Operations on arc diagrams and degenerations for invariant subspaces of linear operators

Justyna Kosakowska, Toruń

A report on a joint project by
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Main aim

α, β, γ — partitions, $\alpha = (\alpha_1, \alpha_2, \ldots)$ such that $\alpha_1 \leq 2$

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Γ :

					1
				1	
		1	1		
	2				
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$$\Gamma : \quad \begin{ytableau} \ & \ & \ & \ & \ & 1 \\ \ & \ & \ & \ & 1 \\ \ & \ & 1 & 1 \\ \ & 2 \\ 2 \end{ytableau} \qquad \Pi : \quad \begin{ytableau} \ & \ & \ & \ & \ & 1 \\ \ & \ & \ & \ & 1 \\ \ & \ & 1 & 1 \\ \ & 2_2 \\ 2_3 \end{ytableau}$$

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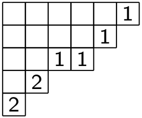
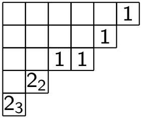


Π :



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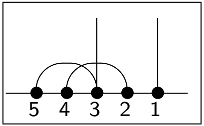

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Group action: $G = \mathrm{Aut}_{K[T]}N(\alpha) \times \mathrm{Aut}_{K[T]}N(\beta)$ acts on $V_{\alpha,\gamma}^{\beta}$.

Categorification

$\mathcal{S}^{\beta}_{\alpha,\gamma}$ — the category consisted of all systems

$$X = (N(\alpha), N(\beta), f)$$

where $f : N(\alpha) \to N(\beta)$ is a monomorphism and $\mathrm{Coker}\, f \cong M(\gamma)$; morph. in $\mathcal{S}^{\beta}_{\alpha,\gamma}$ are defined in a natural way;

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$\mathcal{S}_2$ — the category consisted of all systems

$$(N(\alpha), N(\beta), f)$$

where α and β are partitions, $\alpha_1 \leq 2$, and $f : N(\alpha) \to N(\beta)$ is a monomorphism and $\mathrm{Coker}\, f \cong M(\gamma)$; morphisms in $\mathcal{S}_2$ are def. in a natural way;

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1. **describe** $\leq_{\deg}$ **combinatorically in** $\mathcal{S}_2 \cap \mathcal{S}_{\alpha,\gamma}^{\beta}$
2. **determine** $\dim \mathcal{O}_f$.

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Theorem (Beers, Hunter, Walker, 1983): *The category* $\mathcal{S}_2$ *has the Krull-Remak-Schmidt property.*

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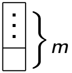
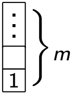
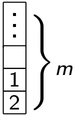
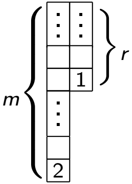
Proposition: *For partitions α, β, γ with $\alpha_1 \leq 2$, there is a one-to-one correspondence*

$$Obj(\mathcal{S}_{\alpha,\gamma}^{\beta})/_{\cong} \stackrel{1-1}{\longleftrightarrow} \{\textit{Klein tableaux of type } (\alpha,\beta,\gamma)\}.$$

Combinatorial invariants

Invariants for the indecomposable objects in $\mathcal{S}_2$				
X :	P_0^m	P_1^m	P_2^m	$B_2^{m,r}$
$\Gamma(X)$:				

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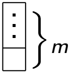
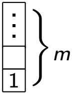
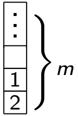
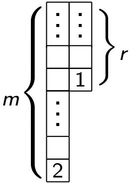
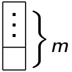
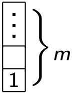
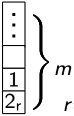
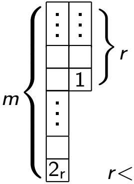
$\Gamma(X)$:				
$\Pi(X)$:			 $r=m-1$	 $r<m-1$

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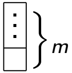
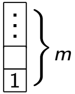
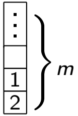
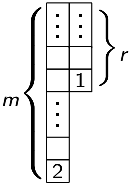
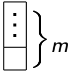
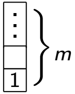
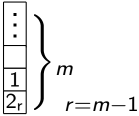
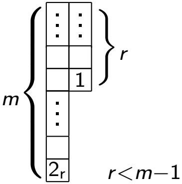
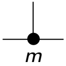
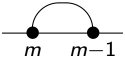
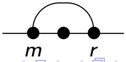
$\Gamma(X)$:				
$\Pi(X)$:				
$\Delta(X)$:	$\emptyset$			

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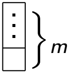
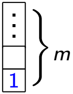
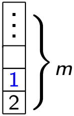
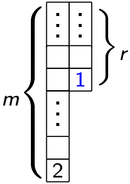
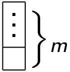
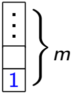
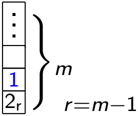
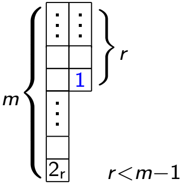
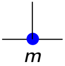
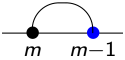
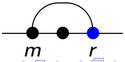
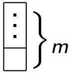
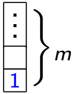
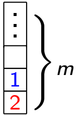
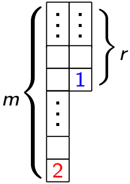
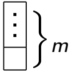
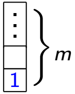
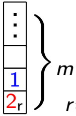
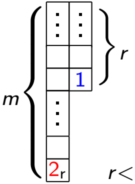
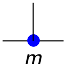
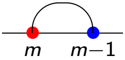
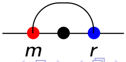
$\Gamma(X)$:				
$\Pi(X)$:				
$\Delta(X)$:	$\emptyset$			

Combinatorial invariants

Invariants for the indecomposable objects in $\mathcal{S}_2$				
X :	P_0^m	P_1^m	P_2^m	$B_2^{m,r}$
$\Gamma(X)$:				
$\Pi(X)$:			 $r=m-1$	 $r<m-1$
$\Delta(X)$:	$\emptyset$			

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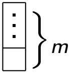
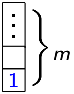
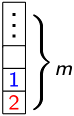
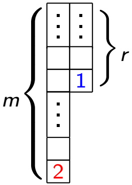
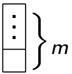
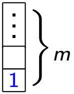
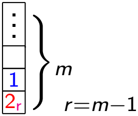
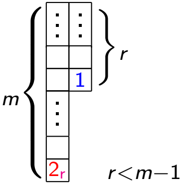
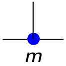
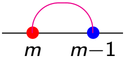
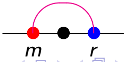
$\Gamma(X)$:				
$\Pi(X)$:				
$\Delta(X)$:	$\emptyset$			

The arc diagram of an object

The Klein tableau for a direct sum $M \oplus M'$ has a diagram given by the union $\beta \cup \beta'$ of the partitions representing the ambient spaces, and in each row the entries are obtained by lexicographically ordering the entries in the corresponding rows in the tableaux for M and M', with empty boxes coming first.

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$$\mathcal{S}_2 \ni X \mapsto \Pi(X) \mapsto \Delta(X)$$

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Example:

$$X \;=\; B_2^{5,3} \oplus B_2^{4,2} \oplus P_1^3 \oplus P_1^1.$$

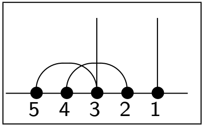
Γ :

					1
				1	
		1	1		
	2				
2					

Π :

					1
				1	
		1	1		
	2_2				
2_3					

Δ :



Arc order

$\mathcal{S}_{\alpha,\gamma}^{\beta} \ni X \qquad \longmapsto \qquad \Delta(X)$ — arc diagram of X

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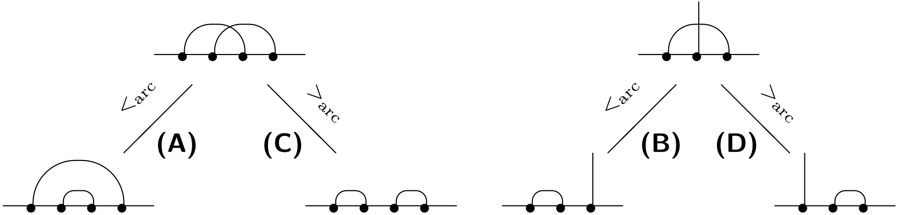

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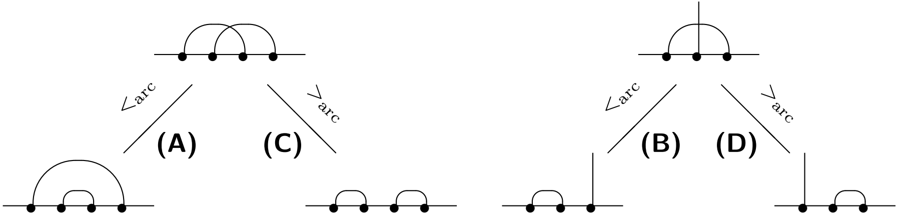

Definition: $X \leq_{\rm arc} Y$ **if and only if** $\Delta(X) \leq_{\rm arc} \Delta(Y)$

Main result

Theorem [K-Schmidmeier 2011]. *Suppose that* K *is an algebraically closed field and that* α, β, γ *are partitions with* $\alpha_1 \leq 2$.

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Definition: $m(\alpha) = \sum_{i=1}^{s} \alpha_i(i-1)$ is the **moment** of the partition $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots$.

Properties of the partial order $\leq_{\text{deg}}$

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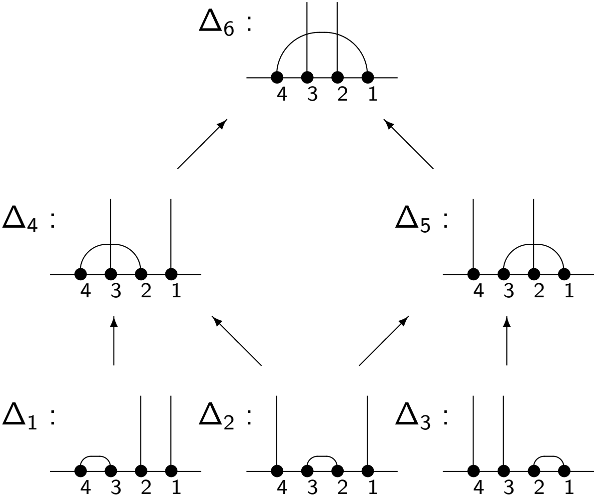
The **Littlewood-Richardson coefficient** $c^{\beta}_{\alpha,\gamma}$ counts the number of LR-tableaux of type (α, β, γ).

Example: The deg-order in $V^{4321}_{211,321}$



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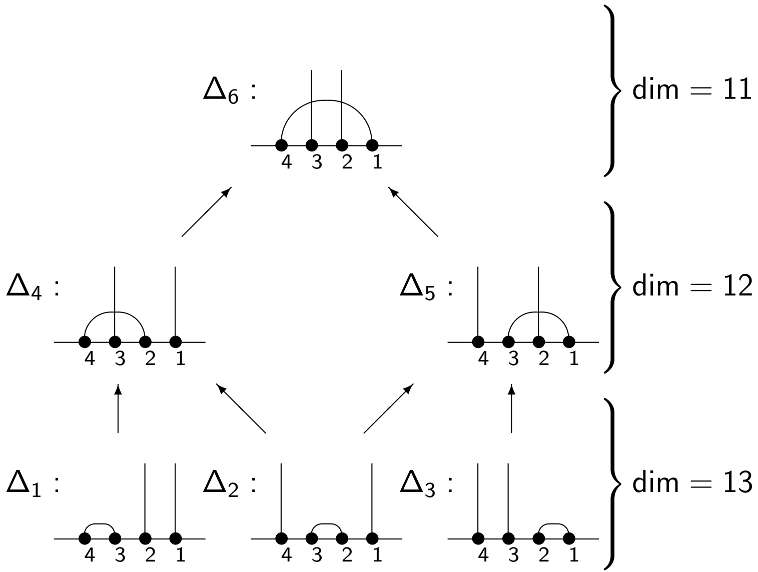

About proof

The proof of Theorem 1.1

- uses properties of AR-quiver of the category $\mathcal{S}_2$
- if $\Delta \leq_{\text{arc}} \Delta'$, gives algorithm that find sequense of moves

$$\Delta \mapsto \Delta_1 \mapsto \ldots \mapsto \Delta'$$

Justyna Kosakowska and Markus Schmidmeier, *Operations on arc diagrams and degenerations for invariant subspaces of linear operators*, arXiv:1202.2813v1 [math.RT].