Topological study of cluster quivers of finite mutation type (I): Genuses of cluster quivers of finite mutation type

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1 Introduction

Two types of cluster algebras are of special interest: finite type and finite mutation type and the former is the special case of the latter. They have been completely classified respectively.

Almost all skew-symmetric cluster algebras (equivalently, cluster quivers) of finite mutation type come from triangulations of surfaces except for 11 exceptional cases.

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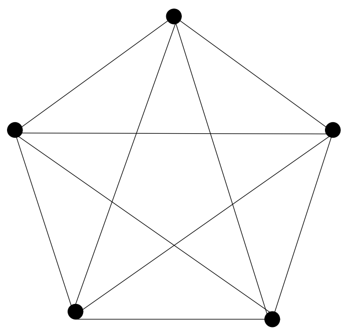
As well-known as we have, genus is a topological invariant for oriented surfaces, as well as for topological graphs.

So, the natural question is how to find out the relation between the genus of a cluster quivers and that of the oriented surfaces that the cluster quiver comes from.

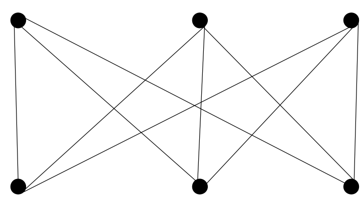
This is the main attempt of this talk

The further motivation is to study the influence of genus of cluster quivers and the coming oriented surfaces for the algebraical characterization of the cluster algebras.

In the theory of topological graphs, the famous Kuratowski's theorem asserts that a graph is non-planar if and only if it contains a subdivision of K_5 or $K_{3,3}$ as its subgraph.



K_5



$K_{3,3}$

Definition

(i) Given an n-regular tree, correspond to each vertex a **seed** (X, Q) with the **cluster** $X = \{x_1, \cdots, x_n\}$ and cluster variables x_i, Q is a quiver with n vertices and without loops and 2-cycles, which is called a **cluster quiver**.

(ii) Given an initial seed (X, Q) and $k \in \{1, \cdots, n\}$, a **mutation** μ_k transforms (X, Q) into a new seed (X', Q') with

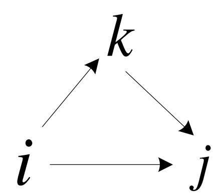
$$X' = (X \setminus \{x_k\}) \cup \{x'_k\}$$

where x'_k is defined by the following **exchange relation**:

$$x_k x'_k = \prod_{\alpha: i \to k} x_i + \prod_{\beta: k \to i} x_i$$

Here, Q' is the **mutation of a quiver** Q defined by the following steps:

(1) $\forall$ each path $i \to k \to j$, add a new arrow:



(2) Reverse all arrows incident with k,

(3) Delate all 2-cycles.

(iii) **Finite mutation type**:

A cluster algebra is of **finite mutation type** if the number of quivers which appear in the seeds is finite. In this case, the cluster quivers are said to be of **finite mutation type**.

Cluster algebras from surface:

Let S be an oriented Riemann surface with boundary $\partial S, M \subset S$ a finite set of points, at least one in each boundary component.

Punctures: the points in $M \cap (S \setminus \partial S)$.

Such pair (S, M) is called simply a **surface**.

An **arc** is the homotopy class of a curve in S that connects two points.

An (**ideal**) **triangulation** T is a maximal set of non-crossing arcs, whose number of arcs is an invariant, denote by n.

T: Triangulation $\longleftrightarrow$ Quiver Q_T.

A cluster algebra is said to **be from a surface** if it contains a seed (X, Q) where the quiver Q is from a triangulation of a surface.

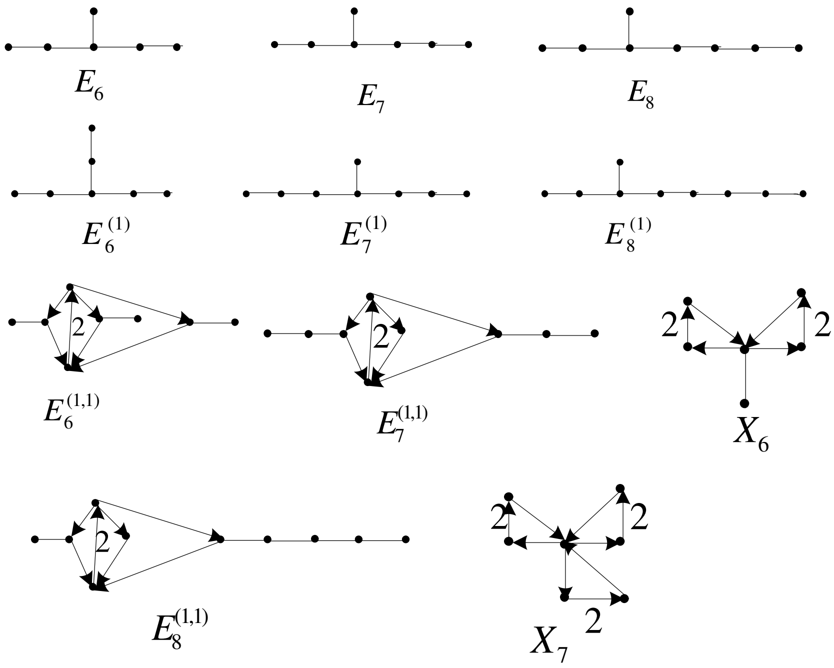
Theorem 1.1. *(Felikson-Shapiro-Tumarkin)*

A cluster algebra $\mathscr{A} = \mathscr{A}(X, Q)$ *is of finite mutation type if and only if:*

(1) $\mathscr{A}$ *is coming from a surface* $(n \geq 3)$*; or*

(2) $(n \leq 2)$*; or*

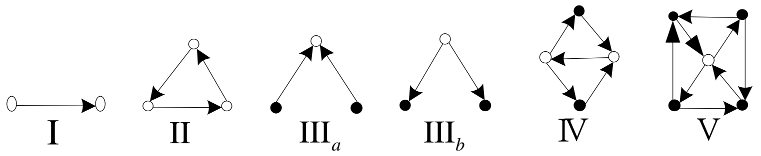
(3) $\mathscr{A}$ *is one of* 11 *exceptional types:*
$E_6, E_7, E_8, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, X_6, X_7, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$*:*



That is, $\mathscr{A}$ has a cluster quiver at one of these types.

Theorem 1.2. *(Fomin-Shapiro-Thurston)*

Suppose a cluster quiver Q is of finite mutation type and comes from triangulation of a surface. Then Q is obtained by gluing from the following six block quivers under certain rules:



It is known (Fomin-Shapiro-Thurston) that for a triangulation of a surface, the formula holds.

$$\boxed{n = 6g + 3b + 3p + c - 6}$$

where n: the number of arcs, which equals the vertex number of the associated cluster quiver;

g: the genus of the surface;

b: the number of boundary components;

p: the number of punctures;

c: the number of marked points on the boundary.

Note that a cluster quiver will be of infinite mutation type if its a full subquiver is so.

Thus, we may use this note to investigate the mutation type of any cluster quiver with genus > 0.

2 Genuses of exceptional cluster quivers

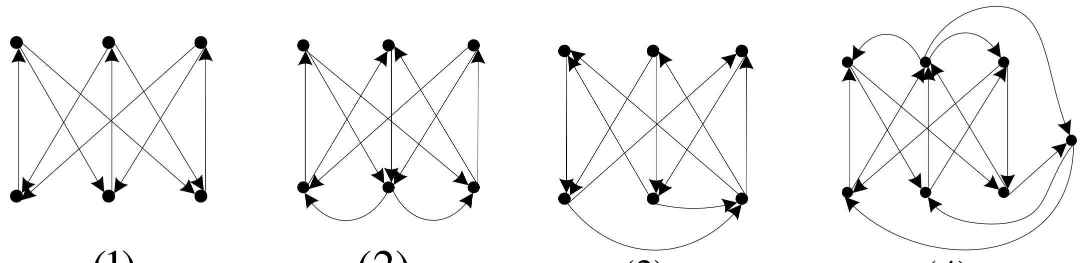
Lemma(see [8]) Any orientations on the same tree are mutation-equivalent.

By the Keller's quiver mutation in java, we give the table of distribution of genuses of 11 exceptional cluster quivers in the classification of finite mutation type, as follows:

cluster quivers	total number	number of genus 0	number of genus 1
E_6	21	21	0
E_7	112	112	0
E_8	391	391	0
$E_6^{(1)}$	52	52	0
$E_7^{(1)}$	338	338	0
$E_8^{(1)}$	1935	1935	0
$E_6^{(1,1)}$	27	27	0
$E_7^{(1,1)}$	217	217	0
$E_8^{(1,1)}$	1886	1886	0
X_6	4	1	3
X_7	2	1	1

For the exceptional quivers with genus 1, we have the following:

Proposition 2.1. *There are exactly* 4 *non-planar cluster quivers of exceptional finite mutation types, which are listed as follows, where the quivers (1), (2), (3) are in the mutation-equivalent*



(1) (2) (3) (4)

Figure 5

class of X_6*, the quiver (4) is in the mutation-equivalent class of* X_7*.*

Their genuses are just 1.

Proof. For the convenience to describe the mutations at X_6 and X_7 to obtain these four quivers, we will label the vertices of X_6 and X_7 as Figure 6.

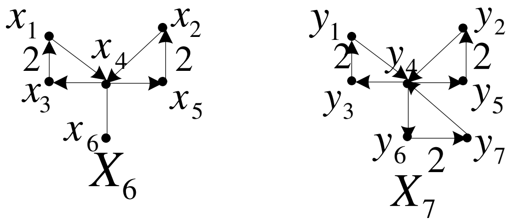


Figure 6

Then, we can get the four quivers in Figure 5 as follows:

the quiver (1) can be obtained from X_6 by mutation on the vertices x_4 and x_6;

the quiver (2) can be obtained from X_6 by mutation on the vertices x_4;

the quiver (3) can be obtained from X_6 by mutation on the vertices x_4 and x_3;

the quiver (4) can be obtained from X_7 by mutation on the vertices y_4. □

3 Relation between the genus of a cluster quiver and that of its surface

Theorem For a triangulation T of a surface S with genus g, let g' be the genus of the cluster quiver Q associated with T. Then, $g' \leq g$.

Outline of Proof

(i) all triangulations that we are interested in can be obtained by gluing together a number of puzzle pieces including three kinds as in Figure 2(II), except for one case, i.e. the triangulation of 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle, see Figure 2(I).

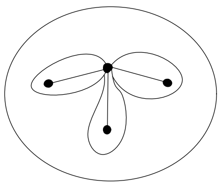


Figure 2(I)

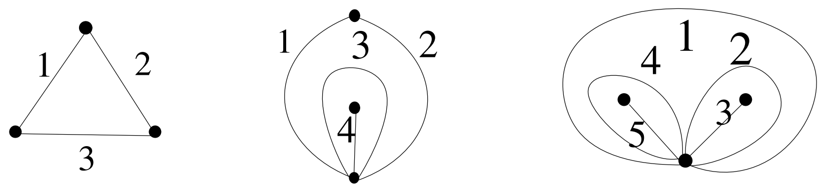
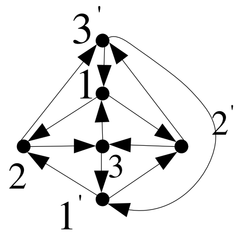


Figure 2(II)

These three types of puzzle pieces correspond to blocks of type I-V depending on whether outer sides lying on the boundary.

(ii) For each puzzle piece, we put its corresponding block into the face bounded by it. If two puzzle pieces have a common edge, then we glue two vertices corresponding to the common edge between these two blocks. Hence we obtain the *cluster quiver* Q of the triangulation in this way and moreover the underlying graph of Q can be drawn without self-crossing on the same surface. Then, we have $g' \leq g$ by the definition of genus of quiver.

(iii) For the only exceptional case whose triangulation cannot be obtained by gluing the puzzle pieces, let T be the triangulation of 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle. The corresponding cluster quiver of T can be obtained by gluing four blocks of type II, which can be shown as the following Figure. Obviously it is a planar quiver. Hence in this case $g' = g = 0$.



Corollary 3.1. *Let S be a surface of genus 0 and M be a set of marked points of S. Given any triangulation T of (S, M), suppose Q is the associated cluster quiver from T. Then, all quivers in the mutation-equivalent class of Q are of genus 0.*

Corollary 3.2. *Let S be a surface of genus g with M the set of marked points. For any triangulation T of (S, M), let Q be its corresponding quiver and let Q' be another cluster quiver of genus g' such that $g' > g$. Then, Q and Q' are not mutation-equivalent.*

4 Existence of cluster quiver with appointed genus

We will prove that *the genus of cluster quiver from surface can be any non-negative integers.*

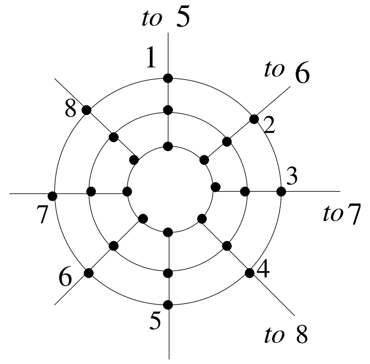
Lemma For an arbitrary non-negative integer n, there always exists a block-decomposable cluster quiver T_n such that the genus $g(T_n)$ of T_n satisfies that $g(T_n) \geq n$.

Sketch of proof: (i) We firstly introduce a class of graphs with arbitrary large genus, which is just the graph R_n for each positive integer n, constructed as follows:

There are $n + 1$ concentric cycles which consists of $4n$ edges. Additionally, there are $4n^2$ inner edges connecting the $n + 1$ cycles to each other and $2n$ outer edges adjoining antipodal vertices on the outermost cycle.

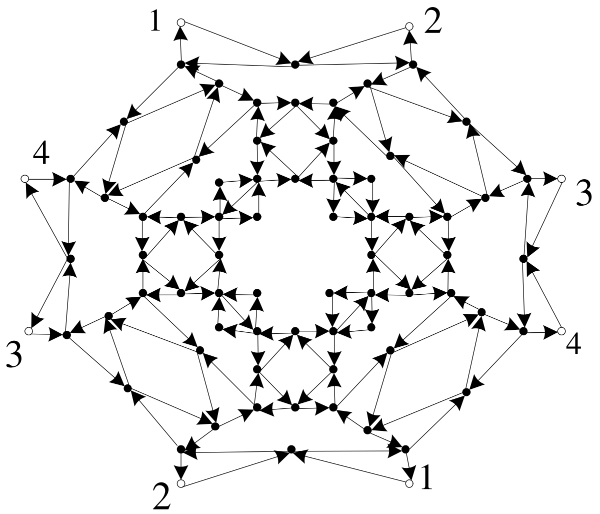
It was shown in the theory of topological graph that R_n is of genus n.

The following Figure gives the example R_2 for such graph in the case $n = 2$.



(ii) R_n is a subgraph of the underlying graph of a cluster quiver T_n, which is obtained by gluing $8n^2 + 4n$ blocks of type II and $2n$ blocks of type IV, which mean T_n is from a surface.

The following figure is the case of T_2 for $n = 2$:



(iii) Hence, we have $g(T_n) \geq g(R_n) = n$.

Theorem For any non-negative integer n, there exists an oriented surface S of genus n with punctures and an ideal triangulation T of S such that the corresponding cluster quiver T_n from T is of genus n.

Outline of Proof: (i) T_n is a uniquely block-decomposable quiver and hence T_n can be uniquely encoded by its corresponding triangulation, that is, blocks of type II are replaced by puzzle pieces of the first type (see the left graph in Figure 2(II)) and blocks of type IV are replaced by puzzle pieces of the second type (see the middle graph in Figure 2(II)).

(ii) When drawing T_n, we shall draw a planar quiver T'_n which has $4n$ unglued outlets. After gluing these $4n$ outlets in pairs (a pair consisting of one outlet and its opposite one), one obtains T_n.

(iii) To construct the corresponding triangulation of T_n, the corresponding triangulation T' of T'_n is from a surface S' with $4n$-gon as its boundary.

(iv) Let S be the resulting quotient space of this $4n$-gon by identifying one edge to its opposite one in pairs. T_n is just from the triangulation T in S which is obtained by gluing pairs of edges in T'.

(v) By the classification theorem of compact surfaces, the genus of S is at most n. Since T_n is obtained from this triangulation of S, by Theorem in section 3, $g(T_n) \leq g(S) \leq n$. By Lemma, $g(T_n) \geq n$. Hence, we obtain $g(T_n) = n$.

(vi) For the genus $g(S)$ of S, since $n = g(T_n) \leq g(S) \leq n$, we also have $g(S) = n$.

5 Relation between the numbers of blocks of types II and V in a cluster quiver

Lemma 5.1. *Any full subquiver of a block-decomposable quiver is also block-decomposable.*

Using this lemma, we get the following:

Proposition 5.2. *Any full subquiver of a cluster quiver of finite mutation type from a triangulation of a surface is also a cluster quiver of finite mutation type from a triangulation of the surface.*

Relying on the above discussion, we have the following:

Theorem 5.3. *Assume there are p puncture points on the oriented surface S and T is a triangulation of S. Let:*

g: the genus of the corresponding cluster quiver Q;

u: the numbers of block II in Q from the triangulation T

w: the numbers of block V in Q from the triangulation T.

Then,

(i) $u \lneqq w \Longrightarrow g = 0 \Longrightarrow w \leq u + 4$;

(ii) $g \geq 1 \Longrightarrow u \geq w$.

In summary, $u \geq w$ *or* $u \lneqq w \leq u + 4$.

Part II:

Topological study of cluster quivers of finite mutation type (II): Non-planar cluster quivers from surfaces

In this part, we will show a complete classification of reduced skeleton quivers of K_5 and $K_{3,3}$ type and thus obtain an analogue of Kuratowski's theorem for cluster quivers from surfaces. This gives a more precise characterization of non-planar cluster quivers from surfaces.

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Thank you a lot!