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A finite set of equations for Bongartz resolutions

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Setting

- Q finite quiver
- kQ path algebra, k field
- I admissible ideal, ($J^t \subseteq I \subseteq J^2$, $J =<$ arrows$>$)
- $\Lambda = kQ/I$ finite dimensional

- $M \in \textbf{mod}\,\Lambda$
  $0 \to P_1 \to P_0 \to M \to 0$ projective presentation over kQ

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$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & P_0 I & = & P_0 I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_\Lambda(M) & \longrightarrow & P_0 / P_0 I & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

$$0 \longrightarrow P_0 I \longrightarrow P_1 \longrightarrow \Omega_\Lambda(M) \longrightarrow 0$$

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$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & P_1 I & = & P_1 I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & P_0 I & \longrightarrow & P_1 & \longrightarrow & \Omega_\Lambda(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^2_\Lambda(M) & \longrightarrow & P_1/P_1 I & \longrightarrow & \Omega_\Lambda(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The filtration

There exists a filtration of P_0

$$\ldots \subseteq P_0 I^{\kappa+1} \subseteq P_1 I^{\kappa} \subseteq P_0 I^{\kappa} \subseteq \cdots \subseteq P_1 I^2 \subseteq P_0 I^2 \subseteq P_1 I \subseteq P_0 I \subseteq P_1 \subseteq P_0$$

such that

$$\Omega_{\Lambda}^{2\kappa}(M) = P_0 I^{\kappa} / P_1 I^{\kappa},$$

and

$$\Omega_{\Lambda}^{2\kappa-1}(M) = P_1 I^{\kappa-1} / P_0 I^{\kappa}.$$

The Bongartz resolution

There exists a projective resolution over Λ

$$\cdots \to P_0 I^2 / P_0 I^3 \to P_1 I / P_1 I^2 \to P_0 I / P_0 I^2 \to P_1 / P_1 I \to P_0 / P_0 I \to M \to 0$$

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Gröbner bases

- $\mathcal{B} = \{\text{paths}\}$, k-basis of kQ
- Length lexicographic order on $\mathcal{B}$
  - $v_1 < \cdots < v_r < \alpha_1 < \cdots < \alpha_s$
  - $p = \beta_1 \cdots \beta_n < \gamma_1 \cdots \gamma_m = q$,
    if $n < m$ or $\beta_{n_0} < \gamma_{n_0}$ and $\beta_i = \gamma_i$, for $i < n_0$.
- $x = \sum_{i=1}^{n} \lambda_i b_i \in kQ$
  $\mathrm{Tip}(x) = b_{i_0}$ the biggest of the b_i's.
- $X \subseteq kQ$, $\mathrm{Tip}(X) = \{\mathrm{Tip}(x), x \in X\}$
- $\mathrm{NonTip}(X) = \mathcal{B} - \mathrm{Tip}(X)$
- $I \subseteq kQ$
  $\mathcal{G} \subseteq I$ right Gröbner basis, if $\langle \mathrm{Tip}(\mathcal{G}) \rangle = \langle \mathrm{Tip}(I) \rangle$ as right ideals.
- $\mathcal{G}$ *uniform* and *reduced*, $I = \coprod_{g \in \mathcal{G}} gkQ$.

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Right uniform and reduced Gröbner bases exist for

- any ideal $I \subseteq kQ$,
- any submodule $P' \subseteq P$ of a sum of vertex projectives modules P.

- They are finite.

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The equations

For all γ in NonTip(I) and g in $\mathcal{G}$, consider the expression

$$\gamma g = \sum_{g' \in \mathcal{G}} g' r_{g'}^{(\gamma,g)}$$

with $r_{g'}^{(\gamma,g)}$ in kQ.

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The algorithm

— $M \in \operatorname{mod} \Lambda$

— $0 \to P_1 \to P_0 \to M \to 0$ projective presentation over kQ such that

- $P_0 = \coprod_{i=1}^{n} v_i kQ$
- $P_1 = \coprod_{j=1}^{m} w_j kQ \subseteq P_0$, with $\{w_j\}_j$ right uniform and reduced Gröbner basis of P_1

Recall the filtration:

$\ldots \subseteq P_0 I^{\kappa+1} \subseteq P_1 I^{\kappa} \subseteq P_0 I^{\kappa} \subseteq \cdots \subseteq P_1 I^2 \subseteq P_0 I^2 \subseteq P_1 I \subseteq P_0 I \subseteq P_1 \subseteq P_0$

$I = \coprod_{g \in \mathcal{G}} gkQ$

- $P_0 I = \coprod_{i,g} (v_i g) kQ$
- $P_1 I = \coprod_{j,g} (w_j g) kQ$

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Proposition 1

Let M be in $\operatorname{mod}\Lambda$ and $0 \to P_1 \to P_0 \to M \to 0$ a projective presentation of M as a right kQ-module. Suppose that

- $P_0 = \coprod_{i=1}^{n} v_i R$, where v_i is in Q_0 for all $i = 1, \ldots, n$, and
- $P_1 = \coprod_{j=1}^{m} w_j kQ \subseteq P_0$, where the elements $w_j = (w_j^1, \cdots, w_j^n)^T$ of P_0 form a uniform, right reduced, right Gröbner basis for P_1.

Then $\boldsymbol{w_j^i}$ **is in Span**$\{$**NonTip(*I*)** $\cup \mathcal{G}\}$, for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

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Proposition 2

Let γ be in NonTip(I) and g be in $\mathcal{G}$. Then in the equation

$$\gamma g = \sum_{g' \in \mathcal{G}} g' r_{g'}^{(\gamma, g)},$$

$\boldsymbol{r_{g'}^{(\gamma,g)}}$ **is in Span{NonTip(*I*) $\cup$ $\mathcal{G}$}**, for all g' in $\mathcal{G}$.

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The matrices

For each γ in NonTip(I) consider the matrix $[\gamma\mathcal{G}]$, where the columns are the vectors $\boldsymbol{r}_{\boldsymbol{g}'}^{(\gamma,\boldsymbol{g})}$, for all $\boldsymbol{g}$'s.

Each entry of $[\gamma\mathcal{G}]$ is in **Span**$\{$**NonTip(*I*)** $\cup\, \mathcal{G}\}$.

— Similarly define the matrices $[\boldsymbol{g}\mathcal{G}]$, for each $\boldsymbol{g}$ in $\mathcal{G}$.

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$$\ldots \subseteq P_0 I^{\kappa+1} \subseteq P_1 I^{\kappa} \subseteq P_0 I^{\kappa} \subseteq \cdots \subseteq P_1 I^2 \subseteq P_0 I^2 \subseteq P_1 I \subseteq P_0 I \subseteq P_1 \subseteq P_0$$

— WI^{κ} the matrix of $P_1 I^{\kappa} \subseteq P_0 I^{\kappa}$

— For $\kappa = 0$, the entries of W are in $\mathrm{Span}\{\mathrm{NonTip}(I) \cup \mathcal{G}\}$.

The algorithm

$WI^{\kappa+1}$ is obtained from WI^{κ}, by replacing each entry, by a linear combination of the matrices $[\gamma\mathcal{G}]$ and $[g\mathcal{G}]$.

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Monomial algebras

— $\Lambda = kQ/I$, I generated by monomials.

— Equations: $\gamma g = g' r_{g'}^{(\gamma, g)}$

- Columns of $[\gamma \mathcal{G}]$ and $[g\mathcal{G}]$: $\begin{pmatrix} 0 \\ \vdots \\ p \\ \vdots \\ 0 \end{pmatrix}$, p path.

— The second syzygy of any module over a monomial algebra is a direct sum of cyclic modules, generated by paths [Z]

- Columns of WI : $\begin{pmatrix} 0 \\ \vdots \\ p \\ \vdots \\ 0 \end{pmatrix}$, p path.

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- $\mathrm{fd}(\Lambda) = \sup\{\mathrm{pd}_\Lambda(M) | M \text{ in } \mathrm{mod}\,\Lambda, \mathrm{pd}_\Lambda(M) < \infty\}$
- $\mathrm{fd}(\Lambda) < \infty$ ([GKK]).

Finitistic dimension

Let Λ be a finite dimensional monomial algebra. Then $\mathrm{fd}(\Lambda) \leq 2(\dim_k \Lambda + 1)$.

- Key: repetition of the non-zero entries of the rows of the matrices WI^κ for $\kappa > \dim_k \Lambda$.

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Global dimension of finite dimensional algebras

$|\mathcal{G}| = n$
v vertex, $\{\gamma_s^v\} = \{$arrows starting at $v\}$.

Elementary Equations degree m

$EE^m(\gamma_s^v, g_t^m)\colon \gamma_s^v g_t^m = \sum_{j=1}^{n^m} g_j^m r_j^{(s,t,m)}$

Minimal Equations of degree m

$ME^m(\gamma_s^v, g_t^m)\colon \{\cdots\} = \sum_{j=1}^{n^m} g_j^m r_j$

$\{\cdots\} = kQ$-combination of right hand sides of EE's
r_j's: **coefficients**

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Theorem

$\operatorname{gldim} \Lambda \leq 2m$ if and only if for any vertex v the coefficients of $ME^m(\gamma_s^v, g_t^m)$ of any pair (γ_s^v, g_t^m) that they are defined on, are in I.

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THANK YOU !

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