Matrix factorisations for domestic singularities

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jt. work with H. Lenzing and H. Meltzer

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$\mathbb{L}$ – the rank one abelian group on generators

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$\mathcal{S}$ is $\mathbb{L}$ – graded algebra, with $\deg X_i = \vec{x}_i$

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$(2, 2, n)$ $(n \geq 2)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.

Motivation

Theorem

(i) $\mathrm{Sing}^{\mathbb{L}}(\mathcal{S}) \cong \underline{\mathrm{CM}}^{\mathbb{L}}(\mathcal{S}) \cong \mathrm{MF}^{\mathbb{L}}(S)$ [Buchweitz, Orlov]

(ii) $\underline{\mathrm{CM}}^{\mathbb{L}}(\mathcal{S}) \cong \underline{\mathrm{vect}}(\mathbb{X})$ [Geigle-Lenzing]

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Goal

Describe those categories using the $\mathbb{L}$–graded matrix factorisations of f.

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$$P_0 \xrightarrow{\phi} P_1 \xrightarrow{\psi} P_0(\vec{c}) \xrightarrow{\phi(\vec{c})} P_1(\vec{c}),$$

such that

$$\psi(\vec{c}) \circ \psi = f \circ 1_{P_1(\vec{c})}, \quad \text{and} \quad \psi \circ \phi = f \circ 1_{P_0(\vec{c})}$$

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These form a category where morphisms are considered up to homotopy. Denote by $\mathrm{MF}^{\mathbb{L}}(f)$.

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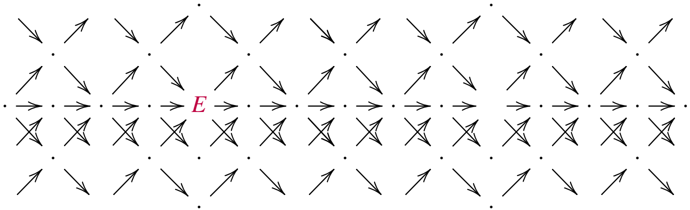

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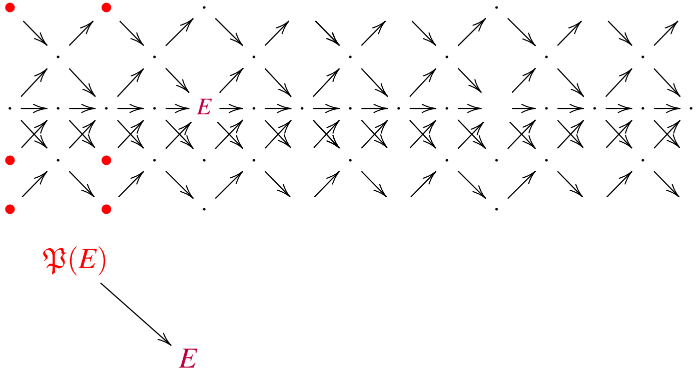

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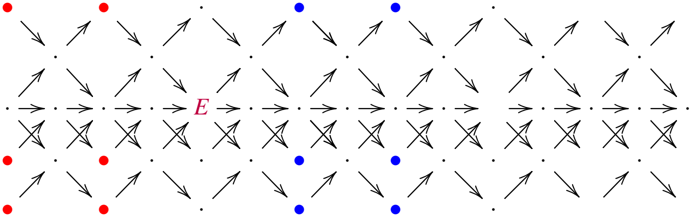

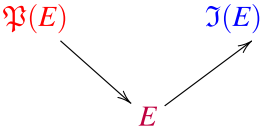

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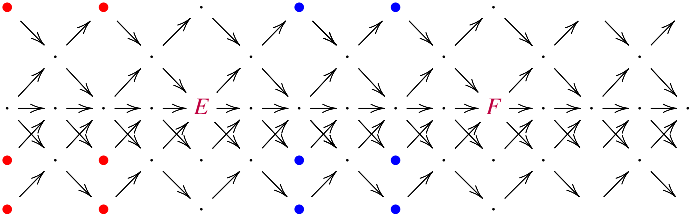

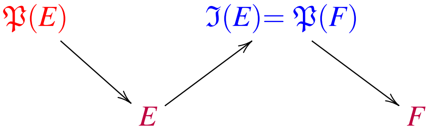

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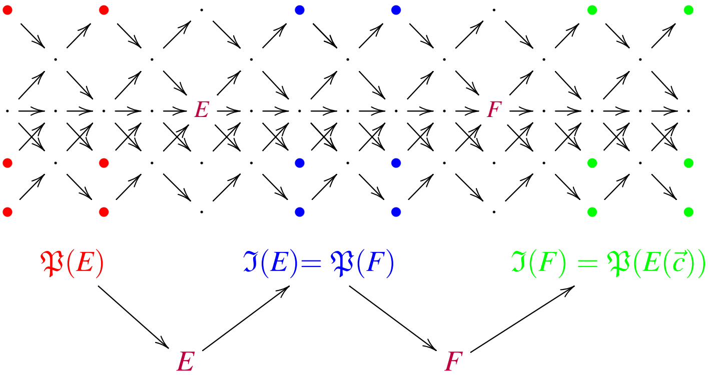

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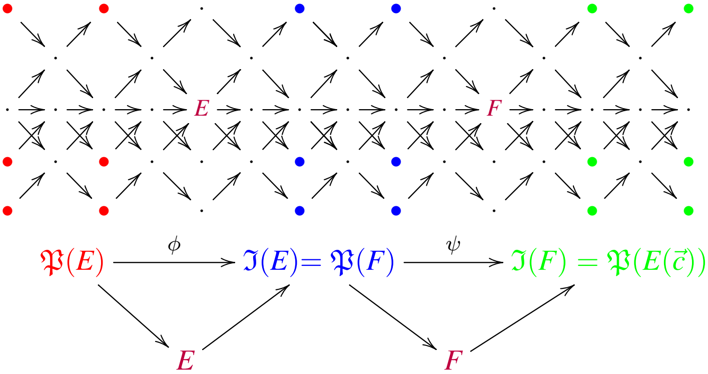

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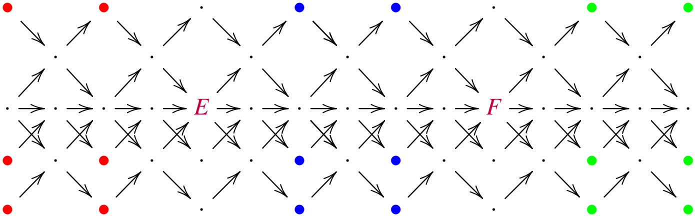

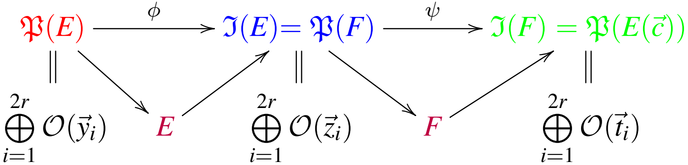

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\| & \searrow & & \nearrow & \| & \searrow & & \nearrow & \| \\
\bigoplus_{i=1}^{2r} \mathcal{O}(\vec{y}_i) & & E & & \bigoplus_{i=1}^{2r} \mathcal{O}(\vec{z}_i) & & F & & \bigoplus_{i=1}^{2r} \mathcal{O}(\vec{t}_i)
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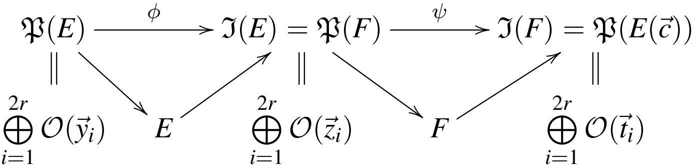

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&\overset{\text{lifting}}{\Longrightarrow} \psi, \phi \in M_{2r}(k[X_1, X_2, X_3])
\end{aligned}$$

Then matrix factorisation (ϕ, ψ) can be represented by two square matrices of $k[X_1, X_2, X_3]$, such that $\psi \circ \phi = fI$ and $\phi \circ \psi = fI$

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Theorem (Kussin-Lenzing-Meltzer)

Let E be an indecomposable vector bundle of rank two.
Then E is the middle term of an non-split sequence

$$0 \longrightarrow L(\vec{\omega}) \longrightarrow E \longrightarrow L(\vec{x}) \longrightarrow 0$$

for some line bundle L and some $\vec{x} = \sum_{i=1}^{3} l_i \vec{x}_i$, *with* $0 \leq l_i \leq p_i - 2$.

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for some line bundle L and some $\vec{x} = \sum_{i=1}^{3} l_i \vec{x}_i$, *with* $0 \leq l_i \leq p_i - 2$. *Moreover*

$$\mathfrak{P}(E) = L(\vec{\omega}) \oplus \bigoplus_{i=1}^{3} L(\vec{x} - (1 + l_i)\vec{x}_i)$$

$$\mathfrak{I}(E) = L(\vec{x}) \oplus \bigoplus_{i=1}^{3} L((1 + l_i)\vec{x}_i + \vec{\omega})$$

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$$\psi = \begin{bmatrix} 0 & x_3^{p_3-k} & x_2^{p_2-i} & x_1^{p_1-j} \\ x_3^{p_3-k} & 0 & -x_1^j & x_2^i \\ x_2^{p_2-i} & x_1^j & 0 & -x_3^k \\ x_1^{p_1-j} & -x_2^i & x_3^k & 0 \end{bmatrix}$$

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for $0 < j < p_1$, $0 < i < p_2$, $0 < k < p_3$.
This work for any weight triple.

rank 2 bundles in the domestic case

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rank 2 bundles in the domestic case

Assume $\mathbb{X}$ has domestic type.
Then, for E we obtain matrix factorisation $\phi^2 = fI$, with

$$\phi = \begin{bmatrix} -x_1 & x_2^{p_2-1} & -x_3^{p_3-k} & 0 \\ x_2 & x_1 & 0 & x_3^{p_3-k} \\ -x_3^k & 0 & x_1 & x_2^{p_2-1} \\ 0 & x_3^k & x_2 & -x_1 \end{bmatrix}$$

for $k = 1, \dots, p_3 - 1$

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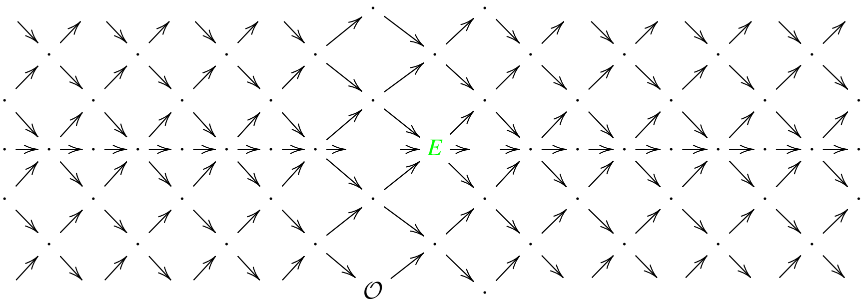

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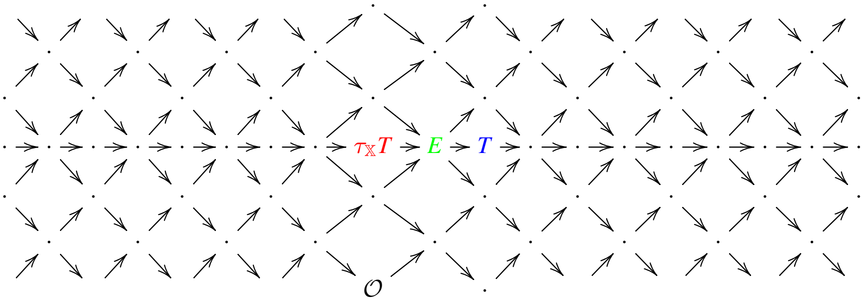

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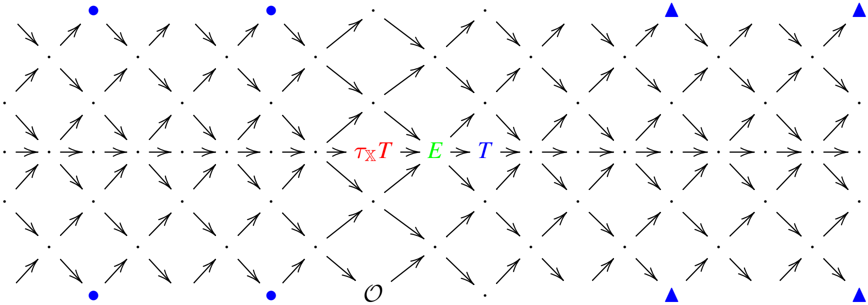

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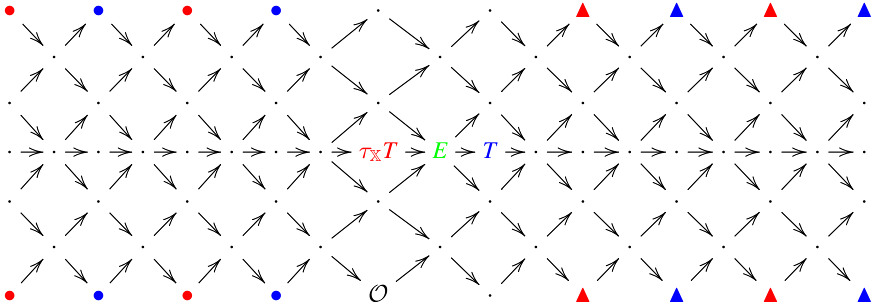

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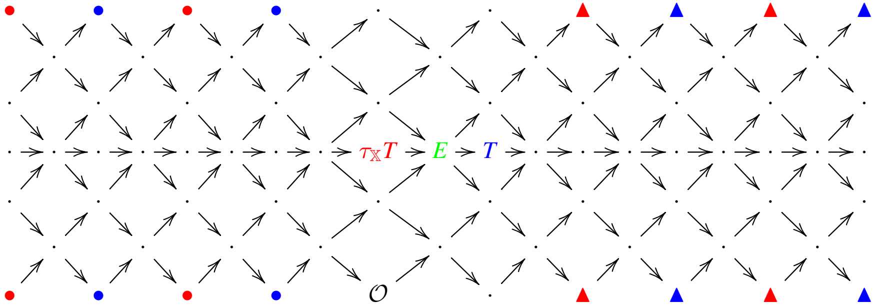

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- $\exists t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T$, $(\mathrm{Ext}^1_{\mathbb{X}}(\mathfrak{P}(T), \tau_{\mathbb{X}} T) = 0)$,

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Then

$$\mathfrak{P}(E) = \mathfrak{P}(\tau_{\mathbb{X}} T) \oplus \mathfrak{P}(T), \qquad \mathfrak{I}(E) = \mathfrak{I}(\tau_{\mathbb{X}} T) \oplus \mathfrak{I}(T)$$

Let $\pi_T : \mathfrak{P}(T) \longrightarrow T$, $q_T : T \longrightarrow \mathfrak{I}(T)$ be a projective cover and injective hull of T

- $\exists t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T$, $(\mathrm{Ext}^1_{\mathbb{X}}(\mathfrak{P}(T), \tau_{\mathbb{X}} T) = 0)$,
- $\exists s : E \longrightarrow \mathfrak{I}(\tau_{\mathbb{X}} T) \mid f \circ s = q_{\tau_{\mathbb{X}} T}$, $(\mathrm{Ext}^1_{\mathbb{X}}(T, \mathfrak{I}(\tau_{\mathbb{X}} T)) = 0)$.

The case $(2, 3, 4)$

$$t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T, s : E \longrightarrow \mathfrak{I}(\tau_{\mathbb{X}} T) \mid f \circ s = q_{\tau_{\mathbb{X}} T}$$

$$0 \longrightarrow \tau_{\mathbb{X}} T \xrightarrow{\quad f \quad} E \xrightarrow{\quad g \quad} T \longrightarrow 0$$

.

The case $(2,3,4)$

$$t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T,\ s : E \longrightarrow \mathfrak{I}(\tau_{\mathbb{X}}T) \mid f \circ s = q_{\tau_{\mathbb{X}}T}$$

$$\begin{array}{ccccccccc}
 & & \mathfrak{P}(\tau_{\mathbb{X}}T) & & & & \mathfrak{P}(T) & & \\
 & & \Big\downarrow {\scriptstyle \pi_{\tau_{\mathbb{X}}T}} & & & & \Big\downarrow {\scriptstyle \pi_T} & & \\
0 & \longrightarrow & \tau_{\mathbb{X}}T & \xrightarrow{\ f\ } & E & \xrightarrow{\ g\ } & T & \longrightarrow & 0
\end{array}$$

.

The case $(2, 3, 4)$

$$t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T, s : E \longrightarrow \mathfrak{I}(\tau_{\mathbb{X}} T) \mid f \circ s = q_{\tau_{\mathbb{X}} T}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) \oplus \mathfrak{P}(T) & \longrightarrow & \mathfrak{P}(T) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{\tau_{\mathbb{X}} T}} & & \downarrow{\scriptstyle \left[\, f \circ \pi_{\tau_{\mathbb{X}} T}, \quad t \,\right]} & & \downarrow{\scriptstyle \pi_T} & & \\ 0 & \longrightarrow & \tau_{\mathbb{X}} T & \xrightarrow{f} & E & \xrightarrow{g} & T & \longrightarrow & 0 \end{array}$$

.

The case $(2, 3, 4)$

$$t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T,\ s : E \longrightarrow \mathfrak{I}(\tau_{\mathbb{X}} T) \mid f \circ s = q_{\tau_{\mathbb{X}} T}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) \oplus \mathfrak{P}(T) & \longrightarrow & \mathfrak{P}(T) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{\tau_{\mathbb{X}} T}} & & \downarrow{\scriptstyle \left[\, f \circ \pi_{\tau_{\mathbb{X}} T}, \quad t \,\right]} & & \downarrow{\scriptstyle \pi_T} & & \\
0 & \longrightarrow & \tau_{\mathbb{X}} T & \xrightarrow{\ f\ } & E & \xrightarrow{\ g\ } & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q_{\tau_{\mathbb{X}} T}} & & & & \downarrow{\scriptstyle q_T} & & \\
 & & \mathfrak{I}(\tau_{\mathbb{X}} T) & & & & \mathfrak{I}(T) & & .
\end{array}$$

The case $(2,3,4)$

$$t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T,\ s : E \longrightarrow \mathfrak{I}(\tau_{\mathbb{X}} T) \mid f \circ s = q_{\tau_{\mathbb{X}} T}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) \oplus \mathfrak{P}(T) & \longrightarrow & \mathfrak{P}(T) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{\tau_{\mathbb{X}} T}} & & \downarrow{\scriptstyle \begin{bmatrix} f \circ \pi_{\tau_{\mathbb{X}} T}, & t \end{bmatrix}} & & \downarrow{\scriptstyle \pi_T} & & \\
0 & \longrightarrow & \tau_{\mathbb{X}} T & \xrightarrow{f} & E & \xrightarrow{g} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q_{\tau_{\mathbb{X}} T}} & & \downarrow{\scriptstyle \begin{bmatrix} s \\ q_T \circ g \end{bmatrix}} & & \downarrow{\scriptstyle q_T} & & \\
0 & \longrightarrow & \mathfrak{I}(\tau_{\mathbb{X}} T) & \longrightarrow & \mathfrak{I}(\tau_{\mathbb{X}} T) \oplus \mathfrak{I}(T) & \longrightarrow & \mathfrak{I}(T) & \longrightarrow & 0.
\end{array}$$

The case $(2,3,4)$

$$t : \mathfrak{P}(T) \longrightarrow E \mid g \circ t = \pi_T,\ s : E \longrightarrow \mathfrak{I}(\tau_{\mathbb{X}} T) \mid f \circ s = q_{\tau_{\mathbb{X}} T}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) & \longrightarrow & \mathfrak{P}(\tau_{\mathbb{X}} T) \oplus \mathfrak{P}(T) & \longrightarrow & \mathfrak{P}(T) & \longrightarrow & 0 \\
 & & \downarrow \pi_{\tau_{\mathbb{X}} T} & & \downarrow \begin{bmatrix} f \circ \pi_{\tau_{\mathbb{X}} T}, & t \end{bmatrix} & & \downarrow \pi_T & & \\
0 & \longrightarrow & \tau_{\mathbb{X}} T & \xrightarrow{f} & E & \xrightarrow{g} & T & \longrightarrow & 0 \\
 & & \downarrow q_{\tau_{\mathbb{X}} T} & & \downarrow \begin{bmatrix} s \\ q_T \circ g \end{bmatrix} & & \downarrow q_T & & \\
0 & \longrightarrow & \mathfrak{I}(\tau_{\mathbb{X}} T) & \longrightarrow & \mathfrak{I}(\tau_{\mathbb{X}} T) \oplus \mathfrak{I}(T) & \longrightarrow & \mathfrak{I}(T) & \longrightarrow & 0.
\end{array}$$

$$\begin{bmatrix} s \\ q_T \circ g \end{bmatrix} \begin{bmatrix} f \circ \pi_{\tau_{\mathbb{X}} T}, & t \end{bmatrix} = \begin{bmatrix} q_{\tau_{\mathbb{X}} T} \circ \pi_{\tau_{\mathbb{X}} T} & s \circ t \\ 0 & q_T \circ \pi_T \end{bmatrix}$$

Therefore we obtain a matrix factorisation for E, such that $\phi^2 = fI$, where

$$\phi = \left[\begin{array}{cccc|cccc}
-x_1 & 0 & -x_2^2 & x_3^2 & 0 & x_2x_3 & 0 & 0 \\
0 & -x_1 & x_3^2 & x_2 & x_3 & 0 & 0 & 0 \\
-x_2 & x_3^2 & x_1 & 0 & 0 & 0 & 0 & -x_3 \\
x_3^2 & x_2^2 & 0 & x_1 & 0 & 0 & -x_2x_3 & 0 \\
\hline
0 & 0 & 0 & 0 & x_1 & 0 & x_2^2 & x_3^2 \\
0 & 0 & 0 & 0 & 0 & x_1 & x_3^2 & -x_2 \\
0 & 0 & 0 & 0 & x_2 & x_3^2 & -x_1 & 0 \\
0 & 0 & 0 & 0 & x_3^2 & -x_2^2 & 0 & -x_1
\end{array}\right]$$

The case $(2,3,3)$:

E –indecomposable of rank 3, $0 \leq \mu(E) < -\delta(\vec{\omega})$
$\mathfrak{P}(E) = \bigoplus_{\vec{y}\in A} \mathcal{O}(\vec{y})$,
$\mathfrak{I}(E) = \tau_{\mathbb{X}}^{-3}\mathfrak{P}(E) = \bigoplus_{\vec{y}\in A} \mathcal{O}(\vec{y} - 3\vec{\omega})$
where $A = \{\vec{0}, \vec{\omega}, \vec{x}_2 + 2\vec{\omega}, \vec{x}_2 + 3\vec{\omega}, \vec{x}_3 + 2\vec{\omega}, \vec{x}_3 + 3\vec{\omega}\}$.
Obtain a matrix factorisation $\phi^2 = fI$, where

$$\phi = \begin{bmatrix} -x_1 & x_2x_3 & 0 & -x_2^2 & 0 & x_3^2 \\ 0 & x_1 & -x_3 & 0 & x_2 & 0 \\ 0 & -x_3^2 & -x_1 & x_2x_3 & 0 & x_2^2 \\ -x^2 & 0 & 0 & x_1 & x_3 & 0 \\ 0 & x_2^2 & 0 & x_3^2 & -x_1 & x_2x_3 \\ x_3 & 0 & x_2 & 0 & 0 & x_1 \end{bmatrix}$$

The case (2,3,4)

rank 3 matrix factorisation $\phi^2 = fI$

$$\phi = \begin{bmatrix} -x_1 & 0 & x_2 & 0 & -x_3 & 0 \\ 0 & -x_1 & 0 & 0 & x_2 & x_3 \\ x_2^2 & x_2x_3 & x_1 & x_3^2 & 0 & 0 \\ 0 & 0 & x_3^3 & -x_1 & 0 & -x_2 \\ -x_3^4 & x_2^2 & 0 & x_2x_3 & x_1 & 0 \\ x_2x_3^3 & x_3^4 & 0 & -x_2^2 & 0 & x_1 \end{bmatrix}$$

The case (2,3,4)

rank 4 matrix factorisation $\phi^2 = fI$

$$\phi = \left[\begin{array}{cccc|cccc} -x_1 & 0 & -x_2^2 & x_3^2 & 0 & x_2x_3 & 0 & 0 \\ 0 & -x_1 & x_3^2 & x_2 & x_3 & 0 & 0 & 0 \\ -x_2 & x_3^2 & x_1 & 0 & 0 & 0 & 0 & -x_3 \\ x_3^2 & x_2^2 & 0 & x_1 & 0 & 0 & -x_2x_3 & 0 \\ \hline 0 & 0 & 0 & 0 & x_1 & 0 & x_2^2 & x_3^2 \\ 0 & 0 & 0 & 0 & 0 & x_1 & x_3^2 & -x_2 \\ 0 & 0 & 0 & 0 & x_2 & x_3^2 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & x_3^2 & -x_2^2 & 0 & -x_1 \end{array}\right]$$

The case (2,3,5)

rank 3 matrix factorisations $\phi_k^2 = fI$ for $k = 1, 2$

$$\phi_1 = \begin{bmatrix} -x_1 & 0 & x_2 & 0 & -x_3 & 0 \\ 0 & -x_1 & 0 & 0 & x_2 & x_3 \\ x_2^2 & x_2x_3 & x_1 & x_3^2 & 0 & 0 \\ 0 & 0 & x_3^3 & -x_1 & 0 & -x_2 \\ -x_3^4 & x_2^2 & 0 & x_2x_3 & x_1 & 0 \\ x_2x_3^3 & x_3^4 & 0 & -x_2^2 & 0 & x_1 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -x_1 & 0 & 0 & -x_2 & 0 & x_3 \\ 0 & -x_1 & 0 & x_3^2 & x_2 & 0 \\ 0 & 0 & -x_1 & 0 & -x_3^2 & x_2 \\ -x_2^2 & x_3^3 & x_2x_3 & x_1 & 0 & 0 \\ x_2x_3^2 & x_2^2 & -x_3^3 & 0 & x_1 & 0 \\ x_3^4 & x_2x_3^2 & x_2^2 & 0 & 0 & x_1 \end{bmatrix}$$

The case (2,3,5)

rank 4 matrix factorisations $\phi_k^2 = fI$ for $k = 1, 2$,

$$\phi_1 = \begin{bmatrix} -x_1 & x_2^2 & -x_3^3 & 0 & 0 & 0 & -x_2x_3^2 & 0 \\ x_2 & x_1 & 0 & x_3^3 & 0 & 0 & 0 & x_2x_3^2 \\ -x_3^2 & 0 & x_1 & x_2^2 & x_2x_3 & 0 & 0 & 0 \\ 0 & x_3^2 & x_2 & -x_1 & 0 & -x_2x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & x_2^2 & -x_3^3 & 0 \\ 0 & 0 & 0 & 0 & x_2 & x_1 & 0 & x_3^3 \\ 0 & 0 & 0 & 0 & -x_3^2 & 0 & x_1 & x_2^2 \\ 0 & 0 & 0 & 0 & 0 & x_3^2 & x_2 & -x_1 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -x_1 & x_2^2 & x_3^3 & 0 & -x_2x_3 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & -x_3^3 & 0 & x_2x_3 & 0 & 0 \\ x_3^2 & 0 & x_1 & x_2^2 & 0 & 0 & x_2x_3 & 0 \\ 0 & -x_3^2 & x_2 & -x_1 & 0 & 0 & 0 & -x_2x_3 \\ 0 & 0 & 0 & 0 & x_1 & x_2^2 & x_3^3 & 0 \\ 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 & -x_3^3 \\ 0 & 0 & 0 & 0 & x_3^2 & 0 & -x_1 & x_2^2 \\ 0 & 0 & 0 & 0 & 0 & -x_3^2 & x_2 & x_1 \end{bmatrix}$$

The case (2,3,5)

rank 5 matrix factorisation $\phi^2 = fI$,

$$\phi = \begin{bmatrix} -x_1 & x_2^2 & -x_3^3 & 0 & 0 & 0 & -x_2x_3^2 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & x_3^3 & 0 & 0 & 0 & 0 & 0 & -x_3^2 \\ -x_3^2 & 0 & x_1 & x_2^2 & 0 & 0 & 0 & 0 & -x_2x_3 & 0 \\ 0 & x_3^2 & x_2 & -x_1 & x_3^3 & x_2x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & -x_3^2 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & x_3^2 & -x_2 \\ 0 & 0 & 0 & 0 & x_2^2 & -x_3^3 & x_2x_3 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2x_3^2 & x_2^2 & x_3^3 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & x_3^4 & x_2x_3^2 & -x_2^2 & 0 & 0 & -x_1 \end{bmatrix}$$

The case (2,3,5)

rank 6 matrix factorisation $\phi^2 = fI$,

$\phi =$

$$\begin{bmatrix}
-x_1 & 0 & 0 & -x_2 & 0 & x_3 & 0 & x_3^2 & -x_2 & 0 & 0 & 0 \\
0 & -x_1 & 0 & x_3^2 & x_2 & 0 & -x_2x_3 & 0 & x_3^2 & 0 & 0 & 0 \\
0 & 0 & -x_1 & 0 & -x_3^2 & x_2 & x_3^3 & x_2x_3 & 0 & 0 & 0 & 0 \\
-x_2^2 & x_3^3 & x_2x_3 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3^2 & x_2 \\
x_2x_3^2 & x_2^2 & -x_3^3 & 0 & x_1 & 0 & 0 & 0 & 0 & x_2x_3 & 0 & x_3^2 \\
x_3^4 & x_2x_3^2 & x_2^2 & 0 & 0 & x_1 & 0 & 0 & 0 & x_3^3 & -x_2x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & x_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & -x_3^2 & x_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & x_3^2 & -x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2^2 & -x_3^3 & x_2x_3 & -x_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2x_3^2 & x_2^2 & x_3^3 & 0 & -x_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_3^4 & x_2x_3^2 & -x_2^2 & 0 & 0 & -x_1
\end{bmatrix}$$

Thank you for your attention!