REALIZING CLUSTER CATEGORIES OF DYNKIN TYPE A_n AS STABLE CATEGORIES OF LATTICES

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Abstract. Cluster tilting objects of the cluster category $\mathcal{C}$ of Dynkin type A_{n-3} are known to be indexed by triangulations of a regular polygon P with n vertices. Given a triangulation of P, we associate a quiver with potential with frozen vertices such that the associated Jacobian algebra has the structure of a $K[\![x]\!]$-order denoted as Λ_n, where $K[\![x]\!]$ is a formal power series ring over a field K. Then we show that $\mathcal{C}$ is equivalent to the stable category of the category of Λ_n-lattices.

Let $n \geq 3$, K be a field and $\mathcal{C}$ be a cluster category of type A_{n-3}.

1. Order

Definition 1.1. Let $R = K[\![x]\!]$ and Λ be an R-algebra. Then Λ is called an *R-order* if it is a finitely generated free R-module.

For an R-order Λ, a left Λ-module L is called a *Λ-lattice* if it is finitely generated as an R-module. We denote by $\mathrm{CM}(\Lambda)$ the category of Λ-lattices.

The order we use to study cluster categories of type A_{n-3} is Λ_n:

$$\Lambda_n = \begin{bmatrix} R & R & R & \cdots & R & x^{-1} \\ x & R & R & \cdots & R & R \\ x^2 & x & R & \cdots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^2 & x^2 & x^2 & \cdots & R & R \\ x^2 & x^2 & x^2 & \cdots & x & R \end{bmatrix}_{n\times n}.$$

As $\mathrm{CM}(\Lambda_n)$ has Auslander-Reiten sequences, we can draw the Auslander-Reiten quiver of Λ_n.

When n is an even number, the Auslander-Reiten quiver of Λ_n is the following:

$$\begin{tikzcd}[column sep=small]
 & \begin{bmatrix} 1 \\ 1 \\ n-2 \end{bmatrix} \arrow[dr] \arrow[rrr, dotted, no head] & & & \begin{bmatrix} \frac{n}{2} \\ 1 \\ \frac{n}{2}-1 \end{bmatrix} \arrow[dr] & \\
\begin{bmatrix} 2 \\ n-2 \\ 0 \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} 1 \\ 2 \\ n-3 \end{bmatrix} \arrow[r, dotted, no head] & \begin{bmatrix} \frac{n}{2}-1 \\ 2 \\ \frac{n}{2}-1 \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} \frac{n}{2} \\ 2 \\ \frac{n}{2}-2 \end{bmatrix} \\
 & \begin{bmatrix} 3 \\ n-3 \\ 0 \end{bmatrix} \arrow[ur] \arrow[rrr, dotted, no head] \arrow[d, dotted, no head] & & & \begin{bmatrix} \frac{n}{2}-1 \\ 3 \\ \frac{n}{2}-2 \end{bmatrix} \arrow[ur] \arrow[d, dotted, no head] & \\
 & \begin{bmatrix} \frac{n}{2} \\ 3 \\ \frac{n}{2}-3 \end{bmatrix} \arrow[dr] \arrow[rrr, dotted, no head] & & & \begin{bmatrix} 2 \\ n-3 \\ 1 \end{bmatrix} \arrow[dr] & \\
\begin{bmatrix} \frac{n}{2} \\ 2 \\ \frac{n}{2}-2 \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} \frac{n}{2}+1 \\ 2 \\ \frac{n}{2}-3 \end{bmatrix} \arrow[r, dotted, no head] & \begin{bmatrix} 1 \\ n-2 \\ 1 \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} 2 \\ n-2 \\ 0 \end{bmatrix} \\
 & \begin{bmatrix} \frac{n}{2}+1 \\ 1 \\ \frac{n}{2}-2 \end{bmatrix} \arrow[ur] \arrow[rrr, dotted, no head] & & & \begin{bmatrix} 1 \\ n-1 \\ 0 \end{bmatrix} \arrow[ur] &
\end{tikzcd}$$

where

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} R \\ \vdots \\ R \\ x \\ \vdots \\ x \\ x^2 \\ \vdots \\ x^2 \end{bmatrix},$$

R appearing m_1 times, x appearing m_2 times, x^2 appearing m_3 times.

This is a Mobius strip, both the first and last row of which consist of $\frac{n}{2}$ projective-injective Λ_n-lattices, with $n-3$ τ-orbits between them.

When n is an odd number, the Auslander-Reiten quiver of Λ_n is similiar.

2. Jacobian Algebra

Let us fix a non-negative integer $n \geq 3$ and a triangulation $\triangle$ of a regular polygon with n vertices (n-gon for short).

Quivers of triangulations are defined in [2] before. And QPs arising from triangulations are defined in [3] and [4]. Extending their definition, we have the following definition which is slightly different.

Definition 2.1. The (colored) quiver $Q_\triangle$ of the triangulation $\triangle$ is a quiver the vertices of which are indexed by all (internal and external) edges of the triangulation. And whenever two edges a and b share a joint vertex, then $Q_\triangle$ contains a red arrow $a \to b$ if a is a predecessor of b with respect to clockwise orientation inside a triangle at the joint vertex of a and b. Moreover, for every vertex of the polygon with at least one internal incident edge in the triangulation, there is a blue arrow $a \to b$ where a and b are its two incident external edges, a being a predecessor of b with respect to clockwise orientation.

In the following we shall denote $Q = Q_\triangle$. We also symbolize a path from i to j of length $l \geq 0$ by $i \rightsquigarrow j$. A *red path* is a path containing only red arrows. A cycle in Q is called a *cyclic triangle* if it contains only three red arrows, and a minimal cycle in Q is called a *big cycle* if it contains red arrows and exactly one blue arrow.

Definition 2.2. We define the set of frozen vertices F of Q as the subset of Q_0 consisting of the n external edges of the n-gon, and the potential

$$W = \sum \textit{cyclic triangles} - \sum \textit{big cycles}.$$

According to [1], the associated Jacobian algebra is defined by

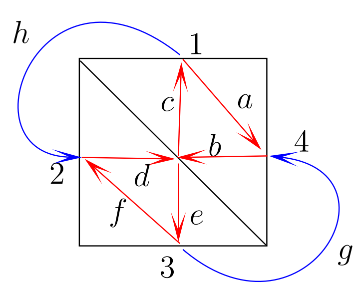
$$\mathcal{P}(Q, W, F) = \widehat{KQ}/\mathcal{J}(W, F),$$

where $\mathcal{J}(W, F)$ is the closure

$$\mathcal{J}(W, F) = \overline{\langle \partial_a W \mid a \in Q_1,\ s(a) \notin F \ \text{ or } \ e(a) \notin F \rangle}$$

with respect to the $\mathcal{J}_{\widehat{KQ}}$-adic topology. Notice that cyclic derivatives associated with arrows between frozen vertices are excluded.

Example 2.3. We illustrate the construction of $Q_\triangle$ and W when $\triangle$ is a triangulation of a square:



In this case $W = abc + def - beg - dch$, $F = \{1, 2, 3, 4\}$ and

$$\mathcal{J}(W, F) = \overline{\langle ca - eg, ab - hd, fd - gb, ef - ch \rangle}.$$

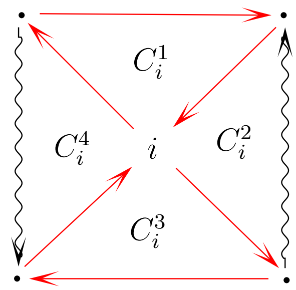
Remark 2.4. All relations in $\mathcal{P}(Q, W, F)$ are commutative relations.

Now we are ready to study the basis of the Jacobian algebra.

3. C-free path

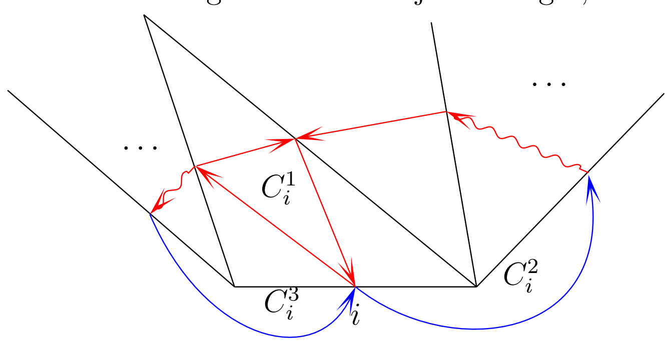
Let $i \in Q_0$. We consider all minimal cycles $C_i^1, \ldots, C_i^k$ passing through i. It is easy to check that in general there are only three possibilities:

1.If i is one of the diagonals of the triangulation, the only possibility is the following:



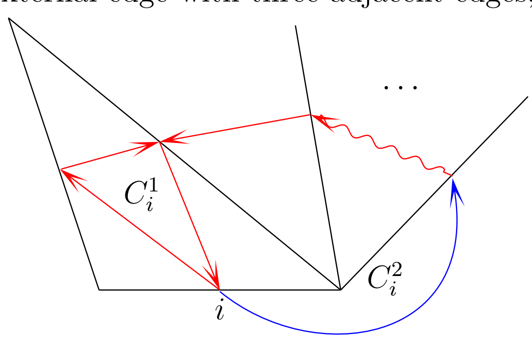
$$C_i := C_i^1 = C_i^2 = C_i^3 = C_i^4 \text{ holds in } \mathcal{P}(Q, W, F).$$

2. If i is an external edge with four adjacent edges,



$$C_i := C_i^1 = C_i^2 = C_i^3 \text{ holds in } \mathcal{P}(Q, W, F).$$

3. If i is an external edge with three adjacent edges,



$$C_i := C_i^1 = C_i^2 \text{ holds in } \mathcal{P}(Q, W, F).$$

It is already known that $\widehat{KQ}$ has a basis P_Q which is the set of all paths on Q (each time we speak about a basis in this text, we refer to a basis which generate a dense subspace, sometimes called countable basis). We say that two paths w_1 and w_2 are *equivalent* ($w_1 \sim w_2$) if $w_1 = w_2$ in $\mathcal{P}(Q, W, F)$. This gives an equivalence relation on P_Q.

Consider the element $C := \sum_{i \in Q_0} C_i$ in the associated Jacobian algebra $\mathcal{P}(Q, W, F)$, then C is in the center of $\mathcal{P}(Q, W, F)$. It turns out that the associated Jacobian algebra $\mathcal{P}(Q, W, F)$ is an R-algebra through $x \mapsto C$. In fact it is an R-order whose set of generators consists of C-free paths we define as follows.

Definition 3.1. For any vertices $i, j \in Q_0$, a path $w : i \rightsquigarrow j$ is called *C-free* if it is not left divisible (or equivalently right divisible) by C.

Considering all the C-free paths, we have the following proposition.

Proposition 3.2. *For any vertices $i, j \in Q_0$,*

(1) *there exists a unique C-free path w_0 from i to j up to $\sim$.*
(2) $\{w_0, w_0C, w_0C^2, w_0C^3, \cdots\}$ *is a (countable) basis of* $e_i\mathcal{P}(Q, W, F)e_j$.

Definition 3.3. We define the modified length of a red arrow from i to j as t ($1 \leq t \leq n-2$) if the angle between the two edges i and j of the triangle in the triangulation of the regular polygon is $t\frac{1}{n}\pi$. The modified length of a blue arrow is always 2.

We denote the modified length of a path w as $l^m(w)$. This modified length induces a degree on $\widehat{kQ}$ which will also be denoted by l^m.

Remark 3.4. We get the following easy facts from the definition of l^m:

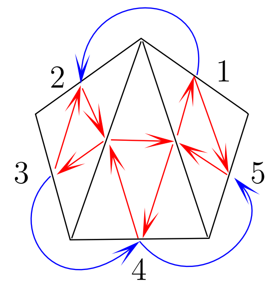
(1) the potential W is homogeneous of degree n for l^m. Thus, l^m induces the structure of a complete graded algebra on $\mathcal{P}(Q, W, F)$;
(2) $l^m(C) = n$;
(3) let i, j be any two vertices in Q_0. The unique C-free element of $\mathcal{P}(Q, W, F)$ has minimal modified length.

4. Main Result

Theorem 4.1. *The cluster category $\mathcal{C}_{A_{n-3}}$ is equivalent to the stable category of the category of Λ_n-lattices.*

Theorem 4.2. *The Jacobian algebra $\mathcal{P}(Q, W, F)$ arising from a triangulation of the n-gon is an R-order which can be explicitely calculated by using the modified length. Moreover $e_F\mathcal{P}(Q, W, F)e_F \cong \Lambda_n$ where e_F is the sum of the idempotents at frozen vertices.*

Example 4.3. The Jacobian algebra associated with the following triangulation of the pentagon:



is isomorphic to the following R-order:

$$\mathcal{P}(Q,W,F) \cong \begin{bmatrix} R & R & R & R & x^{-1} & R & R \\ x & R & R & R & R & R & R \\ x^2 & x & R & R & R & x & x \\ x^2 & x^2 & x & R & R & x & x \\ x^2 & x^2 & x^2 & x & R & x^2 & x \\ x & x & R & R & R & R & R \\ x & x & x & R & R & x & R \end{bmatrix}$$

It is clear that $e_F\mathcal{P}(Q,W,F)e_F = \Lambda_5$ holds in this case. The Auslander-Reiten quiver of Λ_5 is the following:

$$\begin{tikzcd}
\begin{bmatrix}R\\x\\x^2\\x^2\\x^2\end{bmatrix} \arrow[dr] & & \begin{bmatrix}R\\R\\x\\x^2\\x^2\end{bmatrix} \arrow[dr] & & \begin{bmatrix}R\\R\\R\\x\\x^2\end{bmatrix} \arrow[dr] & \\
& \begin{bmatrix}R\\x\\x\\x^2\\x^2\end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix}R\\R\\x\\x\\x^2\end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix}R\\R\\R\\x\\x\end{bmatrix} \\
\begin{bmatrix}R\\R\\R\\x\\x\end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix}R\\x\\x\\x\\x^2\end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix}R\\R\\x\\x\\x\end{bmatrix} \arrow[ur] \arrow[dr] & \\
& \begin{bmatrix}R\\R\\R\\R\\x\end{bmatrix} \arrow[ur] & & \begin{bmatrix}R\\x\\x\\x\\x\end{bmatrix} \arrow[ur] & & \begin{bmatrix}R\\x\\x^2\\x^2\\x^2\end{bmatrix}
\end{tikzcd}$$

As a Λ_5-module, $\mathcal{P}(Q,W,F)$ is isomorphic to

$$\begin{bmatrix} R \\ x \\ x^2 \\ x^2 \\ x^2 \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ x \\ x^2 \\ x^2 \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ R \\ x \\ x^2 \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ R \\ R \\ x \end{bmatrix} \oplus \begin{bmatrix} R \\ x \\ x \\ x \\ x \end{bmatrix} \oplus \begin{bmatrix} R \\ x \\ x \\ x \\ x \end{bmatrix} \oplus \begin{bmatrix} R \\ x \\ x \\ x \\ x^2 \end{bmatrix}$$

which is cluster tilting. As expected, its summands correspond bijectively to the (internal and external) edges of the triangulation, or equivalently to the vertices of the corresponding quiver.

Then $\underline{\mathrm{CM}}(\Lambda_5) \cong \mathcal{C}_{A_2}$ with a cluster tilting object

$$\begin{bmatrix} R \\ x \\ x \\ x \\ x \end{bmatrix} \oplus \begin{bmatrix} R \\ x \\ x \\ x \\ x^2 \end{bmatrix}.$$

References

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[2] P. Caldero, F. Chapoton, and R. Schiffler. Quivers with relations arising from clusters (A_n case). *Trans. Amer. Math. Soc.*, 358(3):1347–1364, 2006.

[3] Sergey Fomin, Michael Shapiro, and Dylan Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[4] Daniel Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. *Proc. Lond. Math. Soc. (3)*, 98(3):797–839, 2009.