Operations on arc diagrams and degenerations for invariant subspaces of linear operators

Justyna Kosakowska, Toruń

A report on a joint project by Justyna Kosakowska, Toruń, and Markus Schmidmeier, FAU

ICRA 2012, Bielefeld

August 13, 2012
Main aim

\( \alpha, \beta, \gamma \) — partitions, \( \alpha = (\alpha_1, \alpha_2, \ldots) \) such that \( \alpha_1 \leq 2 \)
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\[ N(\alpha) = \bigoplus_{i=1}^{s} K[T]/(T^{\alpha_i}) \text{ — nilpotent linear operator} \]
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Combinatorial description (example):

For \( \alpha = (2, 2, 1, 1), \beta = (5, 4, 3, 3, 2, 1), \gamma = (4, 3, 2, 2, 1) \):
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\[
0 \longrightarrow N(\alpha) \xrightarrow{f} N(\beta) \longrightarrow N(\gamma) \longrightarrow 0
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Main aim

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\]

\[\Gamma:\]

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 2 \\
0 & 2 & 2 & 2 \\
\end{array}
\]
\(\alpha, \beta, \gamma\) — partitions, \(\alpha = (\alpha_1, \alpha_2, \ldots)\) such that \(\alpha_1 \leq 2\)

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\(\Gamma:\) 
\[
\begin{array}{ccccc}
\, & \, & \, & \, & 1 \\
\, & \, & \, & 1 \\
\, & 1 \\
2 & 11 \\
2 & 2
\end{array}
\]

\(\Pi:\) 
\[
\begin{array}{ccccc}
\, & \, & \, & \, & 1 \\
\, & \, & \, & 1 \\
\, & 11 \\
2 & 11 \\
2 & 2_2 \\
2 & 2_3
\end{array}
\]

\(\Delta:\) 
\[
\begin{array}{c}
5 \quad 4 \quad 3 \quad 2 \quad 1
\end{array}
\]
\( K \) — an algebraically closed field
Notation

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\[ \alpha, \beta, \gamma \] — partitions
\( K \) — an algebraically closed field

\( \alpha, \beta, \gamma \) — partitions

For a partition \( \alpha = (\alpha_1, \alpha_2, \ldots) \) denote:

\[
N(\alpha) = \bigoplus_{i=1}^{s} K[T] / (T^{\alpha_i})
\]

— nilpotent linear operator

With partitions \( \alpha, \beta \), we associate:

\[
H_{\beta \alpha} = \text{Hom}_K(N(\alpha), N(\beta))
\]

— affine variety (Zariski topology)

\( V_{\beta \alpha, \gamma} \subseteq H_{\beta \alpha} \) — subset consisted of all monomorphisms \( f \) such that there exists a s. e. s.

\[
0 \rightarrow N(\alpha) \xrightarrow{f} N(\beta) \rightarrow N(\gamma) \rightarrow 0
\]

Group action:

\[
G = \text{Aut}_K[\mathcal{T}] \times \text{Aut}_K[\mathcal{T}]
\]

acts on \( V_{\beta \alpha, \gamma} \).
**Notation**

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\[
V^\beta_{\alpha, \gamma} \subset H^\beta_\alpha
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\( V^{\beta}_{\alpha, \gamma} \subset H^{\beta}_{\alpha} \) — subset consisted of all monomorphisms \( f \) such that there exists a s. e. s.

\[
0 \longrightarrow N(\alpha) \xrightarrow{f} N(\beta) \longrightarrow N(\gamma) \longrightarrow 0
\]

Group action: \( G = \text{Aut}_{K[T]} N(\alpha) \times \text{Aut}_{K[T]} N(\beta) \) acts on \( V^{\beta}_{\alpha, \gamma} \).
\( S^\beta_{\alpha,\gamma} \) — the category consisted of all systems

\[ X = (N(\alpha), N(\beta), f) \]

where \( f : N(\alpha) \to N(\beta) \) is a monomorphism and \( \text{Coker } f \cong M(\gamma) \); morph. in \( S^\beta_{\alpha,\gamma} \) are defined in a natural way;
\( S_{\alpha,\gamma}^\beta \) — the category consisted of all systems

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The \( G \)-orbits in \( V_{\alpha,\gamma}^\beta \) are in 1–1-correspondence with the equivalence classes of objects in \( S_{\alpha,\gamma}^\beta \).
Categorification

$S^\beta_{\alpha,\gamma}$ — the category consisted of all systems

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The $G$-orbits in $V^\beta_{\alpha,\gamma}$ are in 1–1-correspondence with the equivalence classes of objects in $S^\beta_{\alpha,\gamma}$.

$S_2$ — the category consisted of all systems

\[ (N(\alpha), N(\beta), f) \]

where $\alpha$ and $\beta$ are partitions, $\alpha_1 \leq 2$, and $f : N(\alpha) \to N(\beta)$ is a monomorphism and $\text{Coker } f \cong M(\gamma)$; morphisms in $S_2$ are def. in a natural way;
Problem

For $f \in V^\beta_{\alpha,\gamma}$ denote by $O_f$ the orbit of $f$ under the action of $G$. 

1. describe $\leq_{\text{deg}}$ combinatorially in $S_2 \cap S^\beta_{\alpha,\gamma}$
2. determine $\dim O_f$. 

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For \( f \in V^\beta_{\alpha, \gamma} \) denote by \( O_f \) the orbit of \( f \) under the action of \( G \).

We define the following partial order (\textit{degeneration order}).

For \( f, g \in V^\beta_{\alpha, \gamma} \) (resp. \( X, Y \in S^\beta_{\alpha, \gamma} \)):
For $f \in V_{\alpha, \gamma}^\beta$ denote by $O_f$ the orbit of $f$ under the action of $G$.

We define the following partial order (degeneration order). For $f, g \in V_{\alpha, \gamma}^\beta$ (resp. $X, Y \in S_{\alpha, \gamma}^\beta$):

$$X \leq_{\text{deg}} Y :\iff f \leq_{\text{deg}} g :\iff O_g \subseteq \overline{O_f}$$
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**Theorem** (Beers, Hunter, Walker, 1983): The category $S_2$ has the Krull-Remak-Schmidt property.

Each indecomposable object in $S_2$ is isomorphic to one of the following.

- $P_m^0: 0 \subseteq \mathbb{N}(m)$ ($m \geq 1$)
- $P_m^1: (T_m-1) \subseteq \mathbb{N}(m)$ ($m \geq 1$)
- $P_m^2: (T_m-2) \subseteq \mathbb{N}(m)$ ($m \geq 2$)
- $B_m^2: ((T_m-2), (T_r-1)) \subseteq \mathbb{N}(m, r)$ ($m-2 \geq r \geq 1$)

**Proposition:** For partitions $\alpha, \beta, \gamma$ with $\alpha_1 \leq 2$, there is a one-to-one correspondence $\text{Obj}(S_{\beta \alpha, \gamma})/\sim \cong \{\text{Klein tableaux of type } (\alpha, \beta, \gamma)\}$. 

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Operations on arc diagrams and degenerations
The category $\mathcal{S}_2$

**Theorem** (Beers, Hunter, Walker, 1983): *The category $\mathcal{S}_2$ has the Krull-Remak-Schmidt property. Each indecomposable object in $\mathcal{S}_2$ is isomorphic to one of the following.*

$$
\begin{align*}
P_0^m : & \quad 0 \subseteq N(m) \quad (m \geq 1) \\
P_1^m : & \quad (T^{m-1}) \subseteq N(m) \quad (m \geq 1) \\
P_2^m : & \quad (T^{m-2}) \subseteq N(m) \quad (m \geq 2) \\
B_{2}^{m,r} : & \quad (T^{m-2}, T^{r-1}) \subseteq N(m, r) \quad (m - 2 \geq r \geq 1)
\end{align*}
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The category $S_2$

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$$\text{Obj}(S^\beta_{\alpha, \gamma}) / \simeq \longleftrightarrow \{ \text{Klein tableaux of type } (\alpha, \beta, \gamma) \}.$$
## Combinatorial invariants

### Invariants for the indecomposable objects in $S_2$

<table>
<thead>
<tr>
<th>$X$ :</th>
<th>$P_0^m$</th>
<th>$P_1^m$</th>
<th>$P_2^m$</th>
<th>$B_2^{m,r}$</th>
</tr>
</thead>
</table>
| $\Gamma(X)$ : | \[
\begin{array}{c}
\vdots \\
\vdots \\
1 \\
2
\end{array}
\] | \[
\begin{array}{c}
\vdots \\
\vdots \\
1
\end{array}
\] | \[
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Combinatorial invariants

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<th>$B_{2,m,r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma(X)$:</td>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
</tr>
<tr>
<td>$\Pi(X)$:</td>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
<td>$r = m-1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$r &lt; m-1$</td>
</tr>
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Operations on arc diagrams and degenerations
### Combinatorial invariants

#### Invariants for the indecomposable objects in $S_2$

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<tr>
<td>$\Gamma(X)$ :</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
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<tr>
<td>$\Pi(X)$ :</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
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<tr>
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<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
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Combinatorial invariants

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### Combinatorial invariants

**Invariants for the indecomposable objects in $S_2$**

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</table>

- **$\Gamma(X)$**:
  - $P_0^m$: $m$ boxes
  - $P_1^m$: $m$ boxes, $1$ and $2$
  - $P_2^m$: $m$ boxes, $1$ and $2$

- **$\Pi(X)$**:
  - $P_0^m$: $m$ boxes
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- **$\Delta(X)$**:
  - $\emptyset$
  - $m$ boxes
  - $m-1$ boxes

---

![Diagram](image)
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\vdots \\
1 \\
2 \\
\vdots \\
\vdots
\end{array}
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\vdots \\
\vdots \\
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2 \\
\vdots \\
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\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
1 \\
2 \\
\vdots \\
\vdots
\end{array}
\] | $m$ \[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
1 \\
2 \\
\vdots
\end{array}
\] $r$ |
| $\Pi(X)$ | \[
\begin{array}{c}
\vdots \\
\vdots \\
1 \\
2r \\
\vdots \\
\vdots
\end{array}
\] | \[
\begin{array}{c}
\vdots \\
\vdots \\
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2r \\
\vdots \\
\vdots
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\vdots \\
\vdots \\
\vdots \\
1 \\
2r \\
\vdots
\end{array}
\] $r=m-1$ |
| $\Delta(X)$ | $\emptyset$ | $m$ | $m \quad m-1$ | $m \quad r$ |
The arc diagram of an object

The Klein tableau for a direct sum $M \oplus M'$ has a diagram given by the union $\beta \cup \beta'$ of the partitions representing the ambient spaces, and in each row the entries are obtained by lexicographically ordering the entries in the corresponding rows in the tableaux for $M$ and $M'$, with empty boxes coming first.
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$$S_2 \ni X \mapsto \Pi(X) \mapsto \Delta(X)$$
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$$S_2 \ni X \mapsto \Pi(X) \mapsto \Delta(X)$$

Example:

$$X = B_2^{5,3} \oplus B_2^{4,2} \oplus P_1^3 \oplus P_1^1.$$
Two arc-diagrams are said to be in arc order if the first is obtained from the second by a sequence of moves of types:

\[ S_{\alpha,\gamma}^\beta \ni X \quad \longrightarrow \quad \Delta(X) \quad \text{— arc diagram of } X \]
Arc order

\[ S^\beta_{\alpha, \gamma} \ni X \quad \longmapsto \quad \Delta(X) \] — arc diagram of \( X \)

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Two arc-diagrams are said to be in **arc order** if the first is obtained from the second by a sequence of moves of types:

- **(A)**: \( < \text{arc} \)
- **(B)**: \( > \text{arc} \)
- **(C)**: \( \quad > \text{arc} \quad \)
- **(D)**: \( \quad < \text{arc} \quad \)

**Definition:** \( X \leq_{\text{arc}} Y \) if and only if \( \Delta(X) \leq_{\text{arc}} \Delta(Y) \)
Theorem [K-Schmidmeier 2011]. Suppose that $K$ is an algebraically closed field and that $\alpha, \beta, \gamma$ are partitions with $\alpha_1 \leq 2$. 

For $Y, Z \in S_{\beta, \alpha, \gamma}$ we have $Y \leq \deg Z$ if and only if $Y \leq \arc Z$.

Suppose the arc diagram $\Delta$ of an invariant subspace $Y = (N_{\alpha}, N_{\beta}, f)$ in $S_{\beta, \alpha, \gamma}$ has $\chi(\Delta)$ crossings. Then $\dim O_f = m(\beta) - m(\alpha) - m(\gamma) - \chi(\Delta) + |\alpha| + 2m(\alpha)$.

Definition: $m(\alpha) = \sum_{i=1}^{s} \alpha_i (i - 1)$ is the moment of the partition $\alpha = (\alpha_1, ..., \alpha_s)$ and $|\alpha| = \alpha_1 + \alpha_2 + ...$. 

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Operations on arc diagrams and degenerations
Main result

**Theorem [K-Schmidmeier 2011].** Suppose that $K$ is an algebraically closed field and that $\alpha, \beta, \gamma$ are partitions with $\alpha_1 \leq 2$.

1. For $Y, Z \in S_{\alpha, \gamma}^\beta$ we have

\[ Y \leq_{\text{deg}} Z \quad \text{if and only if} \quad Y \leq_{\text{arc}} Z. \]
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2. Suppose the arc diagram $\Delta$ of an invariant subspace $Y = (N_\alpha, N_\beta, f)$ in $S_{\alpha, \gamma}^\beta$ has $x(\Delta)$ crossings. Then

$$\dim O_f = m(\beta) - m(\alpha) - m(\gamma) - x(\Delta) + |\alpha| + 2m(\alpha).$$
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Properties of the partial order $\leq_{\text{deg}}$

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The Littlewood-Richardson coefficient $c_{\alpha,\gamma}^\beta$ counts the number of LR-tableaux of type $(\alpha, \beta, \gamma)$. 
Example: The deg-order in $V_{211,321}^{4321}$
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$\Delta_6:$

$\Delta_4:$

$\Delta_5:$

$\Delta_1:$

$\Delta_2:$

$\Delta_3:$

\[$\begin{align*}
\dim &= 11 \\
\dim &= 12 \\
\dim &= 13
\end{align*}\$
The proof of Theorem 1.1

- uses properties of AR-quiver of the category $S_2$
- if $\Delta \leq_{\text{arc}} \Delta'$, gives algorithm that finds sequence of moves

$$\Delta \mapsto \Delta_1 \mapsto \ldots \mapsto \Delta'$$