Triangulated defect categories

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Motivation

Investigate *Gorenstein projective* modules M over a ring R.
(AKA modules of *Gorenstein dimension zero*, or *totally reflexive modules (TR)*)

Definition

$$
\begin{tikzcd}
C: & \cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[rr] \arrow[dr] & & C_{-1} \arrow[r] & C_{-2} \arrow[r] & \cdots \\
& & & & M \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

- C_i projective
- C acyclic, and $\mathrm{Hom}_R(C, P)$ acyclic for every projective P.

Note: Projectives are Gorenstein projective — these are the trivial ones.

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R commutative local.
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Thus: Pevtsova's rationale applies ...

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From the unpublished notes ...

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Definition

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- C is moreover *totally acyclic* if $\operatorname{Hom}_{\mathscr{P}}(C, P)$ is acyclic $\forall P \in \mathscr{P}$.

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We consider certain triangulated subcategories of

$\mathbf{K}\mathscr{P}$ — the homotopy category of complexes in $\mathscr{P}$

$$\begin{aligned}
\mathbf{K}_{\mathrm{tac}}\mathscr{P} &= \{C \in \mathbf{K}\mathscr{P} \mid C \text{ is totally acyclic}\} \\
\mathbf{K}^{-,\mathrm{b}}\mathscr{P} &= \{C \in \mathbf{K}\mathscr{P} \mid C_n = 0 \text{ for } n \ll 0 \text{ and } C \text{ is eventually acyclic}\} \\
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Definition

Define a function

$$\beta : \mathbf{K}_{\mathrm{tac}}\mathscr{P} \to \mathbf{K}^{-,\mathrm{b}}\mathscr{P}$$
$$\beta(C) = C_{\geq 0}$$

Brutal truncation at degree 0

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Not a functor of triangulated categories ...

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Theorem (BJO)

Brutal truncation at degree 0

$$\beta : \mathbf{K}_{\mathrm{tac}}\mathscr{P} \to \mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}$$
$$\beta(C) = C_{\geq 0}$$

is a fully faithful triangle functor.

Define

$$\boxed{\mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathscr{P}) = \mathbf{K}^{-,\mathrm{b}}\mathscr{P}/\mathbf{K}^{\mathrm{b}}\mathscr{P}}$$

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Definition

$$\mathbf{D}^{\mathrm{b}}_{\mathcal{C}}(\mathscr{P}) \overset{\text{def}}{=} \mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathscr{P})/\langle \operatorname{Im}\beta\rangle$$

Where $\langle \operatorname{Im}\beta\rangle$ is the thick closure of the image of β in $\mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(A)$. Thus $\mathbf{D}^{\mathrm{b}}_{\mathcal{C}}(\mathscr{P})$ is a triangulated category, called the *defect category of $\mathcal{C}$*

Overly restrictive applications

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- Theorem is a reformulation of Auslander-Bridger

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Is a reformulation of Gulliksen's result

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Theorem

$\mathbf{D}^{\mathrm{b}}_{\mathcal{C}}(\operatorname{proj} R) = 0 \iff$ *M has finite projective dimension*

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$\dim \mathbf{D}^{\mathrm{b}}_{\mathcal{C}}(\operatorname{proj} R)$ measures relatively how many modules are not TR

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Goal: Compute the dimension of the defect category ...

Dankeschön!