vector bundles with rank 2(5) and 3(1). There doesn't exist tilting object consisting of rank 2(6) such that End$T$ is a canonical algebra.

[CLR] J. Chen, Y. Lin, S. Ruan, Tilting objects in the stable category of vector bundles on the weighted projective line of type $(2,2,2,2;\lambda)$. Submitted

Relations between cluster tilting and tilting

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Note: Let $\mathbb{X}$ be the weighted projective lines of domestic or tubular. By [L2], the stable category $\underline{\mathrm{vect}}\mathbb{X}$ is a triangulated category and derived hereditary. Moreover, it satisfies the conditions of Theorem 1 of Keller [K]. So we can define the cluster category

$$\mathcal{C} := \underline{\mathrm{vect}}\mathbb{X}/(\tau^{-1} \circ [1]).$$

[L2] H. Lenzing. Weighted projective lines and applications. Representations of Algebras and Related Topics, European Mathematical Society, 153-187. DOI: 10.4171/101-1/5

[K] B. Keller. On triangulated orbit categories. Documenta Math. 10(2005), 551-581.

For domestic cases

Let $T = \bigoplus T_i$ with T_i indecomposable and the slope of T_i satisfy $\mu T_i \in [0, \delta(-\overrightarrow{\omega} + \overrightarrow{x_1}))$, where $\overrightarrow{\omega}$ is the dualizing element and $\delta : L(p) \to \mathbb{Z}$ is the homomorphism defined on generators by $\delta(\overrightarrow{x_i}) = \frac{l.c.m.\{p_i\}}{p_i}$. We have the following result.

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T is a cluster tilting object in $\mathcal{C}$ if and only if T is a tilting object in $\underline{\text{vect}}\mathbb{X}$.

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T is a cluster tilting object in $\mathcal{C}$ if and only if T is a tilting object in $\underline{\text{vect}}\mathbb{X}$.

Remark: Since there are only finitely many indecomposable objects with slopes belong to $[0, \delta(-\vec{\omega} + \vec{x_1}))$, by the Theorem, there are only finitely many cluster tiling objects in $\mathcal{C}$.

[CLR] J. Chen, P. Liu and S. Ruan. The relationship between the cluster tilting objects in $\underline{\text{vect}}\mathbb{X}/\tau^{-1}[1]$ and the tilting objects in $\underline{\text{vect}}\mathbb{X}$. Prepared.

Relations between cluster tilting and tilting

For tubular cases

Let $T = \bigoplus T_i$ with T_i indecomposable and the slope of T_i satisfy $\mu T_i \in [0, \alpha(0))$, where α is a bijective, monotonous map on $\mathbb{Q}$ satisfies $\alpha(q) > q$ for all $q \in \mathbb{Q}$ and $\mu(X[1]) = \alpha(\mu(X))$ for each indecomposable $X \in \mathrm{vect}\mathbb{X}$. We have the following result.

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T is a cluster tilting object in $\mathcal{C}$ if and only if T is a tilting object in $\underline{\text{vect}}\mathbb{X}$.

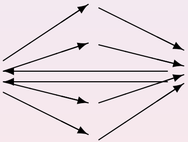
Note: For type $(2, 2, 2, 2)$, we have $\alpha(0) = \frac{4}{3}$. And the slope of each indecomposable direct summand of the tilting object T we have showed in [CLR] belongs to $[0, \frac{4}{3})$ in $\underline{\text{vect}}\mathbb{X}$.

Conclusion

Tilting objects in $\underline{\text{vect}}\mathbb{X}(2,2,2,2)$

Conclusion

Let T be the tilting object we have showed in [CLR]. Then T is corresponding to a cluster tilting object in $\mathcal{C}$ which we still denote it by T. Moreover, $\text{End}_{\mathcal{C}}(T)$ is an algebra given by the following quiver:



It is a self-injective algebra. So by the connectedness of tilting graph, all the cluster tilting objects in $\mathcal{C}$ can be obtained from the cluster mutations of this quiver.

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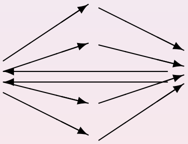

It is a self-injective algebra. So by the connectedness of tilting graph, all the cluster tilting objects in $\mathcal{C}$ can be obtained from the cluster mutations of this quiver. Therefore,

We can get all the tilting objects which satisfies the slope of each indecomposable direct summand belongs to $[0, \frac{4}{3})$.

Prüfer sheaves and generic sheaves

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Our work is based on the study of generic modules, prüfer modules, adic modules in the representation theory of algebras, which we would refer as follows:

[K] H. Krause, Generic modules over artin algebras, Proc. London Math. Soc. 76 (2) (1998) 276306.

[L] H. Lenzing, Generic modules over tubular algebras, in: Advances in Algebra and Model Theory, Gordon and Breach, London, 1997, pp. 375385.

[R] C.M. Ringel, Infinite dimensional representations of finite dimensional hereditary algebras, Ist. Naz. Alta Mat. Symp. Math. 23 (1979) 321412.

[RR] I. Reiten and C.M. Ringel. Infinite dimensional representations of canonical algebras. Canad. J. Math., 58(2006), 180-224.

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Definition([L3])

Let T be a tilting sheaf on $\mathbb{X}$, an indecomposable quasi-coherent sheaf G is called generic if G is not a coherent sheaf, and $\mathrm{Hom}(T, G)$ and $\mathrm{Ext}^1(T, G)$ have finite $\mathrm{End}(G)$-length.

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Theorem([L3])

There exists an unique indecomposable generic sheaf G_q of slope q for each $q \in \mathbb{Q} \bigcup \{\infty\}$ under isomorphism.

[L3] H. Lenzing, Generic modules over tubular algebras, in: Advances in Algebra and Model Theory, Gordon and Breach, London, 1997, pp. 375385.

Prüfer sheaves and generic sheaves

Let $\mathcal{T}$ be a stable tube in $\mathcal{C}^{(q)}$ with rank d. Let S_q be a quasi-simple sheaf, $S_q[i]$ be the indecomposable sheaf of length i in $\mathcal{C}^{(q)}$ satisfies $\mathrm{Hom}(S_q, S_q[i]) \neq 0$. Then there is a sequence of embeddings $S_q \to S_q[2] \to \ldots \to S_q[i] \to \ldots$.

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Denote by $S_q[\infty]$ the corresponding direct limit. We call $S_q[\infty]$ the prüfer sheaf of S_q over the weighted projective line $\mathbb{X}$.

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There is also an indecomposable sheaf $S_q[-i]$ of length i in $\mathcal{C}^{(q)}$ satisfies $\mathrm{Hom}(S_q[-i], S) \neq 0$, and we can obtain a sequence of epimorphisms $\ldots \to S_q[-i] \to \ldots \to S_q[-2] \to S_q$.

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Denote by $S_q[-\infty]$ the corresponding inverse limit. We call $S_q[-\infty]$ the adic sheaf of S_q over $\mathbb{X}$.

Prüfer sheaves and generic sheaves

Proposition

$S[-\infty] = \mathrm{D}((\mathrm{D}S)[\infty])$ for each quasi-simple sheaf S, where $\mathrm{D} = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ the duality.

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Theorem

There are two exact sequences as following in $\mathrm{Qcoh}(\mathbb{X})$:

$$0 \to S_q \to S_q[\infty] \to \tau^{-1} S_q[\infty] \to 0,$$
$$0 \to S[d]_q \to S_q[\infty] \to S_q[\infty] \to 0,$$

which produce two inverse systems $\{\tau^i S_q[\infty] \mid i \in \mathbb{N}\}$ and $\{S_q[\infty] \mid i \in \mathbb{N}\}$. Moreover, we have $\varprojlim \tau^i S_q[\infty] = \bigoplus G_q$ and $\varprojlim S_q[\infty] = \bigoplus G_q$.

[CCL] J. Chen, J. Chen, Y. Lin. Prüfer sheaves and generic sheaves over the weighted projective lines of genus one. Completed

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Corollary

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By describing the morphisms between coherent sheaves and prüfer sheaves, and adic sheaves, and combining with the results of generic sheaves obtained by [L3], we have

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Proposition

Let $q \in \mathbb{Q} \bigcup \{\infty\}$, then

$$({}^{\perp}S_q[-\infty] \bigcap {}^{\perp}S_q[\infty]) \bigcap \mathrm{coh}(\mathbb{X}) = \mathcal{C}^{(q)} = {}^{\perp} G_q \bigcap \mathrm{coh}(\mathbb{X}),$$

where ${}^{\perp}X$ is the left perpendicular category of X in $\mathrm{Qcoh}(\mathbb{X})$.

Thank you !