Auslander-Reiten-quivers of functorially finite subcategories

Matthias Krebs

University of East Anglia

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Preliminaries

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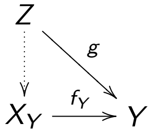
- A a finite dimensional associative K-algebra with multiplicative identity,
- A-mod the category of all finitely generated left A-modules,
- Ω a full subcategory of A-mod closed under direct sums, direct summands and isomorphisms.

Approximations

Definition

A right Ω-approximation of a module Y is a morphism $f_Y : X_Y \to Y$ where X_Y is in Ω such that for all Z in Ω and $g : Z \to Y$, g factors through f_Y.



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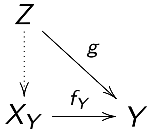

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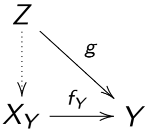

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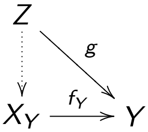

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Definition

$\tau_\Omega(Y)$ is the relative Auslander-Reiten translate of Y in Ω.

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If Ω is contravariantly finite and closed under extensions, then Ω has an Auslander-Reiten-quiver.

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Let Ω be a functorially finite resolving subcategory.

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Brauer-Thrall 1.5

Let Ω be a functorially finite resolving subcategory such that there exist a positive integer n such that there are $N \geq N_0$ non-isomorphic indecomposable modules of Jordan-Hölder length n in Ω, where N_0 is the cardinality of an infinite, countable set. Then there are infinitely many positive integers $n_1, n_2, \ldots$ with N non-isomorphic indecomposable modules of length n_i in Ω for all $i \in \mathbb{N}$.

Sectional paths

Definition

A path

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$$

in the Auslander-Reiten-quiver is called sectional if $X_j \ncong \tau_\Omega(X_{j+2})$ for all $j = 0, \ldots, n-2$.

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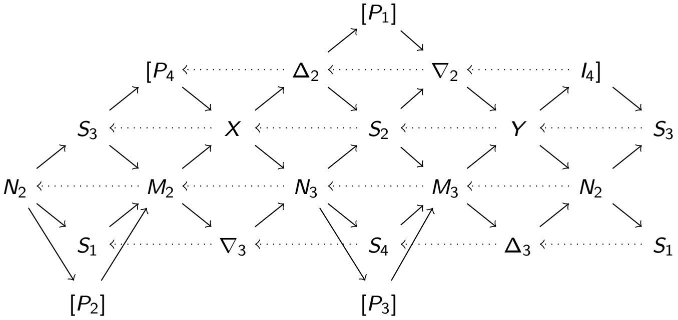

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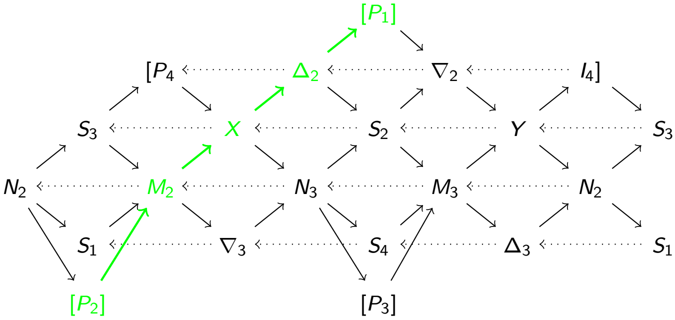

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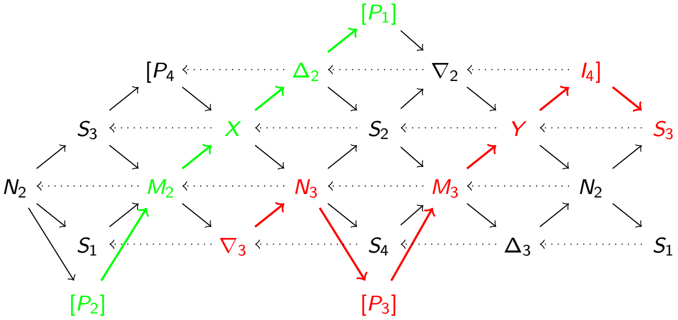

Sectional paths theorem

Theorem

Let

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$$

be a sectional path in the Auslander-Reiten-quiver of A-mod such that X_0 and X_n are in Ω while $X_1, \ldots, X_{n-1}$ are not in Ω. Then $f_n \cdots f_1$ is irreducible in Ω.

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Moreover, the cosets of all sectional paths from X_0 *to* X_n *in A-mod such that all modules along these paths other than* X_0 *and* X_n *are not in* Ω *are linearly independent in* $\mathrm{rad}_\Omega(X_0, X_n)/\mathrm{rad}^2_\Omega(X_0, X_n)$.

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Does the converse hold as well, i.e is a morphism in Ω given by a non-sectional path in A-mod reducible?

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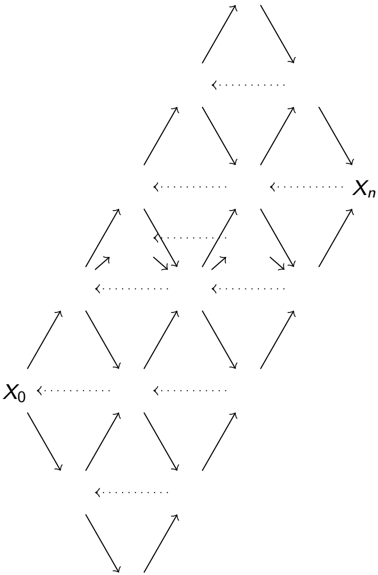

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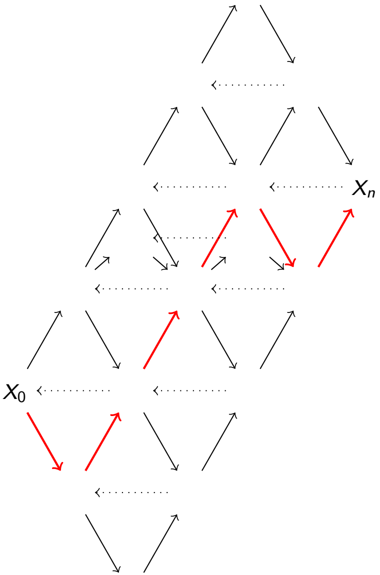

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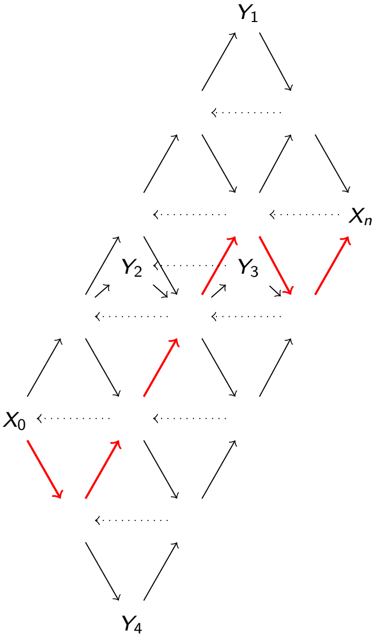

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- There is a short exact sequence

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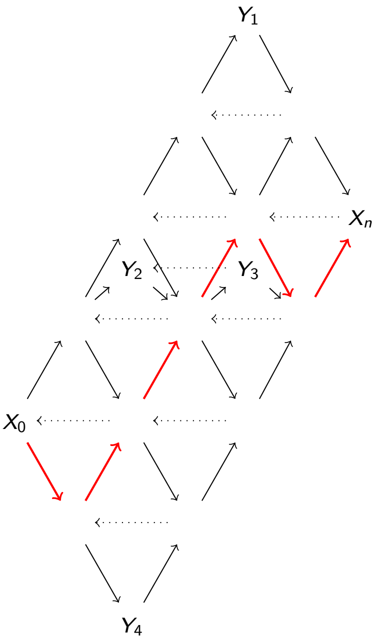

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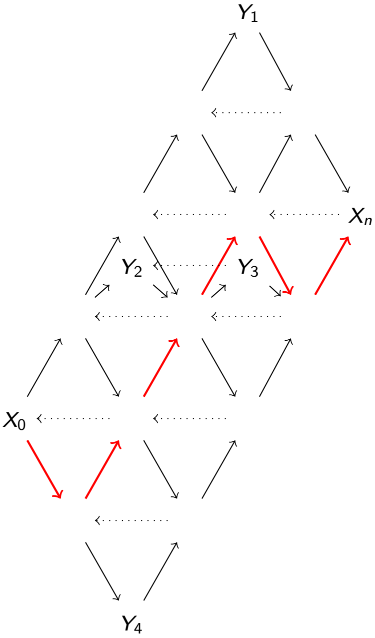

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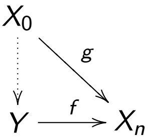

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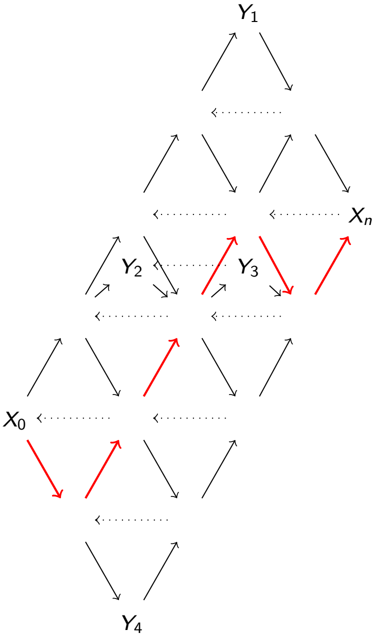

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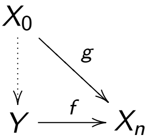

- There are no irreducible morphisms from X_0 to X_n

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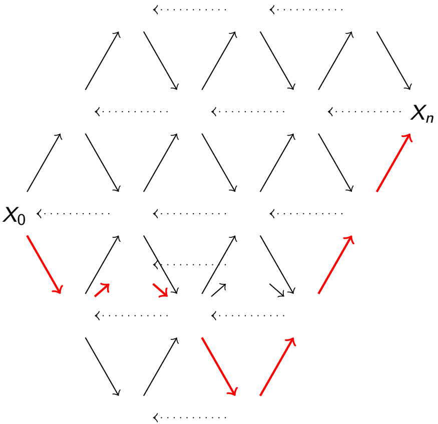

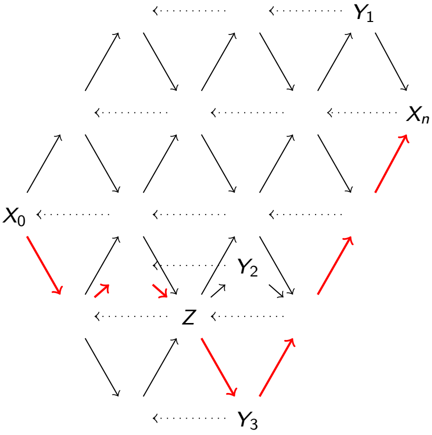

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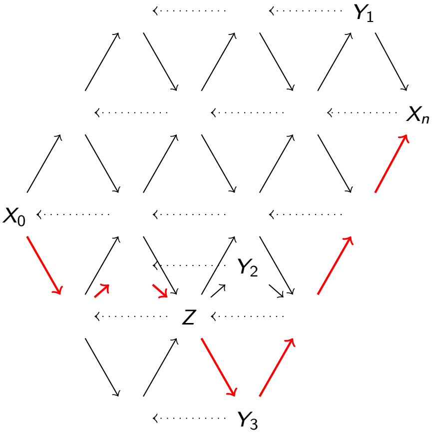

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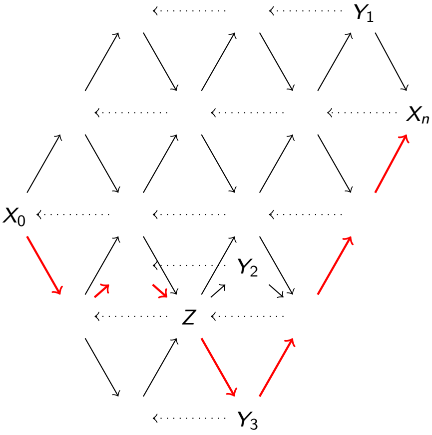

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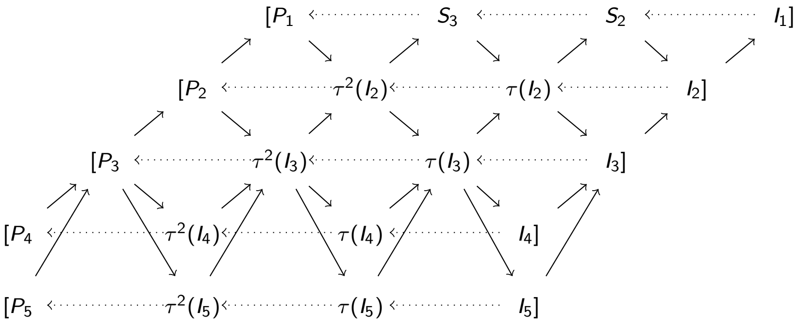
- If X_Z contains only one direct summand that is not Ext-injective, then there are no Ω-irreducible morphisms from X_0 to X_n.

A counterexample



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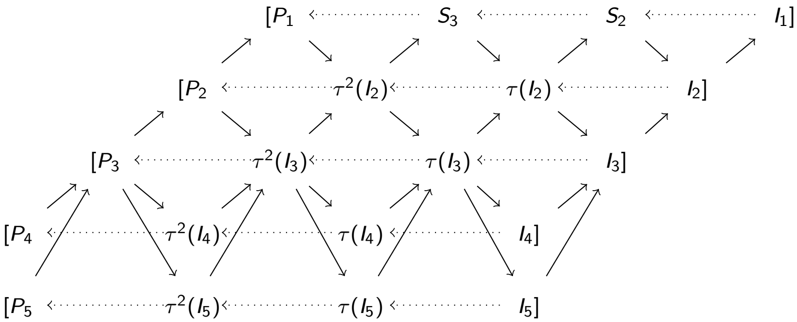

$T = P_1 \oplus P_4 \oplus P_5 \oplus S_2 \oplus I_2$ is a generalized cotilting module and the functorially finite resolving subcategory $\Omega = \mathrm{Ext}^i_A(-, T) = 0$ contains P_2, P_3 and I_1 in addition to the direct summands of T.

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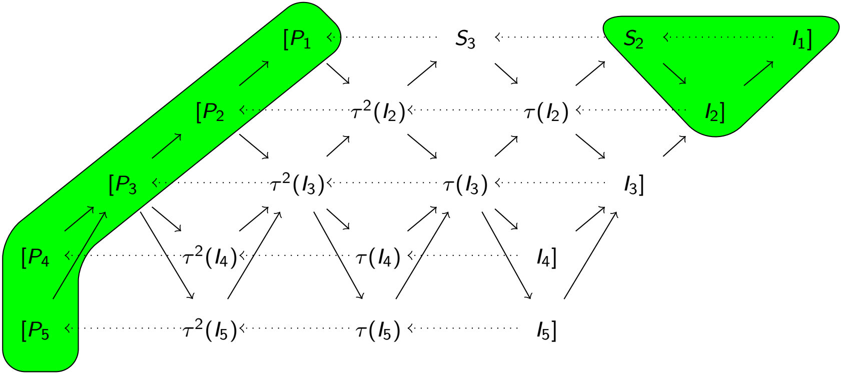

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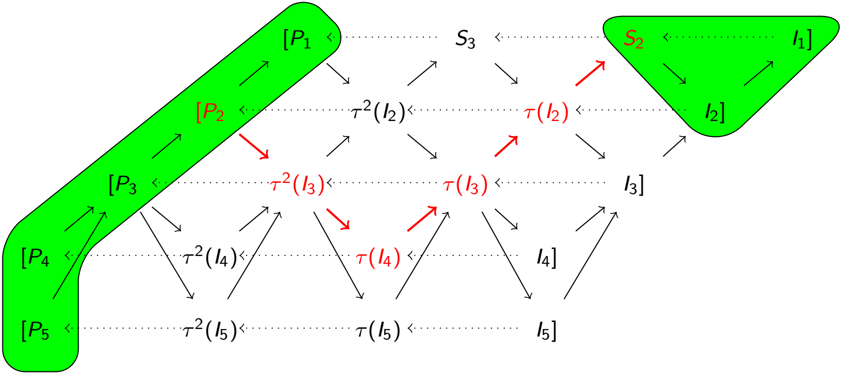

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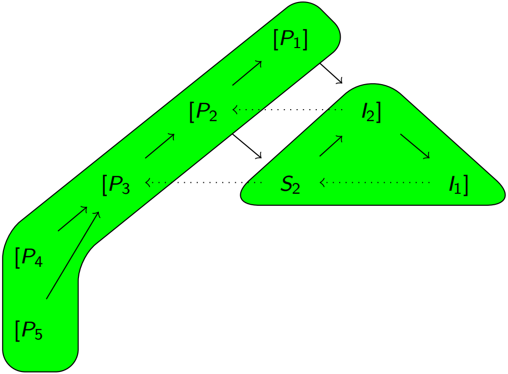

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A sectional subgraph Σ is a connected subgraph of Γ_Ω such that all subpaths in Σ are sectional. Σ is called full, if any connected subgraph Σ' of Γ_Ω such that $\Sigma \subsetneq \Sigma'$ is not a sectional subgraph. The undirected graph $\overline{\Sigma}$ associated to Σ is called the type of Σ.

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For certain left or right stable components the type of a full sectional subgraph is independent from the choice of a path. In these components we define the left and right subgraph type of a component as the type of an arbitrary full sectional subgraph respectively.

The main result

Let Γ_l and Γ_r denote the subquivers of an Auslander-Reiten-quiver consisting of all left stable and right stable modules in Ω respectively.

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- *The right subgraph types of all connected components of Γ_r are given by Dynkin diagrams.*