Semi-tilting modules and mutation

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Introduction

R : a commutative complete local noetherian ring.

A : a module-finite R-algebra.

mod-A : the category of finitely generated A-modules.

$\mathrm{add}(T)$: the full subcategory of mod-A consisting of all direct summands of finite direct sums of copies of $T \in$ mod-A.

$\mathrm{gen}(T)$: the full subcategory of mod-A consisting of modules which are epimorphic images of modules in $\mathrm{add}(T)$.

Definition 1 (Wakamatsu, 1990)**.** $T \in \text{mod-}A$: a Wakamatsu tilting module $\overset{\text{def}}{\Leftrightarrow}$

(1) $\mathrm{Ext}^i_A(T,T) = 0$ for $i \neq 0$.

(2) $\exists A \to T^\bullet$: a right resolution in mod-A with $T^i \in \mathrm{add}(T)$ and $\mathrm{Ext}^j_A(\mathrm{Z}^i(T^\bullet), T) = 0$ for all i and $j \neq 0$.

Definition 2 (Miyashita, 1986). $T \in \text{mod-}A$: a tilting module $\overset{\text{def}}{\Leftrightarrow}$

T : a Wakamatsu tilting module and proj dim $T_A =$ proj dim ${}_{\text{End}_A(T)}T < \infty$.

Example 3.(1) A : a tilting module.

(2) $R = k$: a field,

inj dim ${}_AA =$ inj dim $A_A = n < \infty$

$\Rightarrow DA := \text{Hom}_k(A, k) \in \text{mod-}A$: a tilting module with proj dim $DA_A = n$.

Wakamatsu tilting conjecture

$\forall T \in \text{mod-}A$: a Wakamatsu tilting module,

$\text{proj dim } T_A = \text{proj dim }_{\text{End}_A(T)}T$.

Remark 4. $T \in \text{mod-}A$: a Wakamatsu tilting module. If $\text{proj dim } T_A, \text{proj dim }_{\text{End}_A(T)}T < \infty$, then T is a tilting module.

Semi-tilting modules

Definition 5. $T \in \text{mod-}A$: a semi-tilting module

$\overset{\text{def}}{\Leftrightarrow}$

(1) $\text{Ext}_A^i(T,T) = 0$ for $i \neq 0$.

(2) $\exists 0 \to A \to T^0 \to T^1 \to \cdots \to T^m \to 0$: exact in mod-A with $T^i \in \text{add}(T)$ for all i.

Note that ${}_{\text{End}_A(T)}T$ is a Wakamatsu tilting module of finite projective dimension.

$T \in \text{mod-}A$: a semi-tilting module, $B = \operatorname{End}_A(T)$.

$0 \to A \to T^0 \to T^1 \to \cdots \to T^m \to 0$: a minimal right resolution in mod-A with $T^i \in \operatorname{add}(T)$ for $0 \le i \le m$.

Remark 6.(1) $\operatorname{Ext}^i_A(T, A) = 0$ for $i > m$ and $\operatorname{Ext}^m_A(T, A) \neq 0$.

(2) If $P^\bullet \to T$ is a projective resolution in mod-A, then $\oplus_{i=0}^m P^{-i} \in \mathcal{P}_A$ is a projective generator.

Remark 7. $\mathrm{add}(T^0) \cap \mathrm{add}(T^m) = \{0\}$ unless $m = 0$.

Remark 8. If $T \cong \bigoplus^n X$ with $X \in \text{mod-}A$ indecomposable, then T is projective.

Proposition 9. *Every non-projective semi-tilting module $T \in \text{mod-}A$ admits a decomposition $T = U \oplus X$ with $X \in \mathrm{gen}(U)$ indecomposable.*

Sketch of the proof

In case $m = 0$, $X \in \mathrm{add}(T)$: non-projective indecomposable.

In case $m > 0$, $X \in \mathrm{add}(T^m)$: indecomposable.

$$\begin{array}{ccccccc} \cdots \longrightarrow & T^{m-1} & \longrightarrow & T^m & \longrightarrow & 0 \\ & \Vert & & \downarrow \pi & & \\ & E \oplus (\oplus^n X) & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

where $E \in \mathrm{add}(U)$.

$\Rightarrow X \in \mathrm{gen}(U)$.

Mutation

$T = U \oplus X \in \text{mod-}A$: a basic semi-tilting module with X indecomposable.

Lemma 10. $X \in \text{add}(T^0) \Rightarrow X \notin \text{gen}(U)$.

Theorem 11. *Assume that $X \in \text{gen}(U)$. Then there exists an exact sequence in* mod-A

$$0 \to Y \to E \to X \to 0$$

with $E \in \text{add}(U)$, Y indecomposable and $U \oplus Y$ a semi-tilting module.

Sketch of the proof

Taking a minimal right $\mathrm{add}(U)$-approximation $\varepsilon : E \to X$, we have an exact sequence in mod-A

$$0 \to Y \to E \to X \to 0$$

with Y indecomposable and $T' = U \oplus Y$ selforthogonal.

It is enough to show that $\exists 0 \to A \to T'^0 \to T'^1 \to \cdots \to T'^k \to 0$ in mod-A with $T'^i \in \mathrm{add}(T')$ for all i.

Set $i = \inf\{\ j \mid X \in \mathrm{add}(T^j)\} > 0$. We may assume $T^{i-1} \in \mathrm{add}(U)$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & T^{i-2} & \longrightarrow & T^{i-1} & \longrightarrow & E' \oplus (\bigoplus^n X) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \pi & & \downarrow \\
0 & \longrightarrow & \bigoplus^n Y & \longrightarrow & \bigoplus^n E & \longrightarrow & \bigoplus^n X & \longrightarrow & 0,
\end{array}$$

where $E' \in \mathrm{add}(U)$ and π is the projection.

Taking the mapping cone, we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & T'^{i-1} & \longrightarrow & E'' & \longrightarrow & T^{i+1} & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & T'^{i-1} & \longrightarrow & E'' \oplus (\bigoplus^n X) & \longrightarrow & T^{i+1} \oplus (\bigoplus^n X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \| \\
 & & 0 & \longrightarrow & \bigoplus^n X & = & \bigoplus^n X & \longrightarrow & 0
\end{array}
$$

where $T'^{i-1} \cong T^{i-1} \oplus (\bigoplus^n Y)$ and $E'' \cong E' \oplus (\bigoplus^n E)$. By induction we obtain the desired sequence.

Definition 12. If $X \in \mathrm{gen}(U)$, then $\mu_X(T) := U \oplus Y$ in Theorem 11.

Remark 13. Let $T \cong U' \oplus X'$ with X' indecomposable. If $X \in \mathrm{gen}(U)$ and $X' \in \mathrm{gen}(U')$ then $X \cong X'$ if and only if $\mu_X(T) \cong \mu_{X'}(T)$.

Lemma 14. *Assume that* $X \in \mathrm{gen}(U)$*. Then* $\mu_X(T)$ *is a tilting module if and only if so is* T*.*

We will define a quiver K as follows:

Vertices : Isomorphism classes of basic semi-tilting modules.

Arrows : $V \to W \overset{\text{def}}{\Leftrightarrow} W = [T']$ and $V = [\mu_{X'}(T')]$.

Proposition 15 (Riedtmann-Schofield, 1991)**.** *The quiver K contains no oriented cycles.*

Theorem 16. *If the connected component of K including T contains a tilting module then T itself is a tilting module.*

R : a Cohen-Macaulay ring, A : a maximal Cohen-Macaulay R-module,

T : a maximal Cohen-Macaulay R-module.

$\mathcal{L}({}^{\perp}T) := \{\ M \in \text{mod-}A \mid \text{Ext}^i_A(M,T) = 0\ \forall i \neq 0$ and M : a maximal Cohen-Macaulay$R$-module$\}$.

Corollary 17. *Assume that $\mathcal{L}({}^{\perp}T)$ contains only a finite number of non-isomorphic indecomposable modules. Then T is a tilting module.*