Representation-finite algebras over algebraically closed fields form open $\mathbb{Z}$-schemes

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A representation-finite algebra over an algebraically closed field of positive characteristic remains representation-finite after change of the characteristic of the base field to 0.

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KA= the K-algebra $A \otimes_V K$,

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Theorem 2. *Given $d \in \mathbb{N}$, there are polynomials $H_1, ..., H_r$ with integral coefficients such that:*

if L is an algebraically closed field and Λ is a d-dimensional L-algebra then

$$\Lambda \text{ is representation-finite} \Leftrightarrow H_i(\gamma) \neq 0 \text{ for some } i \in \{1, ..., r\},$$

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Outline of the proof of Theorem 1.

Given an algebra Λ, Λ-module E and natural numbers m, l consider

$$\mathrm{Sub}_\Lambda(m, t, E)$$

= the projective variety of m-dimensional Λ-submodules of E with at lest t-dimensional endomorphism algebra.

Λ is representation-infinite $\Leftrightarrow$ $\dim \mathrm{Sub}_\Lambda(m, t, E) > md^2 - t$ for some m, t and an injective E, where $\dim \Lambda = d$.

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Assume KA is representation-infinite.

Choose suitable m, t and injective E defined over V. Prove that

$$\dim \mathrm{Sub}_{KA}(m, t, E) \leq \dim \mathrm{Sub}_{\overline{A}}(m, t, \overline{E}),$$

hence $\overline{A}$ is representation-infinite.

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Main lemma:

Let $F_1, ..., F_r$ be homogeneous polynomials with coefficients in V. The dimension of the projective variety defined by $F_1, ..., F_r$ over K is less than or equal to the dimension of projective variety defined by $F_1, ..., F_r$ over R.

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An idea of the proof:
consider Gelfand-Kirillov dimension.