Constructions of Auslander-Gorenstein local rings

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Motivations

Auslander-Gorenstein rings appear in various areas of current research. However, little is known about constructions of Auslander-Gorenstein rings. Today, generalizing the notion of crossed product, we will provide systematic constructions of Auslander-Gorenstein local rings starting from an arbitrary Auslander-Gorenstein local ring.

Notensions

1. R : a ring
2. $\mathrm{rad}(R)$: the Jacobson radical of R
3. $R^{\times}$: the set of units in R
4. $\mathrm{Z}(R)$: the center of R
5. $\mathrm{Aut}(R)$: the group of ring automorphisms of R
6. R^{σ}: the subring of R consisting of all $x \in R$ with $\sigma(x) = x$ for $\sigma \in \mathrm{Aut}(R)$
7. $\mathrm{M}_n(R)$: the ring of $n \times n$ full matrices over R for $n \geq 2$

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 2.1 (Auslander)

Let R be a left and right noetherian ring. Then for any $n \geq 0$ the following are equivalent.

(1) In a minimal injective resolution $I^\bullet$ of R in $\mathrm{Mod}\text{-}R$, $\mathrm{flat\ dim}\ I^i \leq i$ for all $0 \leq i \leq n$.

(2) In a minimal injective resolution $J^\bullet$ of R in $\mathrm{Mod}\text{-}R^{\mathrm{op}}$, $\mathrm{flat\ dim}\ J^i \leq i$ for all $0 \leq i \leq n$.

(3) For any $1 \leq i \leq n+1$, any $M \in \mathrm{mod}\text{-}R$ and any submodule X of $\mathrm{Ext}^i_R(M, R) \in \mathrm{mod}\text{-}R^{\mathrm{op}}$ we have $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X, R) = 0$ for all $0 \leq j < i$.

(4) For any $1 \leq i \leq n+1$, any $X \in \mathrm{mod}\text{-}R^{\mathrm{op}}$ and any submodule M of $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X, R) \in \mathrm{mod}\text{-}R$ we have $\mathrm{Ext}^j_R(M, R) = 0$ for all $0 \leq j < i$.

Definition 2.2 (Björk)

For a left and right noetherian ring R we say that R satisfies the Auslander condition if it satisfies the equivalent conditions in Proposition 2.1 for all $n \geq 0$, and that R is an Auslander-Gorenstein ring if $\mathrm{inj\ dim}\ R = \mathrm{inj\ dim}\ R^{\mathrm{op}} < \infty$ and if it satisfies the Auslander condition.

Next, we recall the notion of Frobenius extensions of rings due to [Nakayama and Tsuzuku] which we modify as follows.

Definition 2.3

A ring A is said to be an extension of R if A contains R as a subring, and the notation A/R is used to denote that A is an extension of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:
(F1) $A \in \mathrm{Mod}\text{-}R$ and $A \in \mathrm{Mod}\text{-}R^{\mathrm{op}}$ are finitely generated projective; and
(F2) $A \cong \mathrm{Hom}_R(A, R)$ in $\mathrm{Mod}\text{-}A$ and $A \cong \mathrm{Hom}_{R^{\mathrm{op}}}(A, R)$ in $\mathrm{Mod}\text{-}A^{\mathrm{op}}$.

It should be noted that if A/R is a Frobenius extension then so is $A^{\mathrm{op}}/R^{\mathrm{op}}$.

Definition 2.4

A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 2.5 (Abe, Hoshino)

For any Frobenius extension A/R the following hold.

(1) If R is an Auslander-Gorenstein ring then so is A with $\mathrm{inj\ dim}\ A \leq \mathrm{inj\ dim}\ R$.

(2) Assume that A/R is split. If A is an Auslander-Gorenstein ring then so is R with $\mathrm{inj\ dim}\ R = \mathrm{inj\ dim}\ A$.

Structure system

Throughout the rest of this note, we set $I(n) = \{1, \dots, n\}$ with $n \geq 2$ and fix a cyclic permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \end{pmatrix}$$

of $I(n)$. Then $\pi^{-i}(j) = \pi^{-j}(i)$ for all $i, j \in I(n)$ and the law of composition

$$I(n) \times I(n) \to I(n), (i, j) \mapsto \pi^{-i}(j)$$

makes $I(n)$ a cyclic group.

Example 3.1

Let $n = 4$.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$\pi^{-1}(2) = 3,\ \pi^{-4}(3) = 3,\ \pi^{-3}(4) = 3,\ \cdots$

We denote by $\Omega(n)$ the set of mappings $\omega : I(n) \times I(n) \to \mathbb{Z}$ satisfying the following conditions:
(W1) $\omega(i,i) = 0$ for all $i \in I(n)$;
(W2) $\omega(i,j) + \omega(j,k) \geq \omega(i,k)$ for all $i,j,k \in I(n)$;
(W3) $\omega(i,j) + \omega(j,i) \geq 1$ unless $i = j$; and
(W4) $\omega(i,j) + \omega(j,\pi(i)) = \omega(i,\pi(i))$ for all $i,j \in I(n)$.

Example 3.2

Let $n = 4$.

$$\omega(i,j)_{\{i,j\in I(4)\}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 3 & 0 \end{pmatrix}$$

Then $\omega \in \Omega(4)$.

Lemma 3.3

For any $\omega \in \Omega(n)$ the following hold.

(1) $\omega(\pi(i), \pi(j)) = \omega(i, j) - \omega(i, \pi(i)) + \omega(j, \pi(j))$ for all $i, j \in I(n)$.

(2) $\omega(1, i) = 0$ for all $i \in I(n)$ if and only if $\omega(i, n) = 0$ for all $i \in I(n)$.

We denote by $\Omega_+(n)$ the subset of $\Omega(n)$ consisting of $\omega \in \Omega(n)$ such that $\omega(1, i) = \omega(i, n) = 0$ for all $i \in I(n)$ (cf. Lemma 3.3(2)).

Example 3.4

Just before example's ω is in $\Omega_+(4)$.

We denote by $X_+(n)$ the set of mappings $\chi : I(n) \to \mathbb{Z}$ satisfying the following conditions:
(X1) $\chi(1) < \chi(2) < \cdots < \chi(n)$;
(X2) $\chi(i) + \chi(n-i+1) = \chi(n)$ for all $i \in I(n)$; and
(X3) $\chi(j-i) \leq \chi(j) - \chi(i) \leq \chi(j-i+1)$ for all $i, j \in I(n)$ with $i < j$.

Remark 3.5

For any $\chi : I(n) \to \mathbb{Z}$ satisfying the condition (X2) we have $\chi(1) + \chi(n) = \chi(n)$ and hence $\chi(1) = 0$.

Example 3.6

If $\chi(1) = 0, \chi(2) = 3, \chi(3) = 5, \chi(4) = 8$. Then

$$0 < 3 < 5 < 8$$

is satisfies (X1), (X2), (X3).

Proposition 3.7

For any $\omega \in \Omega_+(n)$, setting $\chi(i) = \sum_{k=1}^{i} \omega(k, \pi(k))$ for $i \in I(n)$, we have $\chi \in X_+(n)$ and

$$\omega(i,j) = \begin{cases} \chi(i) - \chi(j) + \chi(j-i+1) & \text{if } i \leq j, \\ \chi(i) - \chi(j) - \chi(i-j) & \text{if } i > j \end{cases}$$

for all $i, j \in I(n)$, so that we have a bijection $\Omega_+(n) \xrightarrow{\sim} X_+(n), \omega \mapsto \chi$.

Example 3.8

$\omega(4,2) = \chi(4) - \chi(2) - \chi(2)$
$\omega(1,3) = \chi(1) - \chi(3) + \chi(3)$
$\omega(2,3) = \chi(2) - \chi(3) + \chi(2)$

Throughout the rest of this note, we fix a ring R together with a pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition

$$(*) \quad \sigma(c) = c \quad \text{and} \quad xc = c\sigma(x) \text{ for all } x \in R.$$

Group rings

Throughout this section, we fix $\omega \in \Omega_+(n)$ and, setting $\chi(i) = \sum_{k=1}^{i} \omega(k, \pi(k))$ for $i \in I(n)$, assume that $\sigma^{\chi(n)} = \mathrm{id}_R$.

Let A be a free right R-module with a basis $\{v_i\}_{i \in I(n)}$ and define a multiplication on A subject to the following axioms:
(G1) $v_i v_j = v_{\pi^{-j}(i)} c^{\omega(\pi^{-j}(i), j)}$ for all $i, j \in I(n)$; and
(G2) $x v_i = v_i \sigma^{-\chi(i)}(x)$ for all $x \in R$ and $i \in I(n)$.

Denoting by $\{\beta_i\}_{i \in I(n)}$ the dual basis of $\{v_i\}_{i \in I(n)}$ for the free left R-module $\mathrm{Hom}_R(A, R)$, we have $a = \sum_{i \in I(n)} v_i \beta_i(a)$ for all $a \in A$. For any $a, b \in A$ and $i \in I(n)$ we have

$$\beta_i(ab) = \sum_{j \in I(n)} c^{\omega(i,j)} \sigma^{-\chi(j)}(\beta_{\pi^j(i)}(a)) \beta_j(b).$$

Theorem 4.1

The following hold.

(1) *A is an associative ring with* $1 = v_n$ *and contains R as a subring via the injective ring homomorphism* $R \to A, x \mapsto v_n x$.

(2) A/R *is a split Frobenius extension of first kind.*

(3) $v_i v_j = v_j v_i$ *for all* $i, j \in I(n)$*. In particular, A is commutative if R is commutative and* $\sigma^{\chi(i)} = \mathrm{id}_R$ *for all* $i \in I(n)$*. Furthermore, for any* $i \in I(n)$ *with* $i \neq n$ *we have* ${v_i}^r = c^s$ *for some* $2 \leq r \leq n$ *and* $s \geq 1$.

(4) *There exists an injective ring homomorphism*

$$\rho : A \to \mathrm{M}_n(R), a \mapsto (c^{\omega(i,j)}\sigma^{-\chi(j)}(\beta_{\pi^j(i)}(a)))_{i,j \in I(n)}$$

such that $a \in A^\times$ *for all* $a \in A$ *with* $\rho(a) \in \mathrm{M}_n(R)^\times$.

(5) *If* $c \in \mathrm{rad}(R)$ *then* $\beta_n(a) \in R^\times$ *for all* $a \in A^\times$ *and* $R/\mathrm{rad}(R) \xrightarrow{\sim} A/\mathrm{rad}(A)$ *canonically, so that if R is local then so is A.*

Remark 4.2

$\sigma = \mathrm{id} \Rightarrow A = R * G$: crossed product
In general, A is group rings and may not be crossed product.

Example

Example 5.1

We assume

$$0 < 3 < 5 < 8$$

since (G1) $v_i v_j = v_{\pi^{-j}(i)} c^{\omega(\pi^{-j}(i),j)}$ $(\forall i, j \in I(n))$ and Theorem 4.1

$$v_2 v_2 = v_4 c^{\omega(4,2)}, \quad v_3 v_3 = v_2 c^{\omega(2,3)}$$
$$v_2 v_3 = v_1 c^{\omega(1,3)}$$

$$v_2 v_2 = v_4 c^2, \quad v_3 v_3 = v_2 c$$
$$v_2 v_3 = v_1$$

Thus $A \cong R[X, Y]/(X^2 - c^2, Y^2 - Xc)$.

$$0 < 3 < 5 < 8$$

$$\Updownarrow$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 3 & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$A \cong R[X, Y]/(X^2 - c^2, Y^2 - Xc)$$

Moreover we denote generaly

$\chi(1) = 0,\ \chi(2) = p,\ \chi(3) = p + q,\ \chi(4) = 2p + q$

$\omega \in \Omega_+(4)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ p & 0 & p-q & 0 \\ q & q & 0 & 0 \\ p & q & p & 0 \end{pmatrix}$$

$$1 \leq q \leq p$$

1. $p - q = 0$
$$R[X]/(X^4 - c^p)$$
2. $p - q \geq 1$
$$R[X, Y]/(X^2 - c^q, Y^2 - Xc^{p-q})$$

Before example is the case $p = 3,\ q = 2$.

$$0 < 10 < 19 < 27 < 30 < 38 < 47 < 57$$

$$\Updownarrow$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 1 & 2 & 7 & 2 & 1 & 0 \\ 9 & 9 & 0 & 2 & 8 & 8 & 2 & 0 \\ 8 & 7 & 8 & 0 & 7 & 8 & 7 & 0 \\ 3 & 1 & 1 & 3 & 0 & 2 & 2 & 0 \\ 8 & 1 & 0 & 1 & 8 & 0 & 1 & 0 \\ 9 & 7 & 1 & 1 & 7 & 9 & 0 & 0 \\ 10 & 9 & 8 & 3 & 8 & 9 & 10 & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$R[X, Y, Z, W, U]/I$$

$$\begin{aligned} I = (&X^2 - Zc^7, XY - Zc, XZ - Y^2c, XW - Uc^7, \\ &YZ - Uc, YW - c^8, YU - Xc, Z^2 - c^3, ZU - Yc^2, \\ &W^2 - Xc^7, WU - Zc^7, U^2 - Y^2c, XU - ZW, ZW - Y^3) \end{aligned}$$

$$0 < 13 < 17 < 26 < 34 < 39 < 47 < 56 < 60 < 73$$

$$\Updownarrow$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13 & 0 & 9 & 4 & 5 & 8 & 5 & 4 & 9 & 0 \\ 4 & 4 & 0 & 4 & 0 & 4 & 5 & 0 & 4 & 0 \\ 9 & 0 & 9 & 0 & 5 & 4 & 5 & 4 & 5 & 0 \\ 8 & 4 & 4 & 8 & 0 & 8 & 5 & 4 & 8 & 0 \\ 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 \\ 8 & 0 & 8 & 4 & 0 & 8 & 0 & 4 & 4 & 0 \\ 9 & 4 & 5 & 4 & 5 & 4 & 9 & 0 & 9 & 0 \\ 4 & 0 & 4 & 0 & 0 & 4 & 0 & 4 & 0 & 0 \\ 13 & 4 & 9 & 8 & 5 & 8 & 9 & 4 & 13 & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$R[X, Y, Z]/(XY - c^4, Z^2 - c^5, X^3 - Y^2)$$