

# CLASSIFYING SERRE SUBCATEGORIES VIA ATOM SPECTRUM

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ABSTRACT. This report is based on the paper [K12]. We introduce the atom spectrum of an abelian category as a topological space consisting of all the equivalence classes of monoform objects. In terms of the atom spectrum, we give a classification of Serre subcategories of an arbitrary noetherian abelian category.

## 1. INTRODUCTION

Classification of subcategories has been studied by a number of authors, for example, [G62], [H97], [K97], [T08], and [BIK11]. Subcategories themselves are interesting objects. Moreover we expect that the structure of subcategories reflects some important properties of the whole category.

Throughout this report, we fix an abelian category  $\mathcal{A}$ . First of all, we recall the definition of a Serre subcategory.

**Definition 1.1.** A full subcategory  $\mathcal{X}$  of  $\mathcal{A}$  is called a *Serre subcategory* if it is closed under subobjects, quotient objects, and extensions.

**Remark 1.2.** This condition is equivalent to that for any short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in  $\mathcal{A}$ ,  $M$  belongs to  $\mathcal{X}$  if and only if  $L$  and  $N$  belong to  $\mathcal{X}$ .

The prototype of classifications of subcategories is the following theorem shown by Gabriel [G62]. For a ring  $R$ , denote by  $\text{Mod } R$  the category of all the  $R$ -modules and by  $\text{mod } R$  the category of finitely generated  $R$ -modules. We say that a subset  $\Phi$  of  $\text{Spec } R$  is *closed under specialization* if for any  $\mathfrak{p}, \mathfrak{q} \in \text{Spec } R$ ,  $\mathfrak{p} \subset \mathfrak{q}$  and  $\mathfrak{p} \in \Phi$  imply  $\mathfrak{q} \in \Phi$ .

**Theorem 1.3** (Gabriel [G62]). *Let  $R$  be a commutative noetherian ring. Then we have the following bijection*

$$\begin{aligned} \{\text{Serre subcategories of } \text{mod } R\} &\rightarrow \{\Phi \subset \text{Spec } R \mid \Phi \text{ is closed under specialization}\} \\ \mathcal{X} &\mapsto \bigcup_{M \in \mathcal{X}} \text{Supp } M. \end{aligned}$$

In this report, we generalize this theorem to any abelian category with some noetherian property.

## 2. MONOFORM OBJECTS

The key notion of this report is that of monoform objects, which was introduced by Storrer [S72].

**Definition 2.1.** A nonzero object  $H$  in  $\mathcal{A}$  is called *monoform* if for any nonzero subobject  $L$  of  $H$ , there does not exist a nonzero subobject of  $H$  which is isomorphic to a subobject of  $H/L$ .

The following theorem states an important relationship between monoform objects and Serre subcategories.

**Theorem 2.2.** *Let  $M$  be an object in  $\mathcal{A}$ .  $M$  is monoform if and only if  $M$  does not belong to the smallest Serre subcategory containing all the objects of the form  $M/N$  where  $N$  is a nonzero subobject of  $M$ .*

**Proposition 2.3.** *Let  $H$  be a monoform object in  $\mathcal{A}$ . Then the following hold.*

- (1) *Any nonzero subobject of  $H$  is also monoform.*
- (2)  *$H$  is uniform, that is, for any nonzero subobjects  $L_1$  and  $L_2$  of  $H$ ,  $L_1 \cap L_2 \neq 0$ .*

**Definition 2.4.** For monoform objects  $H$  and  $H'$  in  $\mathcal{A}$ , we say that  $H$  is *atom-equivalent* to  $H'$  if there exists a nonzero subobject of  $H$  which is isomorphic to a subobject of  $H'$ .

**Remark 2.5.** In fact, the relation of atom equivalence is an equivalence relation between monoform objects in  $\mathcal{A}$  since any monoform object is uniform.

Now we define the notion of atoms, which was originally introduced by Storrer [S72] in the case of module categories.

**Definition 2.6.** Denote by  $\text{ASpec } \mathcal{A}$  the quotient set (or quotient class) of the set of monoform objects in  $\mathcal{A}$  by atom equivalence. We call it the *atom spectrum* of  $\mathcal{A}$ . Elements of  $\text{ASpec } \mathcal{A}$  are called *atoms* in  $\mathcal{A}$ . The equivalence class of a monoform object  $H$  in  $\mathcal{A}$  is denoted by  $\overline{H}$ .

In section 4, we see that there exists a bijection between  $\text{ASpec}(\text{Mod } R)$  and  $\text{Spec } R$ . Hence the atom spectrum is a generalization of the prime spectrum in the commutative ring theory.

**Definition 2.7.** Let  $M$  be an object in  $\mathcal{A}$ .

- (1) Define the *atom support* of  $M$  by

$$\text{ASupp } M = \{\overline{H} \in \text{ASpec } \mathcal{A} \mid H \text{ is a subquotient of } M\}.$$

- (2) Define the set of *associated atoms* of  $M$  by

$$\text{AAss } M = \{\overline{H} \in \text{ASpec } \mathcal{A} \mid H \text{ is a subobject of } M\}.$$

The following proposition is a generalization of a proposition which is well-known in the commutative ring theory.

**Proposition 2.8.** *Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a short exact sequence in  $\mathcal{A}$ . Then the following hold.*

- (1)  $\text{ASupp } M = \text{ASupp } L \cup \text{ASupp } N$ .
- (2)  $\text{AAss } L \subset \text{AAss } M \subset \text{AAss } L \cup \text{AAss } N$ .

### 3. MAIN THEOREM

In order to generalize Gabriel's theorem (Theorem 1.3), we need to consider a generalized condition of "closed under specialization". This condition is given by the following topology.

**Definition 3.1.** Define a topology on  $\text{ASpec } \mathcal{A}$  as follows: we say that a subset (or subclass)  $\Phi$  of  $\text{ASpec } \mathcal{A}$  is open if for any  $\alpha$ , there exists  $H \in \alpha$  such that  $\text{ASupp } H \subset \Phi$ .

**Proposition 3.2.** *Open subsets of  $\text{ASpec } \mathcal{A}$  define a topology on  $\text{ASpec } \mathcal{A}$  which has an open basis  $\{\text{ASupp } M \mid M \in \mathcal{A}\}$ .*

We recall the definition of noetherian abelian categories.

- Definition 3.3.**
- (1) An object  $M$  in  $\mathcal{A}$  is called *noetherian* if for any ascending chain  $L_0 \subset L_1 \subset \dots$  of subobjects of  $M$ , there exists  $n \geq 0$  such that  $L_n = L_{n+1} = \dots$ .
  - (2) An abelian category  $\mathcal{A}$  is called *noetherian* if it is skeletally small (that is, the class of isomorphism classes forms a set), and any object in  $\mathcal{A}$  is noetherian.

**Remark 3.4.** The skeletally smallness is just a set-theoretical assumption. It  $\mathcal{A}$  ensures that  $\text{ASpec } \mathcal{A}$  is a set. We do not need to assume it if we allow  $\text{ASpec } \mathcal{A}$  to be a proper class.

**Theorem 3.5.** [K12] *Let  $\mathcal{A}$  be a noetherian abelian category. Then there exists a bijection*

$$\begin{aligned} \{\text{Serre subcategories of } \mathcal{A}\} &\rightarrow \{\text{open subsets of } \text{ASpec } \mathcal{A}\} \\ \mathcal{X} &\mapsto \bigcup_{M \in \mathcal{X}} \text{ASupp } M \\ \{M \in \mathcal{A} \mid \text{ASupp } M \subset \Phi\} &\leftarrow \Phi. \end{aligned}$$

### 4. IN THE CASE OF MODULE CATEGORIES

In the case of module categories, the atom spectrum is described in terms of one-sided ideals.

**Proposition 4.1.** *Let  $R$  be a ring. Then any atom in  $\text{Mod } R$  is represented by a monofrom object of the form  $R/\mathfrak{p}$ , where  $\mathfrak{p}$  is a right ideal of  $R$ . Moreover if  $R$  is right noetherian, then  $\text{ASpec}(\text{mod } R)$  is homeomorphic to  $\text{ASpec}(\text{Mod } R)$ .*

**Proposition 4.2.** *Let  $R$  be a commutative ring. Then the following hold.*

- (1) *For any ideal  $\mathfrak{a}$  of  $R$ ,  $R/\mathfrak{a}$  is monofrom in  $\text{Mod } R$  if and only if  $\mathfrak{a}$  is a prime ideal of  $R$ .*

- (2) For any prime ideals  $\mathfrak{p}$  and  $\mathfrak{q}$  of  $R$ ,  $R/\mathfrak{p}$  is atom-equivalent to  $R/\mathfrak{q}$  if and only if  $\mathfrak{p} = \mathfrak{q}$ . Therefore the correspondence  $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$  gives a bijection  $\text{Spec } R \rightarrow \text{ASpec}(\text{Mod } R)$ .
- (3) For any  $R$ -module  $M$ ,  $\text{ASupp } M = \text{Supp } M$ , and  $\text{AAss } M = \text{Ass } M$ .
- (4) For any subset  $\Phi$  of  $\text{Spec } R$ ,  $\Phi$  is open in the sense of  $\text{ASpec}(\text{Mod } R)$  if and only if  $\Phi$  is closed under specialization.

**Remark 4.3.** In the case where  $R$  is noetherian, we can formulate these claims by using  $\text{ASpec}(\text{mod } R)$  instead of  $\text{ASpec}(\text{Mod } R)$ . Then new claims which we obtain also hold.

In the case of artinian rings, moniformness is stated in terms of composition factors.

**Proposition 4.4.** *Let  $R$  be a right artinian ring. Then a finitely generated  $R$ -module  $M$  is moniform if and only if it has simple socle  $S$  such that there exists no other composition factor of  $M$  which is isomorphic to  $S$ .*

**Proposition 4.5.** *Let  $R$  be a right artinian ring and  $\{S_1, \dots, S_n\}$  be a maximal set of pairwise nonisomorphic simple modules. Then  $\text{ASpec } R = \{\overline{S_1}, \dots, \overline{S_n}\}$  with the discrete topology.*

**Example 4.6.** Let  $R$  be the ring of lower triangular matrices over a field  $K$ , that is,

$$R = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}.$$

Then all the right ideals of  $R$  are

$$0, \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}, \mathfrak{p}_a = K \begin{bmatrix} 1 & 0 \\ a & 0 \end{bmatrix} (a \in K), \mathfrak{m}_1 = \begin{bmatrix} 0 & 0 \\ K & K \end{bmatrix}, \mathfrak{m}_2 = \begin{bmatrix} K & 0 \\ K & 0 \end{bmatrix}, R.$$

All the comoniform right ideals of  $R$  are

$$\mathfrak{p}_a (a \in K), \mathfrak{m}_1, \mathfrak{m}_2.$$

Since

$$\frac{R}{\mathfrak{p}_a} \cong [K \ K], \frac{R}{\mathfrak{m}_1} \cong [K \ 0], \frac{R}{\mathfrak{m}_2} \cong \frac{[K \ K]}{[K \ 0]},$$

we have  $\widetilde{\mathfrak{p}}_a = \widetilde{\mathfrak{m}}_1 \neq \widetilde{\mathfrak{m}}_2$ . Therefore all the Serre subcategories of  $\text{mod } R$  are  $\{\text{zero objects}\}$ ,  $\langle R/\mathfrak{m}_1 \rangle_{\text{Serre}}$ ,  $\langle R/\mathfrak{m}_2 \rangle_{\text{Serre}}$ , and  $\text{mod } R$ .

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