Relative Singularity Categories

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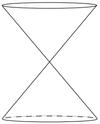
Motivation



$X = \mathbb{C}^2/\mathbb{Z}_2$

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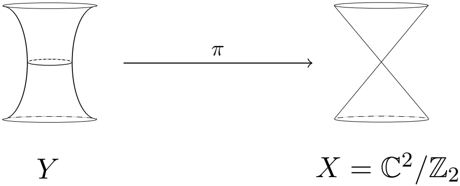

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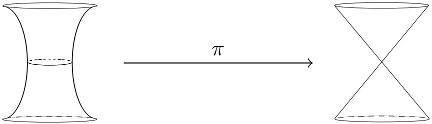

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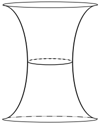
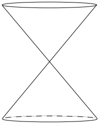


$$\xrightarrow{\pi}$$



$$\mathcal{D}^b\Big(\Pi(\circ \rightrightarrows \circ)\Big) \xleftarrow[\text{Kapranov \& Vasserot}\\ \text{Bridgeland, King \& Reid}]{\cong} \mathcal{D}^b(Y) \xhookleftarrow{\mathbb{L}\pi^*} \mathsf{Perf}(X)$$

Kapranov & Vasserot
Bridgeland, King & Reid

Derived McKay Correspondence

Introduction

Idea (Van den Bergh)

Replace $\mathcal{D}^b(Y)$ *by* $\mathcal{D}^b(A)$ *for a "nice" algebra* A *(e.g.* $\mathrm{gl.dim}(A) < \infty$*) and consider it as* **categorical resolution** *of* X *if there is an embedding*

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Let $M \in \mathrm{MCM}(R) := \left\{ N \in \mathrm{mod}-R \,|\, \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i > 0 \right\}$ be a **maximal Cohen–Macaulay module** and $A := \mathrm{End}_R(R \oplus M)$.

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- R: Gorenstein Singularity

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Buchweitz
Orlov

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Classical Singularity Category

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  where $n = n(S_i)$ is given by the length of the τ–orbit of M_i, by [KY].

A natural question

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$$\mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(\text{mod}-R)}{K^b(\text{proj}-R)} \xleftrightarrow{\text{Relation ?}} \Delta_R(A) := \frac{\mathcal{D}^b(\text{mod}-A)}{K^b(\text{proj}-R)}$$

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Remark

Knörrer's Periodicity yields a wealth of non-trivial examples for (i):

$$\mathcal{D}_{sg}\big(S/(f)\big) \xrightarrow{\sim} \mathcal{D}_{sg}\big(S[\![x,y]\!]/(f+xy)\big),$$

where $S = k[\![z_0, \ldots, z_d]\!]$, $f \in (z_0, \ldots, z_d)$ and $d \geq 0$.

Example

Let $R = \mathbb{C}[\![x]\!]/(x^2)$ and $R' = \mathbb{C}[\![x, y, z]\!]/(x^2 + yz)$. **Knörrer's equivalence** and our theorem above yield a triangle equivalence

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which may be written explicitly as

$$\frac{\mathcal{D}^b\left(\begin{tikzcd} 1 \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "i"] \end{tikzcd} \Big/ (pi) \right)}{K^b(\mathsf{add}\, P_1)} \xrightarrow{\sim} \frac{\mathcal{D}^b\left(\begin{tikzcd} 1 \arrow[r, bend left=40, "x"] \arrow[r, bend left=20, "y"] & 2 \arrow[l, bend left=20, "y"] \arrow[l, bend left=40, "x"] \end{tikzcd} \Big/ (xy-yx) \right)}{K^b(\mathsf{add}\, P_1)}.$$

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The quiver algebra on the right hand side is the **completion** of the preprojective algebra of the Kronecker quiver $\Pi(\circ \rightrightarrows \circ)$.

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Key Statement (Kalck & Yang)

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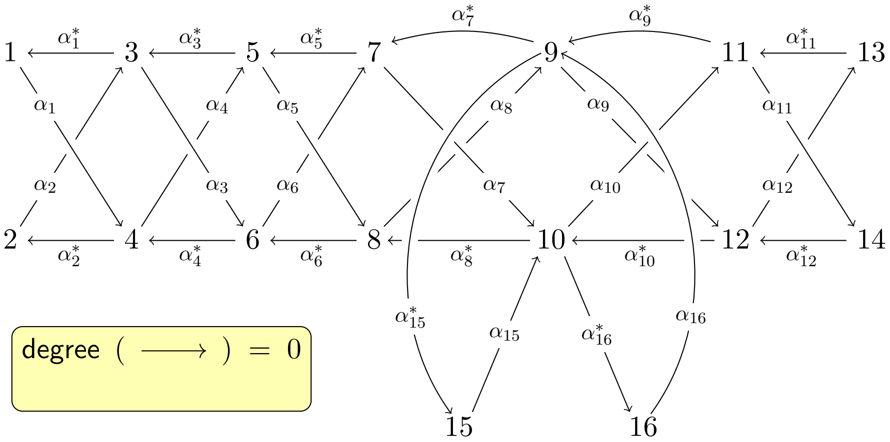
Corollary

$\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$ *implies* $\Delta_R(\mathsf{Aus}(R)) \cong \Delta_{R'}(\mathsf{Aus}(R'))$.

Example: The dg-Auslander algebra of an odd-dimensional $\mathbb{E}_8$-singularity: $f = z_0^3 + z_1^5 + z_2^2 + \ldots + z_d^2$



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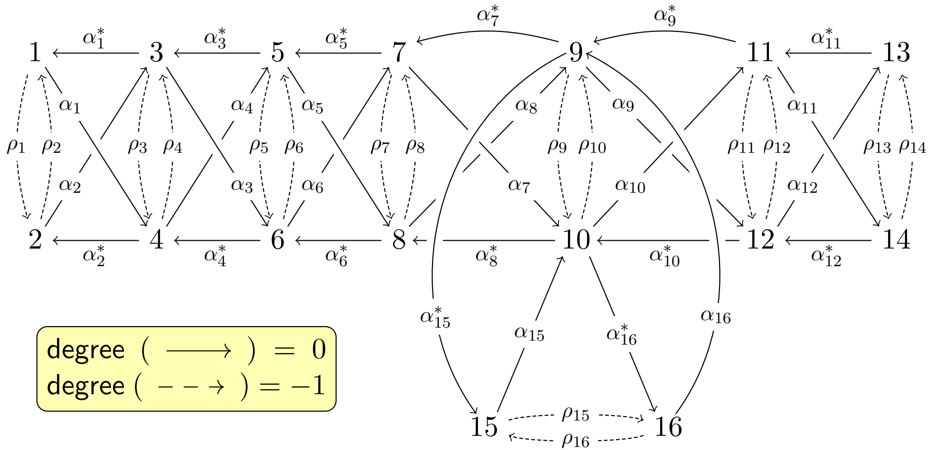

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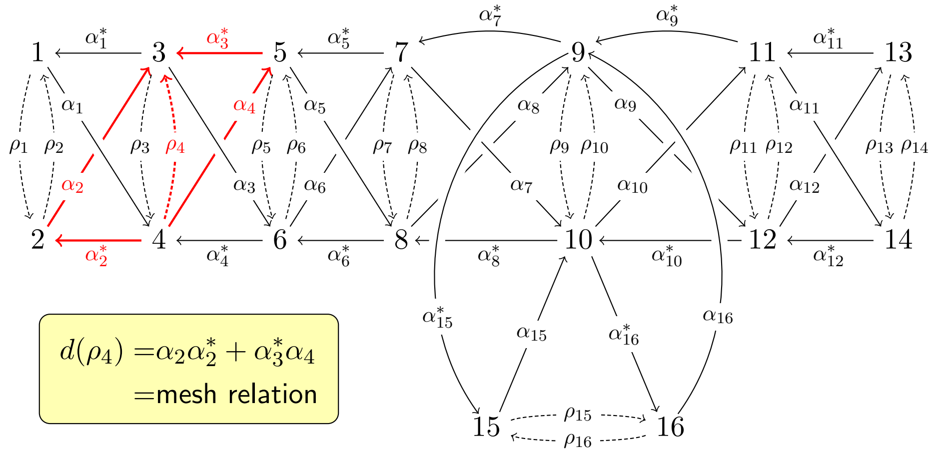

$$d(\rho_4) = \alpha_2\alpha_2^* + \alpha_3^*\alpha_4$$
$$= \text{mesh relation}$$

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A “purely commutative” application

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This is joint work with Osamu Iyama, Michael Wemyss & Dong Yang.

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Let R be a rational surface singularity with minimal resolution Y.

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- $\mathsf{gl.dim}(\Lambda) < \infty \qquad \Longrightarrow \qquad \mathcal{D}_{sg}(e\Lambda e) \cong \underline{\underline{\mathsf{SCM}}}(R)$

Example

Let $G \subseteq \mathrm{GL}(2, \mathbb{C})$ be the cyclic group of order 27 generated by

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{19} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}),$$

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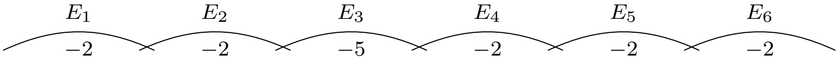

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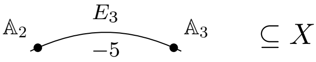

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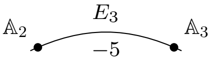

$\subseteq X$

Our Theorem yields a description of the stable category of SCMs:

$$\underline{\underline{\mathrm{SCM}}}(R_{27,19}) \cong \mathcal{D}_{sg}(X) \cong \underline{\mathrm{MCM}}\left(\frac{\mathbb{C}[\![x, y, z]\!]}{(x^3 + yz)}\right) \oplus \underline{\mathrm{MCM}}\left(\frac{\mathbb{C}[\![x, y, z]\!]}{(x^4 + yz)}\right)$$

Thank you!