Toric moduli spaces of quivers

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Introduction

- A *quiver setting* is a pair (Q, α) consisting of a quiver $Q = (V, A, s, t)$ and a dimension vector $\alpha : V \to \mathbb{N}$.
- An α dimensional *representation* of Q assigns an $\alpha(v)$ dimensional vector space to each vertex v and a linear map to each arrow, going from the vector space at its source to the vector space at its target. $Rep_\alpha Q$ denotes the vector space of all α dimensional representations.
- We have the action of $GL_\alpha := \bigoplus_{v \in V} GL_{\alpha(v)}(\mathbb{C})$ on $Rep_\alpha Q$ by basechange. The orbits under this action are isomorphism classes of representations.

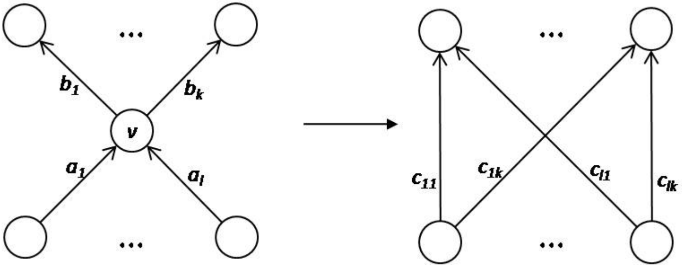
- The affine quotient: $iss_\alpha Q = Rep_\alpha Q//GL_\alpha = Spec(\mathbb{C}[Rep_\alpha Q]^{GL_\alpha})$
- The points of $iss_\alpha Q$ are in bijection with the isomorphism classes of semisimple representations.
- (LeBruyn - Procesi) $\mathbb{C}[iss_\alpha Q] = [Rep_\alpha Q]^{GL_\alpha}$ is generated by traces of products taken along cycles of bounded length $(\sum_i \alpha_i^2 + 1)$
- The aim is to describe the properties of this moduli space in terms of the "combinatorial input".

Smooth quiver settings

- If $iss_\alpha Q$ is smooth, it is an affine space.
- R. Bocklandt characterized the smooth cases via an algorithmic method.
- Combinatorial reduction steps: Remove vertices or arrows from the quiver in a way that $\mathbb{C}[iss_\alpha Q]$ remains the same (up to some free variables).
- Provide a list of the smooth cases where none of these steps can be applied

Reduction Steps

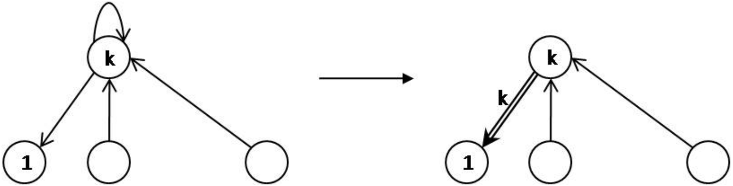
Reduction step I (RI)



If $\sum \alpha(s(a_i)) \leq \alpha(v)$ or $\sum \alpha(t(b_i)) \leq \alpha(v)$

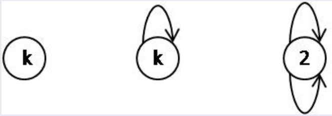
Reduction Steps

Reduction step II (RII): Remove loops from vertices of dimension 1.
Reduction step III (RIII):



Theorem (R. Bocklandt)

Let (Q, α) be a genuine strongly connected quiver setting, to which RI-III can not be applied. Then $iss_\alpha Q$ is smooth if and only if (Q, α) is one of the three quiver settings below:



Toric moduli spaces

- We studied the special case $\alpha = (1, ..., 1)$
- In this case $\mathbb{C}[iss_\alpha Q]$ is a monomial algebra, $iss_\alpha Q$ is a toric variety
- $\mathbb{C}[iss_\alpha Q]$ is generated by products of variables that correspond to the arrows of some primitive cycle of Q
- The ideal of relations can be generated by binomials that all have a degree 2 term.

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Theorem

For a strongly connected, Eulerian digraph G and two primitive cycles c_1 and c_2 in G, such that $G - c_1$ and $G - c_2$ contain more than one pr. cycle, there exists a sequence $c_1 = d_1, d_2, ..., d_k = c_2$ of pr. cycles such that d_i is arrow-disjoint from d_{i+1} for $1 \leq i \leq k - 1$

Complete Intersections in the toric case

- The complete intersection quiver settings to which RI-III can not be applied are very difficult to list even in the $\alpha = (1, ..., 1)$ case
- Reduction Step IV: If there is arrows both ways between two vertices, and the number of additional paths in between them is small enough (either 0 in one direction or no more than 1 in both directions) we contract them into a new vertex
- This new reduction step changes the coordinate ring, but leaves the complete intersection property invariant

Theorem

If (Q, α) is a strongly connected, complete intersection quiver setting with $\alpha = (1, ..., 1)$ and none of the reduction steps RI-RIV can be applied, then Q consists of a single vertex with no loops.

Characterizing via forbidden descendants

Definition

Q' is a *descendant* of Q if it can be obtained from Q by repeteadly applying reduction steps RI-III, taking subquivers or contracting strongly connected components.

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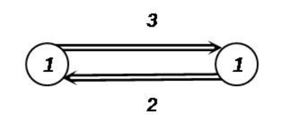
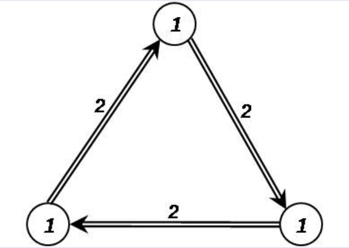
Theorem

A quiver setting (Q, α) *with* $\alpha = (1, ..., 1)$ *is C.I. if and only if it has none of the following descendants:*





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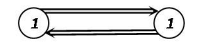
Theorem

A quiver setting (Q, α) is smooth if and only if it has no descendant of the following form:



A conjecture about the GIT moduli spaces

- Given a weight vector θ on the vertices, one can obtain a moduli space as the projective spectrum of the relative invariants whose weight is a rational multiple of θ
- In the $\alpha = (1, ..., 1)$ case these relative invariants correspond to weighted matchings of the quiver
- When the quiver contains no directed cycles, these moduli spaces will be projective toric varieties.

- Our conjecture is that, if $\alpha = (1, ..., 1)$ and Q contains no oriented cycles, then the vanishing ideals of the projective toric varieties obtained as GIT moduli spaces are generated in degree 3.
- It would be enough to see this conjecture for the special case when the quiver is a full bipartite quiver on $2 * n$ vertices and $\theta = -1$ on sources and $\theta = 1$ on targets. In this case the generators for the relative invariants are indexed by the symmetric group S_n.
- This algebra has been studied by Diaconis and Erikkson in an algebraic statistics paper. They formulated the same conjecture. Until now no bound is proven that is independent from the number of vertices.

Thank you for your attention!