On Jacobian algebras from closed surfaces

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Motivation – QP from triangulations

Labardini associated quivers with potentials (QP) to ideal triangulations of surfaces with marked points, linking:

- cluster algebras from surfaces [Fomin-Shapiro-Thurston]
- theory of quivers with potentials [Derksen-Weyman-Zelevinsky]

For surfaces with non-empty boundary, the QP are rigid and the Jacobian algebras are finite-dimensional [Labardini].

**Question.** What happens for empty boundary (i.e. closed surfaces)?

Known cases:

- Torus with one puncture [Labardini]
- Spheres [Barot-Geiss, Trepode-Valdivieso-Diaz]
Motivation – derived equivalences

**Problem.** Find the mutation classes of QP such that all their Jacobian algebras are *derived equivalent*.

**Non-example:** acyclic quivers with more than 2 vertices.

**Example:** 3-Calabi-Yau [Keller-Yang] (infinite-dimensional).

**More instances [L.]:**

- Unpunctured surfaces with exactly one marked point on each boundary component (finite-dimensional).
- Once-punctured closed surfaces with “non-standard” potentials (infinite-dimensional, locally gentle).

**Questions.** What happens for the standard potentials? more punctures?
(\(S, M\)) – surface with marked points and empty boundary.

**Theorem [L.]**

- If \((S, M)\) is not a sphere with 4 punctures, then the QP associated to any ideal triangulation of \((S, M)\) is *not rigid* and its (completed) Jacobian algebra is *finite-dimensional* and *symmetric*.

- If \((S, M)\) is a sphere with 4 punctures, then the same holds when the product of the scalars defining the potential is not 1.

**Corollary.** There is a Hom-finite triangulated 2-Calabi-Yau category \(\mathcal{C}_{(S,M)}\) with a cluster-tilting object for each ideal triangulation.
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**Results – derived equivalences**

$\mathcal{P}(Q,W)$ – the Jacobian algebra of a QP $(Q,W)$.

**Proposition [L.].** If $\mathcal{P}(Q,W)$ is *(weakly) symmetric*, then $\mathcal{P}(\mu_k(Q,W))$ is *(weakly) symmetric* [Herschend-Iyama] and *derived equivalent* to $\mathcal{P}(Q,W)$.

**Corollary 1.** *All* the Jacobian algebras associated to the triangulations of a closed surface are derived equivalent.

**Corollary 2.** Let $A$ be $\frac{2n-2}{n}$-CY with $\text{gl.dim } A \leq 2$. Write $T_A(\text{Ext}_A^2(DA, A)) = \mathcal{P}(Q,W)$. If $(Q,W)$ is non-degenerate, then all the Jacobian algebras in its mutation class are *symmetric* and *derived equivalent*.

**Example.** $A = KD_4 \otimes KD_4$ is $\frac{4}{3}$-CY $\implies$ *infinite* mutation class of finite-dimensional, symmetric, derived equivalent Jacobian algebras.
Combinatorial model for the quivers

\( T \) – a fixed triangulation of \((S, M)\) such that:
there are at least 3 arcs of \( T \) incident to each puncture.

**Proposition [L.].** Let \((Q, W)\) be the QP associated to \( T \). Then:

- \( Q \) is connected without any loops or 2-cycles.
- For any \( i \in Q_0 \), there are exactly two arrows in \( Q_1 \) starting at \( i \) and two arrows ending at \( i \).
- There are invertible maps \( f, g : Q_1 \rightarrow Q_1 \) with the following properties:
  - For any \( \alpha \in Q_1 \), the set \( \{f(\alpha), g(\alpha)\} \) consists of the two arrows that start at the vertex which \( \alpha \) ends at;
  - \( f^3 \) is the identity on \( Q_1 \).
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**PSL\(_2(\mathbb{Z})\)**-action on the set of arrows

The definition of the maps \( f \) and \( g \):

For an arrow \( \alpha \in Q_1 \), denote by \( \bar{\alpha} \) the other arrow starting at the same vertex as \( \alpha \). In particular, \( g(\bar{\alpha}) = f(\alpha) \).

**Proposition [L.]** \(\text{PSL}_2(\mathbb{Z})\) acts transitively on \( Q_1 \).

**Remark.** Non-trivial path in \( Q = \text{arrow} + \text{word in } f, g \).
The potentials

Two kinds of cycles in $Q$: $f$-cycles and $g$-cycles.

$f$-cycles are 3-cycles corresponding to the triangles of $T$, $g$-cycles arise from traversing arcs around a puncture.

**Proposition [L.]** The potential $W$ is given by

$$W = \sum \alpha \cdot f(\alpha) \cdot f^2(\alpha) - \sum c_{\beta} \beta \cdot g(\beta) \cdot \ldots \cdot g^{n\beta-1}(\beta)$$

where $c : Q_1 \to K^\times$ is $g$-invariant.
Finite-dimensionality

Let \( \Lambda = \mathcal{P}(Q, W) \).

By computing cyclic derivatives of the potential we get:

**Lemma.** For any \( \beta \in Q_1 \),
\[
\beta \cdot f(\beta) = c_\beta \bar{\beta} \cdot g(\bar{\beta}) \cdot \ldots \cdot g^{n_\beta-2}(\bar{\beta}).
\]

**Lemma.** For any \( \alpha \in Q_1 \),
\[
\begin{align*}
\alpha \cdot f(\alpha) \cdot f^2(\alpha) &= c_\alpha \alpha \cdot g(\alpha) \cdot g^2(\alpha) \cdot \ldots \cdot g^{n_\alpha-1}(\alpha) \\
&= c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot g^2(\bar{\alpha}) \cdot \ldots \cdot g^{n_{\bar{\alpha}}-1}(\bar{\alpha}) \\
&= \bar{\alpha} \cdot f(\bar{\alpha}) \cdot f^2(\bar{\alpha}) \\
\alpha \cdot g(\alpha) \cdot f g(\alpha) &= c_{f(\alpha)} \alpha \cdot f(\alpha) \cdot g f(\alpha) \cdot g^2 f(\alpha) \cdot \ldots \cdot g^{n f(\alpha)-2} f(\alpha) \\
\alpha \cdot f(\alpha) \cdot g f(\alpha) &= c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot \ldots \cdot g^{n_{\bar{\alpha}}-3}(\bar{\alpha}) \cdot g^{n_{\bar{\alpha}}-2}(\bar{\alpha}) \cdot f g^{n_{\bar{\alpha}}-2}(\bar{\alpha})
\end{align*}
\]
Finite-dimensionality (continued)

Assume further that:

any arc of $T$ has an endpoint with $\geq 4$ arcs incident to it

(such $T$ always exists if $(S,M) \neq$ sphere with 4 punctures)

$\Rightarrow \Lambda$ has a finite basis consisting of the paths

$$\{e_i\}_{i \in Q_0} \cup \{\alpha \cdot g(\alpha) \cdot \ldots \cdot g^r(\alpha)\}_{\alpha \in Q_1, 0 \leq r < n_{\alpha}-1} \cup \{z_i\}_{i \in Q_0}$$

where $z_i$ is a $g$-cycle starting at $i$.

**Remark.** $T$ gives rise also to a Brauer graph algebra (via the data of a graph + cyclic order at nodes):

- Same quiver as $\Lambda$,
- Different defining relations, but same basis.
Symmetry, non-rigidity and more

- $\Lambda$ is *symmetric*:

  The isomorphism $D\Lambda \simeq \Lambda$ as $\Lambda$-$\Lambda$-bimodules follows from the “duality”

  $$\alpha \cdot g(\alpha) \cdot \ldots \cdot g^{r-1}(\alpha) \leftrightarrow c_\alpha g^r(\alpha) \cdot \ldots \cdot g^{n_\alpha-1}(\alpha)$$

  for $0 \leq r \leq n_\alpha$.

- $(Q, W)$ is *not rigid*:

  The image of any cycle $z_i$ in $\Lambda/[\Lambda, \Lambda]$ is not zero.

- Can compute the *Cartan matrix* of $\Lambda$, its *center*, ...