# Representation type and Auslander-Reiten theory of Frobenius-Lusztig kernels

Julian Külshammer

University of Kiel, Germany

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- $\ell \geq 3$ an odd integer
- $\zeta$ a primitive $\ell$-th root of unity

# Definition of $u_\zeta(\mathfrak{g})$

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The Lusztig form of the quantised enveloping algebra is obtained in three steps:

(i) Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$ with generators $E_i, F_i, K_i^{\pm 1}$, $i = 1, \ldots, n$ and quantum Serre relations

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The **small quantum group** $U_\zeta(G_0)$ is now the $k$-subalgebra generated by $E_i, F_i, K_i^{\pm 1}$

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(s) $E^\ell = F^\ell = K^\ell - 1 = 0$

# Why Frobenius-Lusztig kernel?

In positive characteristic:

$$k \to U_\zeta(G_0) \to U_\zeta(G_r) \stackrel{\mathrm{Fr}_\zeta}{\to} \mathrm{Dist}(G_r) \to k$$

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- symmetric

# Normal Hopf subalgebras

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- *All simple $A$-modules are of the form $M \otimes L$, where $M$ is a simple $C$-module and $L$ is a simple $B$-module*
- *Assume simple $B$-modules have no self-extensions. Then:*

$$\mathrm{Ext}^1(M_1 \otimes L_1, M_2 \otimes L_2) \cong \begin{cases} \mathrm{Ext}^1(M_1, M_2) & L_1 \cong L_2 \\ \mathrm{Hom}(M_1, M_2 \otimes \mathrm{Ext}^1(L_1, L_2)) & L_1 \ncong L_2 \end{cases}$$

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- *Assume $B$ has a simple projective module* St. *Then: There is a block embedding* $\mathrm{mod}\, C \to \mathrm{mod}\, A$, $M \mapsto M \otimes \mathrm{St}$.

# Support varieties

If certain finiteness assumptions on cohomology hold, we can define a theory of support varieties for Hopf algebras, i.e. to each module $M$ one can associate a variety $\mathcal{V}(M)$ reflecting certain properties of $M$.

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Proven for $r = 0$ and $r = 1$ (in special cases). From now on assumed in general.

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  - $r = 1$*: all but one of the block in the image of* $- \otimes \mathrm{St}$.
- **wild** *in all other cases.*

# Only $SL_2$ with $r = 0, 1$ has tame blocks

Proof.

1. If $B$ is a block of an (fg)-Hopf algebra $A$ and there is $M$ with $\dim \mathcal{V}(M) \geq 3$, then $B$ is wild. [Farnsteiner 2007, Feldvoss-Witherspoon 2009]
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2. For $\mathbb{A}_1$, one can compute the quiver of $U_\zeta(G_r)$ by considering the quiver of $U_\zeta(G_0)$ and the quiver of $\mathrm{Dist}(G_r)$, i.e. using the Ext-group result from before.
   Then one can identify a wild subquiver $\circ \rightrightarrows \circ \longleftarrow \circ$

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# Webb's Theorem

**Theorem (Kerner-Zacharia 2009)**

*Let $A$ be an (fg)-Hopf algebra. Then the components of the stable Auslander-Reiten quiver $\Gamma_s(A)$ are of the form:*

periodic: *finite or $\mathbb{Z}[A_\infty]/\tau^m$ (tubes)*

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*The following hold for* $\Gamma_s(U_\zeta(G_r))$ *not associated to* $\mathbb{A}_1$*:*

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- *There are no Euclidean components.*
- *For $r = 0$ the periodic components are of the form $\mathbb{Z}[A_\infty]/\tau$.*
- *For $r = 0$ the components containing $U_\zeta(\mathfrak{g})$-modules, e.g. simples, are of the form $\mathbb{Z}[A_\infty]$.*

# Application

**Corollary**

*If $S$ is a simple module for $U_\zeta(G_r)$ in a $\mathbb{Z}[A_\infty]$-component, then $\operatorname{ht} P(S) = \operatorname{rad} P(S)/\operatorname{soc} P(S)$ is indecomposable.*

# Indecomposable Heart

Proof.

- $0 \to \operatorname{rad} P \to \operatorname{ht} P \oplus P \to P/\operatorname{soc} P \to 0$ is the standard almost split sequence, i.e. if $P/\operatorname{soc} P$ has only one predecessor in $\Gamma_s(U_\zeta(G_r))$, then $\operatorname{ht} P$ is indecomposable.

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- [Kawata] Symmetric algebras with a non-quasi-simple simple module $S$ in a $\mathbb{Z}[A_\infty]$-component have a very special shape of projective modules.

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- [Kawata] Symmetric algebras with a non-quasi-simple simple module $S$ in a $\mathbb{Z}[A_\infty]$-component have a very special shape of projective modules.
- This contradicts a certain Ext-symmetry for $U_\zeta(G_r)$.

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# The tame small quantum group

**Theorem (...)**

*The tame blocks are as follows:*

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- *special biserial*
- *stable Auslander-Reiten quiver: 2 components of type* $\mathbb{Z}[\tilde{A}_{12}]$ *and two* $\mathbb{P}^1$*-families of homogeneous tubes.*

# Stable Auslander-Reiten quiver of $u_\zeta(\mathfrak{sl}_2)$