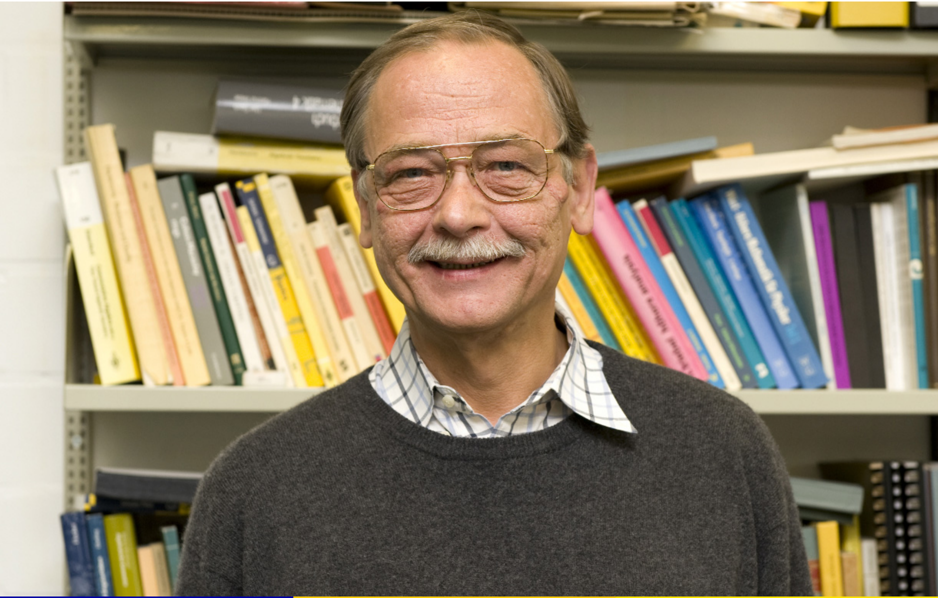
The $\mathbb{E}$-series, Happel-Seidel symmetry, and Orlov's theorem

Helmut Lenzing

Paderborn

ICRA 2012, Bielefeld, August 15, 2012

Happy Birthday!



Birthday Cake, Paderborn, March 18. 2010

Secret Paper!

Greuel 1

MAXIMAL COHEN-MACAULAY MODULES AND TATE-COHOMOLOGY

OVER

GORENSTEIN RINGS

Ragnar-Olaf BUCHWEITZ *
(Hannover)

The main theme of this article is :

Why should one consider Maximal Cohen-Macaulay Modules ?

Although there has been a lot of work and success lately in the theory of such modules, of which this conference witnessed, it has remained mysterious - at least to the present author - why these modules provide such a powerful tool in studying the algebra and geometry of singularities for example.

We try to give one answer here, at least for the case of Gorenstein rings. Their role is special as over such rings "maximal Cohen-Macaulay" and "being a syzygy module of arbitrarily high order" are synonymous.

It turns out, that these modules, in a very precise sense, describe all stable homological features of such rings.

The motif was the observation that maximal Cohen-Macaulay modules - at least up to projective modules - carry a natural triangulated structure which implies that there is a naturally defined cohomology-theory attached to these modules - the Tate-cohomology.

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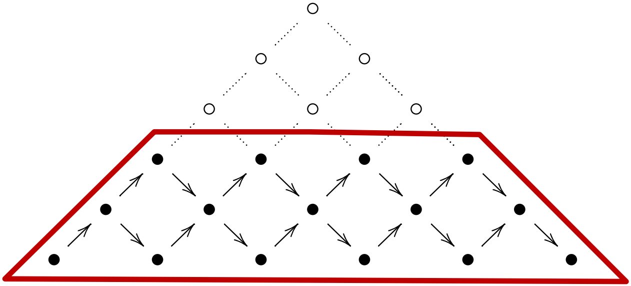
The Topic

We are interested in the *Nakayama k-algebras*, $k = \bar{k}$,

$$A_n(r):\ 1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} \cdots \xrightarrow{x} n-1 \xrightarrow{x} n$$

given by the equioriented A_n-quiver with all relations $x^r = 0$.
Here is the *Auslander-Reiten quiver* (n=6,r=3)



Analysis

Bad luck! The matter does not really look exciting!

Passage to Triangulated Categories

$$A_n(r):\ 1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} \cdots \xrightarrow{x} n-1 \xrightarrow{x} n\ , \quad x^r = 0$$

Put $\mathcal{T}_n(r) = \mathrm{D}^b(\mathrm{mod}\text{-}A_n(r))$. Thus $n =$ rank of $\mathrm{K}_0(\mathcal{T}_n(r))$.
Let P_1, $P_2, \ldots, P_n$ be the indecomposable projectives over $A_n(r)$, viewed as members of $\mathcal{T}_n(r)$.

Observation

The right *perpendicular category* $P_n^{\perp}$, formed in $\mathcal{T}_n(r)$, is equivalent to $\mathcal{T}_{n-1}(r)$.

For *fixed nilpotency degree r* we thus obtain a *tower of triangulated categories*

$$\mathcal{T}_1(r) \subset \mathcal{T}_2(r) \subset \cdots \subset \mathcal{T}_{n-1}(r) \subset \mathcal{T}_n(r) \cdots$$

where each inclusion $\mathcal{T}_{n-1}(r) \subset \mathcal{T}_n(r)$ is nice (adjoints!).

The Aim: Identify the Triangulated Types

Initiated by Happel-Seidel [2010], restricting to piecewise hereditary case.
Some triangulated types of relevance:

- $[a, b, c]$ Derived category of representations of oriented *tree* $[a, b, c]$
- (a, b, c) Derived category of coherent sheaves on *weighted projective line*
- $\langle a, b, c]$ Singularity category of *Kleinian or Fuchsian singularity*) or Derived category of *extended canonical algebra* [-,JAP,2011]
- $\langle a, b, c\rangle$ Singularity category of universally graded *triangle singularity* $f = x^a + y^b + z^c$ (or *stable category of vector bundles* for weighted projective line) [KLM,2012]

Note

- The types are invariant under permutation.
- Typically, the last two types are *not* piecewise hereditary.

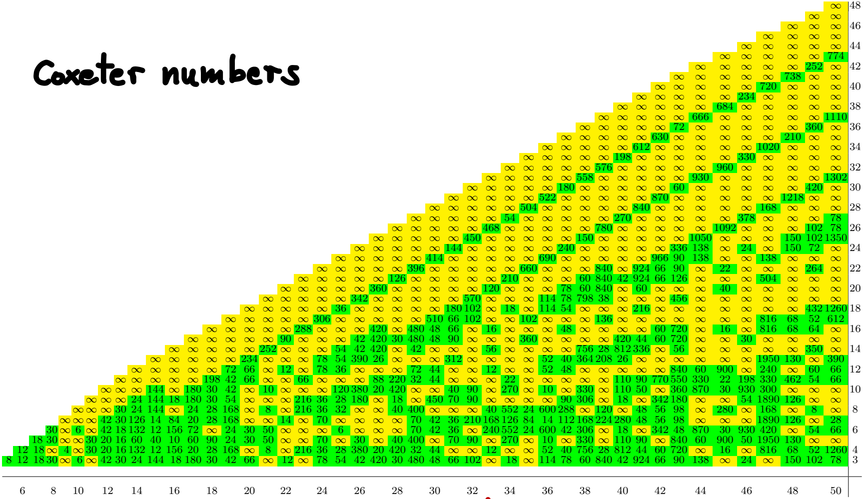
Algebraic Analysis of Singularities

Start with $f = x^a + y^b + z^c$, $a, b, c \geq 2$, *universally graded*, by rank one abelian grading group $\mathbb{L}$.
We follow Buchweitz ('86) and Orlov ('05) in forming the singularity category $\mathrm{Sing}^{\mathbb{L}}(S)$

$$
\begin{tikzcd}
 & S = k[x,y,z]/(f) \arrow[dl, squiggly] \arrow[dr, squiggly] & \\
\mathrm{D}^b\left(\frac{\mathrm{mod}^{\mathbb{L}}\text{-}S}{\mathrm{mod}_0^{\mathbb{L}}\text{-}S}\right) \arrow[rr, squiggly, leftrightarrow, "\text{Orlov}"] & & \mathrm{Sing}^{\mathbb{L}}(S)
\end{tikzcd}
$$

Spectral analysis

Coxeter numbers



r

n

Examples

(i) The triangulated types of $A_6(3)$, $A_7(3)$ and $A_8(3)$ are $\mathbb{E}_6 = [2,3,3]$, $\mathbb{E}_7 = [2,3,4]$, and $\mathbb{E}_8 = [2,3,5]$.

(ii) The type of $A_8(6)$ is also $\mathbb{E}_8 = \langle\langle 2,3,5\rangle\rangle = [2,3,5]$.

(iii) The type of $A_{12}(3)$ equals $\langle\langle 2,3,7\rangle\rangle = \langle 2,3,7]$.

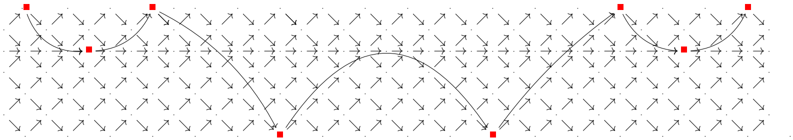


Figure: Realisation of $A_8(6)$ by a tilting object in $[2,3,5] = \langle\langle 2,3,5\rangle\rangle$

The $\mathbb{E}$-series $(r = 3)$

														$[2,3,16]$	$[2,3,17]$	$(2,3,17)$	$\langle 2,3,17]$	∞	306	17
													$[2,3,15]$	$[2,3,16]$	$(2,3,16)$	$\langle 2,3,16]$	∞	288	∞	
												$[2,3,14]$	$[2,3,15]$	$(2,3,15)$	$\langle 2,3,15]$	∞	90	∞	∞	15
											$[2,3,13]$	$[2,3,14]$	$(2,3,14)$	$\langle 2,3,14]$	∞	252	∞	∞	∞	
										$[2,3,12]$	$[2,3,13]$	$(2,3,13)$	$\langle 2,3,13]$	∞	234	∞	∞	∞	$\langle\langle 2,3,13\rangle\rangle$	13
									$[2,3,11]$	$[2,3,12]$	$(2,3,12)$	$\langle 2,3,13]$	∞	72	66	∞	$\langle\langle 2,3,12\rangle\rangle$	∞	78	
								$[2,3,10]$	$[2,3,11]$	$(2,3,11)$	$\langle 2,3,11]$	∞	198	42	$\langle\langle 2,3,11\rangle\rangle$	∞	∞	66	∞	11
							$[2,3,9]$	$[2,3,10]$	$(2,3,10)$	$\langle 2,3,10]$	∞	180	$\langle\langle 2,3,10\rangle\rangle$	42	∞	10	∞	∞	∞	
						$[2,3,8]$	$[2,3,9]$	$(2,3,9)$	$\langle 2,3,9]$	144	$\langle\langle 2,3,9\rangle\rangle$	180	30	54	∞	∞	∞	216	$\langle\langle 2,4,9\rangle\rangle$	9
					$[2,3,7]$	$[2,3,8]$	$(2,3,8)$	$\langle 2,3,8]$	$\langle\langle 2,3,8\rangle\rangle$	144	∞	24	28	168	∞	$\langle\langle 2,8,4\rangle\rangle$	∞	216	36	
				$[2,3,6]$	$[2,3,7]$	$(2,3,7)$	$\langle 2,3,7]$	30	126	14	84	20	$\langle\langle 2,4,7\rangle\rangle$	168	∞	∞	14	∞	$\langle\langle 2,5,7\rangle\rangle$	7
			$[2,3,5]$	$[2,3,6]$	$(2,3,6)$	$(2,3,7)$	$\langle\langle 2,3,7\rangle\rangle$	18	132	$\langle\langle 2,6,4\rangle\rangle$	156	72	∞	24	$\langle\langle 2,6,5\rangle\rangle$	50	∞	∞	∞	
		$[2,3,4]$	$[2,3,5]$	$(2,3,5)$	$[2,3,7]$	30	$\langle\langle 2,5,4\rangle\rangle$	16	60	40	$\langle\langle 2,5,5\rangle\rangle$	60	90	24	$\langle\langle 2,5,6\rangle\rangle$	50	∞	∞	$\langle\langle 2,5,7\rangle\rangle$	5
	$[2,3,3]$	$[2,3,4]$	$(2,3,4)$	$(2,4,4)$	$(2,4,5)$	30	$\langle\langle 2,4,5\rangle\rangle$	16	132	$\langle\langle 2,4,6\rangle\rangle$	156	20	$\langle\langle 2,4,7\rangle\rangle$	168	∞	$\langle\langle 2,4,8\rangle\rangle$	∞	216	$\langle\langle 2,4,9\rangle\rangle$	
$[2,2,3]$	$[2,3,3]$	$[2,3,4]$	$[2,3,5]$	$(2,3,5)$	$(2,3,6)$	$(2,3,7)$	$\langle\langle 2,3,7\rangle\rangle$	30	$\langle\langle 2,3,8\rangle\rangle$	144	$\langle\langle 2,3,9\rangle\rangle$	180	$\langle\langle 2,3,10\rangle\rangle$	42	$\langle\langle 2,3,11\rangle\rangle$	∞	$\langle\langle 2,3,12\rangle\rangle$	∞	$\langle\langle 2,3,13\rangle\rangle$	3
	6		8		10		12		14		16		18		20		22		24	

Note

(i) The sequence $(\mathcal{T}_n(3))$, $n \geq 6$, extends the exceptional types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ to an infinite sequence: *the $\mathbb{E}$-series*.

(ii) *For n even*, the categories $\mathcal{T}_n(3)$ are singularity categories of type $\langle\langle a, b, c\rangle\rangle$, marked in green. [KLM, 2012]

The $\mathbb{E}$-series II

1. The $\mathbb{E}$-series $(\mathcal{T}_n(3))$ plays a special role within the family $\mathcal{T}_n(r)$
   - the only *sheaf types* occurring are $(2, 3, p)$;
   - the only *tree types* occurring are $[2, 3, p]$;
   - the only *Fuchsian types* occurring are $\langle 2, 3, p]$.

   By contrast, all *triangle types* $\langle\langle 2, b, c\rangle\rangle$ occur.
2. The $\mathbb{E}$-series $\mathcal{T}_n(3)$ extends the types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, and contains *further exceptional types* from singularity theory like $\mathbb{E}_{12}$.

The $\mathbb{E}$-series III, Spectral Analysis

3 The *Coxeter polynomial* of each $\mathcal{T}_n(3)$ factors into *cyclotomics*.

4 The *Coxeter transformation* of $\mathcal{T}_n(3)$ is *periodic*, except for $n = 9, 11$ (mod 12). [Generally speaking, one has some *'weak periodicity'* modulo 12.]

5 *Explicit factorizations* are available for the Coxeter polynomial of each $\mathcal{T}_n(3)$, compare [Hille-Müller,'12].

Happel-Seidel symmetry

														[2, 3, 16]	[2, 3, 17]	(2, 3, 17)	$\langle 2,3,17]$	∞	306
													[2, 3, 15]	[2, 3, 16]	(2, 3, 16)	$\langle 2,3,16]$	∞	288	∞
												[2, 3, 14]	[2, 3, 15]	(2, 3, 15)	$\langle 2,3,15]$	∞	90	∞	∞
											[2, 3, 13]	[2, 3, 14]	(2, 3, 14)	$\langle 2,3,14]$	∞	252	∞	∞	∞
										[2, 3, 12]	[2, 3, 13]	(2, 3, 13)	∞	∞	234	∞	∞	∞	$\langle\langle 2,3,13\rangle\rangle$
									[2, 3, 11]	[2, 3, 12]	(2, 3, 12)	∞	∞	72	66	∞	$\langle\langle 2,3,12\rangle\rangle$	∞	78
								[2, 3, 10]	[2, 3, 11]	(2, 3, 11)	$\langle 2,3,11]$	∞	198	42	$\langle\langle 2,3,11\rangle\rangle$	∞	∞	66	∞
							[2, 3, 9]	[2, 3, 10]	(2, 3, 10)	$\langle 2,3,10]$	∞	180	$\langle\langle 2,3,10\rangle\rangle$	42	∞	10	∞	∞	∞
						[2, 3, 8]	[2, 3, 9]	(2, 3, 9	$\langle\langle 2,3,8\rangle\rangle$	144	$\langle\langle 2,3,9\rangle\rangle$	180	$\langle\langle 2,3,10\rangle\rangle$	54	∞	∞	∞	216	$\langle\langle 2,4,9\rangle\rangle$
					[2, 3, 7]	[2, 3, 8]	(2, 3, 8)	$\langle 2,3,8]$	$\langle\langle 2,3,8\rangle\rangle$	144	∞	24	28	168	∞	$\langle\langle 2,8,4\rangle\rangle$	∞	216	36
				[2, 3, 6]	[2, 3, 7	(2, 3, 7)	$\langle\langle 2,3,7\rangle\rangle$	30	126	14	84	20	$\langle\langle 2,4,7\rangle\rangle$	168	∞	∞	14	∞	$\langle\langle 2,5,7\rangle\rangle$
			[2, 3, 5]	[2, 3, 6]	(2, 3, 6)	(2, 3, 7)	42	18	132	$\langle\langle 2,6,4\rangle\rangle$	156	72	∞	24	$\langle\langle 2,6,5\rangle\rangle$	50	∞	∞	∞
		[2, 3, 4]	[2, 3, 5]	(2, 3, 5)	[2, 3, 7]	30	$\langle\langle 2,5,4\rangle\rangle$	16	60	40	$\langle\langle 2,5,5\rangle\rangle$	60	90	24	$\langle\langle 2,5,6\rangle\rangle$	50	∞	∞	$\langle\langle 2,5,7\rangle\rangle$
	[2, 3, 3]	[2, 3, 4]	(2, 3, 4)	(2, 4, 4)	(2, 4, 5)	30	$\langle\langle 2,4,5\rangle\rangle$	16	132	$\langle\langle 2,4,6\rangle\rangle$	156	20	$\langle\langle 2,4,7\rangle\rangle$	168	∞	$\langle\langle 2,4,8\rangle\rangle$	∞	216	$\langle\langle 2,4,9\rangle\rangle$
[2, 2, 3]	[2, 3, 3]	[2, 3, 4]	[2, 3, 5]	(2, 3, 5)	(2, 3, 6	(2, 3, 7)	$\langle\langle 2,3,7\rangle\rangle$	30	$\langle\langle 2,3,8\rangle\rangle$	144	$\langle\langle 2,3,9\rangle\rangle$	180	$\langle\langle 2,3,10\rangle\rangle$	42	$\langle\langle 2,3,11\rangle\rangle$	∞	$\langle\langle 2,3,12\rangle\rangle$	∞	$\langle\langle 2,3,13\rangle\rangle$

Happel-Seidel symmetry II

Theorem (KLM 2012)

Assume $a, b, \geq 2$. Put $n = (a-1)(b-1)$. Then

(i) $A_n(a)$ and $A_n(b)$ are derived equivalent to $\langle\langle 2, a, b\rangle\rangle$.

(ii) $A_{n-1}(a)$ and $A_{n-1}(b)$ are derived equivalent.

(iii) $A_{n+1}(a)$ and $A_{n+1}(b)$ are derived equivalent.

Sketch. (i) Realize $A_n(3)$ by a *tilting object* in $\langle\langle 2, a, b\rangle\rangle$ and use symmetry of the symbol.
For (ii), (iii) use *explicit form of tilting object* representing $A_n(3)$, and combine with perpendicular calculus resp. 1-point extension [Barot-L].

Remark

1. *Ladkani* has related results (different method).
2. Often HS-symmetry extends to *intervals with 5 members* (not just 3).
Proof is similar.

An instance of Orlov's theorem

														[2, 3, 16]	[2, 3, 17]	(2, 3, 17)	⟨2, 3, 17]	∞	306
													[2, 3, 15]	[2, 3, 16]	(2, 3, 16)	⟨2, 3, 16]	∞	288	∞
												[2, 3, 14]	[2, 3, 15]	(2, 3, 15)	⟨2, 3, 15]	∞	90	∞	∞
											[2, 3, 13]	[2, 3, 14]	(2, 3, 14)	⟨2, 3, 14]	∞	252	∞	∞	∞
										[2, 3, 12]	[2, 3, 13]	(2, 3, 13)	∞	∞	234	∞	∞	∞	⟨⟨2, 3, 13⟩⟩
									[2, 3, 11]	[2, 3, 12]	(2, 3, 12)	∞	∞	72	66	∞	⟨⟨2, 3, 12⟩⟩	∞	78
								[2, 3, 10]	[2, 3, 11]	(2, 3, 11)	⟨2, 3, 11]	∞	198	42	⟨⟨2, 3, 11⟩⟩	∞	∞	66	∞
							[2, 3, 9]	[2, 3, 10]	(2, 3, 10)	⟨2, 3, 10]	∞	180	⟨⟨2, 3, 10⟩⟩	42	∞	10	∞	∞	∞
						[2, 3, 8]	[2, 3, 9]	(2, 3, 9)	⟨⟨2, 3, 8⟩⟩	144	⟨⟨2, 3, 9⟩⟩	180	30	54	∞	∞	∞	216	⟨⟨2, 4, 9⟩⟩
					[2, 3, 7]	[2, 3, 8]	(2, 3, 8)	⟨2, 3, 8]	⟨⟨2, 3, 8⟩⟩	144	∞	24	28	168	∞	⟨⟨2, 8, 4⟩⟩	∞	216	36
				[2, 3, 6]	[2, 3, 7]	(2, 3, 7)	⟨⟨2, 3, 7⟩⟩	30	126	14	84	20	⟨⟨2, 4, 7⟩⟩	168	∞	∞	14	∞	⟨⟨2, 5, 7⟩⟩
			[2, 3, 5]	[2, 3, 6]	(2, 3, 6)	(2, 3, 7)	42	18	132	⟨⟨2, 6, 4⟩⟩	156	72	∞	24	⟨⟨2, 6, 5⟩⟩	50	∞	∞	∞
		[2, 3, 4]	[2, 3, 5]	(2, 3, 5)	[2, 3, 7]	30	⟨⟨2, 5, 4⟩⟩	16	60	40	⟨⟨2, 5, 5⟩⟩	60	90	24	⟨⟨2, 5, 6⟩⟩	50	∞	∞	⟨⟨2, 5, 7⟩⟩
	[2, 3, 3]	[2, 3, 4]	(2, 3, 4)	(2, 4, 4)	(2, 4, 5)	30	⟨⟨2, 4, 5⟩⟩	16	132	⟨⟨2, 4, 6⟩⟩	156	20	⟨⟨2, 4, 7⟩⟩	168	∞	⟨⟨2, 4, 8⟩⟩	∞	216	⟨⟨2, 4, 9⟩⟩
[2, 2, 3]	[2, 3, 3]	[2, 3, 4]	[2, 3, 5]	(2, 3, 5)	(2, 3, 6)	(2, 3, 7)	⟨⟨2, 3, 7⟩⟩	30	⟨⟨2, 3, 8⟩⟩	144	⟨⟨2, 3, 9⟩⟩	180	⟨⟨2, 3, 10⟩⟩	42	⟨⟨2, 3, 11⟩⟩	∞	12	∞	78

Global Picture: Cofinality

Theorem

The triangle singularities $\langle\langle 2, a, b\rangle\rangle$, $a, b \geq 2$, are *cofinal* in the set of all $\mathcal{T}_n(r)$ with $r \geq 3$.
That is, for each such category $\mathcal{T}_n(r)$ there exists a singularity category $\mathcal{T}$ of type $\langle\langle 2, a, b\rangle\rangle$ and an explicit *exceptional sequence* $\mathcal{E} = (E_1, \ldots, E_s)$ in $\mathcal{T}$ such that $\mathcal{T}_n(r)$ equals the *perpendicular category* $\mathcal{E}^\perp$, formed in $\mathcal{T}$.

To a certain extend this reduces the study of the $\mathcal{T}_n(r)$ to *singularity theory*.

Conjecture

'Most' categories $\mathcal{T}_n(r)$ are themselves singularity categories.

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