Lie powers and Lie modules

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Notation

- p is a prime number.
- G is a finite group.
- $\mathfrak{S}_n$ is the symmetric group acting on the set $\{1, \ldots, n\}$.
- K is a field, preferably of infinite order.
- All modules are 'left' and have finite dimensional over the defining field K.

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In the case V is a KG-module, $V^{\otimes n}$ is naturally a KG-module via the diagonal action and $L^n(V)$ is a submodule of $V^{\otimes n}$.

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- Suppose that E is the natural $\mathrm{GL}_m(K)$-module and $m \geq n$. Then $f_n(E^{\otimes n}) = K\mathfrak{S}_n$. Under the assumption, we define the Lie module
$$\mathrm{Lie}(n) := f_n(L^n(E)).$$

It is well-known that

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$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \mathrm{Lie}(n) \cong K\mathfrak{S}_{n-1}, \;\; \dim_K \mathrm{Lie}(n) = (n-1)!.$$

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$$L^{p^r s}(V) = L^{p^r}(B_s) \oplus L^{p^{r-1}}(B_{ps}) \oplus \cdots \oplus L^1(B_{p^r s}).$$

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Applying to the case of $G = \mathrm{GL}_m(K)$ and $V = E$, we have

- a decomposition of the Lie module $\mathrm{Lie}(p^r s)$ (given that $m \geq p^r s$), and

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Theorem (L. and Tan 2012)

Let $r, s \geq 1$ be arbitrary positive integers and V be an object of $M_K(m, r)$ such that $m \geq rs$. Then

$$f_{rs} L^s(V) \cong \mathrm{Ind}_{\mathfrak{S}_r \wr \mathfrak{S}_s}^{\mathfrak{S}_{rs}} ((f_r V)^{\otimes s} \otimes_K \mathrm{Lie}(s))$$

where $\mathfrak{S}_r \wr \mathfrak{S}_s$ acts on $(f_r V)^{\otimes s}$ in the obvious way and $(\mathfrak{S}_r)^s$ acts trivially on $\mathrm{Lie}(s)$.

Block components of the Lie module

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Let A be a finite-dimensional K-algebra and $b_1, \ldots, b_m$ be the mutually orthogonal primitive central idempotents of A such that $\sum_{i=1}^m b_i = 1$ associated to the indecomposable two-sided ideals decomposition

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For any A-module M,

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From now on, we suppose that K has characteristic $p > 0$ and denote it by k.

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$$b\,\mathrm{Lie}(n) = \bigoplus P(D)^{\oplus m_D}$$

where the sum is over all non-isomorphic simple $k\mathfrak{S}_n$*-modules* D *lying in the block* b, $P(D)$ *is the projective cover of* D *and* m_D *is some explicit positive integer.*

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We would expect the same hold for the p-power case.

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- M is periodic; namely $\Omega^r M \cong \Omega^s M \neq 0$ for some $r \neq s$, if and only if its complexity is one.
- The complexity of M is bounded above by the p-rank of G.

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- *The complexity of $\mathrm{Lie}(p^r s)$ is bounded above by r.*
- *If $s > 1$ then*

$$c(\mathrm{Lie}(p^r s)) = \max\{c(\mathrm{Lie}(p^i)) \mid 1 \leq i \leq r\}.$$

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Corollary

If $p \nmid s$ then the complexity of $\mathrm{Lie}(ps)$ is one.