On the category of modules with Δ-filtration

Dag Oskar Madsen

August 14, 2012

This is work in progress together with Vanessa Miemietz, *University of East Anglia*.

References

- W. L. Burt, *Almost split sequences and BOCSES*, manuscript, August 2005.
- B. Keller, *A-infinity algebras in representation theory*, Representations of algebra. Vol. I, II, 74–86, Beijing Norm. Univ. Press, Beijing, 2002.
- S. Ovsienko, *Exact Borel subalgebras, Ringel duality and generalizations of quasi-hereditary algebras*, 36irnyk prats' Instytutu matematyky NAN Ukraïny 2005, t.2, No. 3, 108-122.

Standard modules

$\mathbb{k}$ field
B finite dimensional $\mathbb{k}$-algebra
Order the simple B-modules $S_1, \ldots, S_r$

Definition

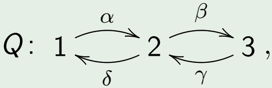
For each $0 \leq i \leq r$, define the **standard module** Δ_i *to be the largest quotient of the projective module P_i having no simple composition factors S_j with $j > i$.*

$$\Delta = \bigoplus_{i=1}^{r} \Delta_i$$

The standard modules depend on the chosen ordering of the simple modules.

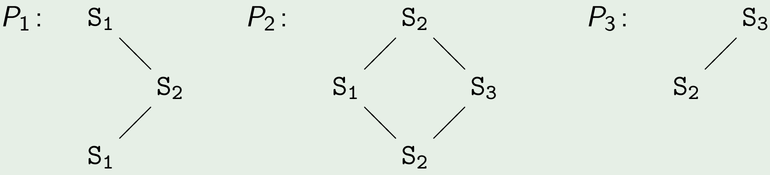
Example

$B = \Bbbk Q/I$ is given by

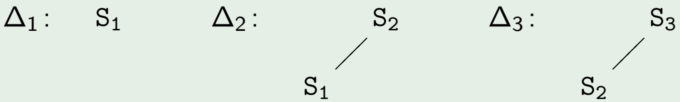


$I = (\beta\alpha, \beta\gamma, \delta\gamma, \alpha\delta - \gamma\beta)$

The projective modules are:



The standard modules are:



Quasi-hereditary algebras

Definition

We say that B is a **quasi-hereditary algebra** *if*
(i) B admits a Δ-filtration, i.e., there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t = B$$

where the subfactors M_j/M_{j-1} are standard modules for all and $1 \leq j \leq t$,
and
(ii) $\mathrm{End}_B(\Delta_i)$ *is a division ring for all* $1 \leq i \leq r$.

Quasi-hereditary algebras have finite global dimension.
If Λ is quasi-hereditary, then

$$\mathrm{Ext}^1_B(\Delta_i, \Delta_j) = 0$$

whenever $i \geq j$.

The algebra in the previous example was quasi-hereditary.

The question whether an algebra is quasi-hereditary or not might depend on the chosen ordering of simple modules.

Hereditary algebras are quasi-hereditary, independent of the ordering of the simple modules. (why?)

The category $\mathcal{F}(\Delta)$

Suppose B is a quasi-hereditary algebra.

Let $\mathcal{F}(\Delta)$ denote the category of B-modules which admit a Δ-filtration.

$\mathcal{F}(\Delta)$ contains the projective B-modules, is closed under direct summands, extensions and kernels of epimorphisms.

$\mathcal{F}(\Delta)$ has AR-sequences.

$\mathcal{F}(\Delta)$ is usually not abelian. (In some sense, there are too many morphisms.)

Two natural questions

1. What is the representation type of $\mathcal{F}(\Delta)$? If finite, find all the indecomposables.

2. How to find the AR-sequences in $\mathcal{F}(\Delta)$?

Burt and Ovsienko

Theorem (Burt, Ovsienko)

There is a finite dimensional algebra A and an exact functor $F\colon \operatorname{mod} A \to \operatorname{mod} B$ sending simple A-modules to standard B-modules in such a way that $F\colon \operatorname{mod} A \to \mathcal{F}(\Delta)$ is dense and faithful. The functor F sends projective modules to projective modules and injective modules to direct sums of tilting modules.

There is an algebra R, Morita equivalent to B, such that $A \hookrightarrow R$.

$$\operatorname{mod} A \xrightarrow{R\otimes_A -} \operatorname{mod} R \xrightarrow{Q\otimes_R -} \operatorname{mod} B$$

So $F = Q \otimes_A -\colon \operatorname{mod} A \to \mathcal{F}(\Delta)$

How to find A and F? Burt and Ovsienko use bocses. (R stands for *right* algebra.)

A and F via twisted stalks

Keller: $\mathcal{F}(\Delta)$ is equivalent to the category of *twisted stalks* $\mathcal{E} = \mathcal{E}(\Delta)$.

The arrows of the quiver of A are given by $\mathrm{Ext}^1_B(\Delta, \Delta)$.

The relations are given by the A_∞-structure maps

$$m_n\colon\ \mathrm{Ext}^1_B(\Delta, \Delta)^{\otimes n} \to \mathrm{Ext}^2_B(\Delta, \Delta)$$

Twisted stalks are modules over A with some extra morphisms.

There is an inclusion

$$A \cong [\mathrm{End}_A(A)]^{\mathrm{op}} \hookrightarrow [\mathrm{End}_{\mathcal{E}}(A)]^{\mathrm{op}} \cong R$$

We also have $R \cong [\mathrm{End}_B(F(A))]^{\mathrm{op}}$

Example (cont.)

In the example the quiver of A is given by

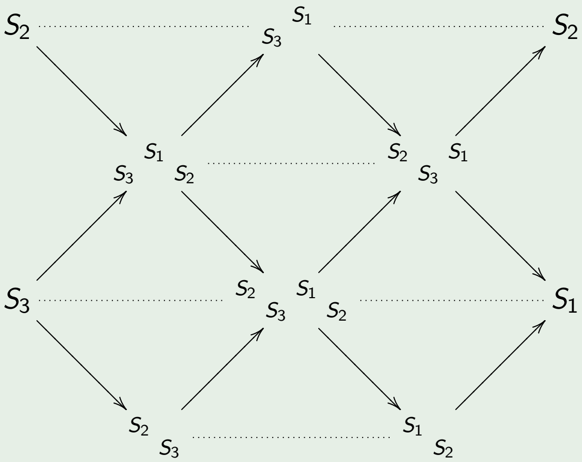
$$Q'\colon\ \xymatrix{1 \ar[r]^{\alpha'} \ar@/_1pc/[rr]_{\gamma'} & 2 \ar[r]^{\beta'} & 3}$$

The algebra B is Koszul with respect to Δ (see my paper *"Quasi-hereditary algebras and generalized Koszul duality"*), so the higher multiplications in the Ext algebra vanish. The product of the two arrows are non-zero, so for A we get a zero relation.

$$A = \Bbbk Q'/I', \quad I = (\beta'\alpha')$$

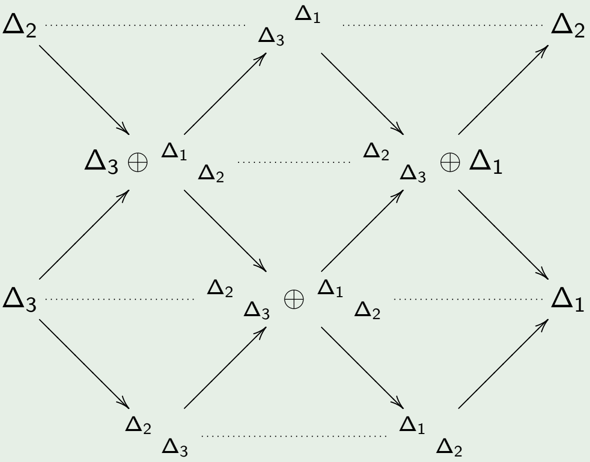
Example (cont.)

AR-quiver for mod A:



Example (cont.)

Apply F:



There are 6 indecomposables in $\mathcal{F}(\Delta)$.
$R \cong [\text{End}_B(P_1 \oplus P_2 \oplus P_3 \oplus P_3)]^{\text{op}}$.

AR-sequences

Theorem (Burt)

Let

$$e\colon 0 \to X \to Y \to Z \to 0$$

be an AR-sequence in mod A *and assume there is an indecomposable summand in* $F(Z)$ *such that* $F(Y) \to F(Z)$ *followed by a projection to that summand is not a split epimorphism. Then there is a direct summand of the extension*

$$F(e)\colon 0 \to F(X) \to F(Y) \to F(Z) \to 0$$

which is an AR-sequence in $\mathcal{F}(\Delta)$*.*

Burt also has cleaner statements starting with an AR-sequence in mod B.