Exceptional components of wild hereditary algebras

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Basic setup

k	algebraically closed field
Q	tame or wild quiver without oriented cycles
$\mathrm{rep}_k Q$	category of finite dimensional k-representations of Q
Γ	its Auslander-Reiten quiver
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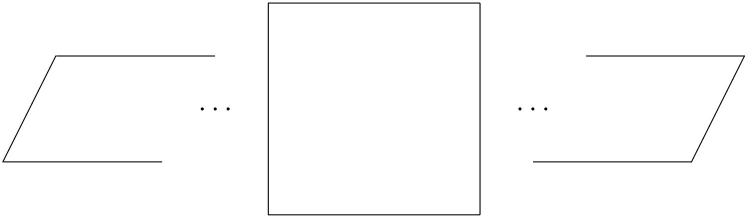
The Auslander-Reiten quiver has infinitely many components:



one preprojective ∞ many regular one preinjective

Outline

1. Structure of regular components

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2. Introduction to exceptional components
3. Question 1 of Kerner

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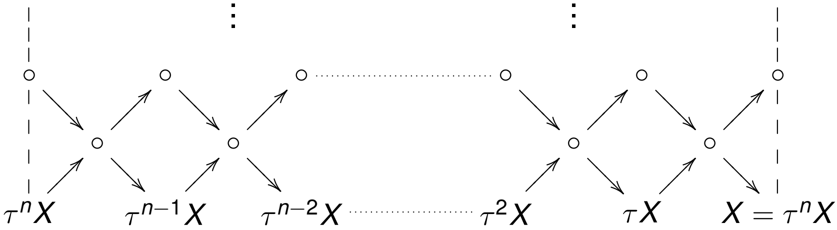

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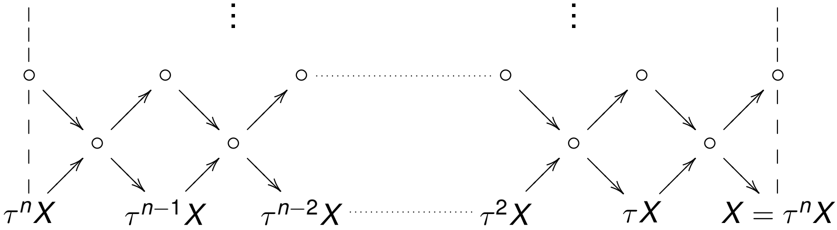

where the left and right dashed lines are identified. The quasi-simple representations $\tau^m X$ are Hom-orthogonal bricks.

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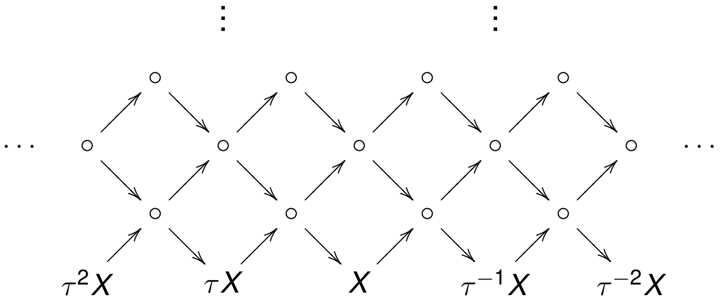

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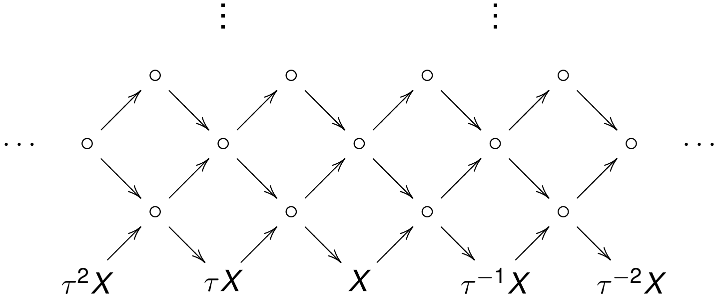

We will now consider morphisms between the quasi-simple representations $\mathrm{Hom}(X, \tau^m X)$.

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Example:

m	1	2	3	4	5	6	7	8
$\dim \operatorname{Hom}(X, \tau^m X)$	0	0	1	0	0	1	1	2

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Question (Kerner)

Is $s = l$ for each exceptional component?

Answer

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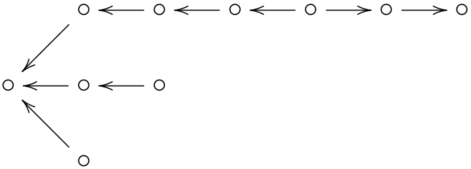

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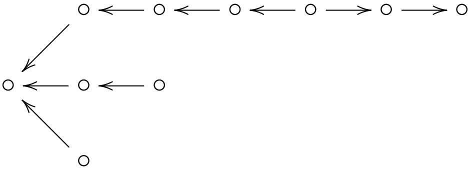

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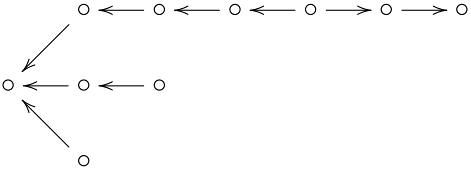

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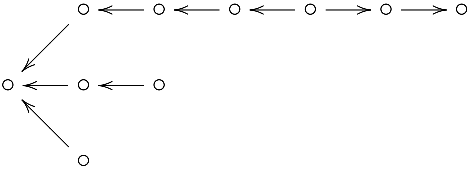

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Theorem

X lies in an exceptional component with $l = 4$ and $s = 7$.

Sketch of proof

- Calculate $\underline{\dim}\,\tau^m X$ for $m = 1, \ldots, 8$.

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m	1	2	3	4
$\underline{\dim}\, \tau^m X$	321100 321 1	211110 321 2	222211 321 1	222110 210 1
m	5	6	7	8
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For $Y, Z \in \mathrm{rep}_k Q$*:* $\langle \underline{\dim}\,Y, \underline{\dim}\,Z\rangle = \dim \mathrm{Hom}(Y, Z) - \dim \mathrm{Ext}^1(Y, Z)$.

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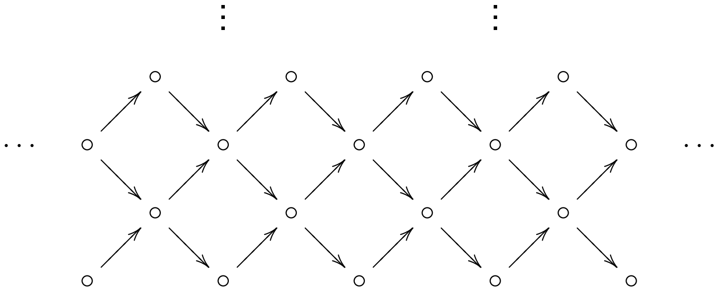

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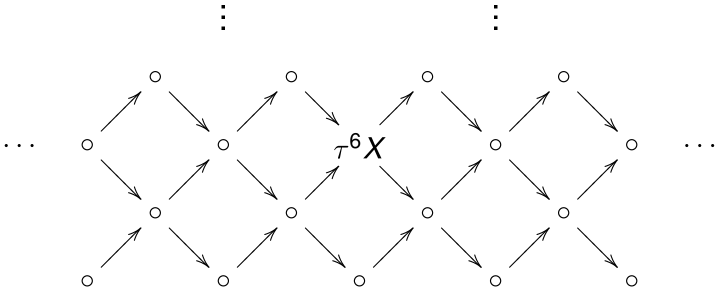

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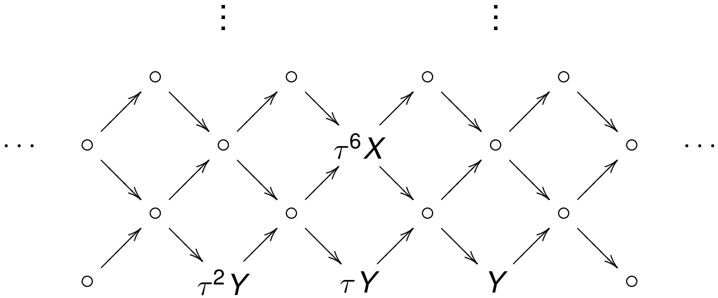

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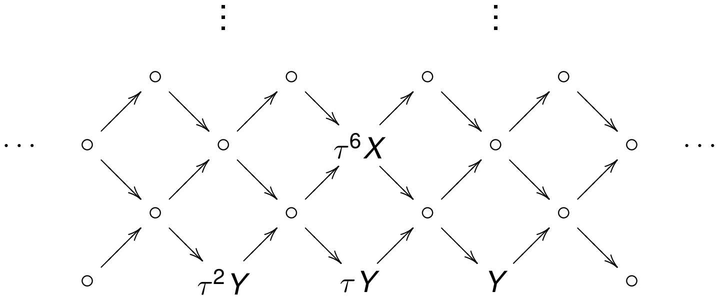

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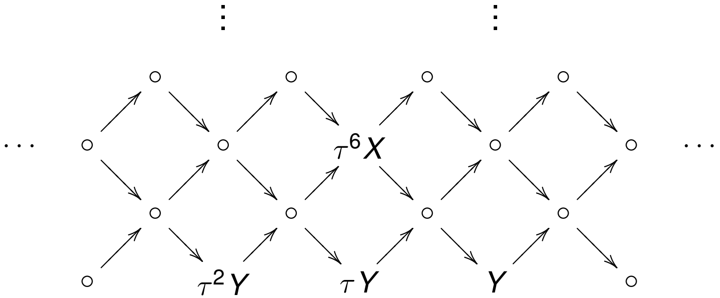

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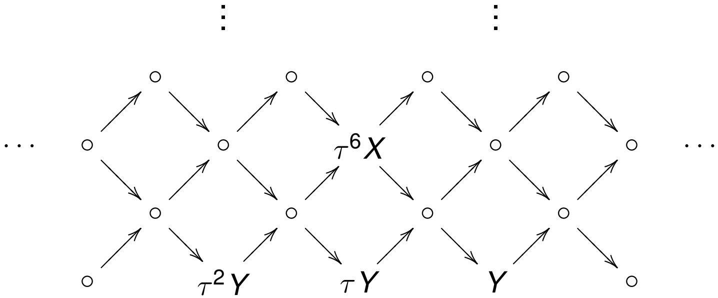

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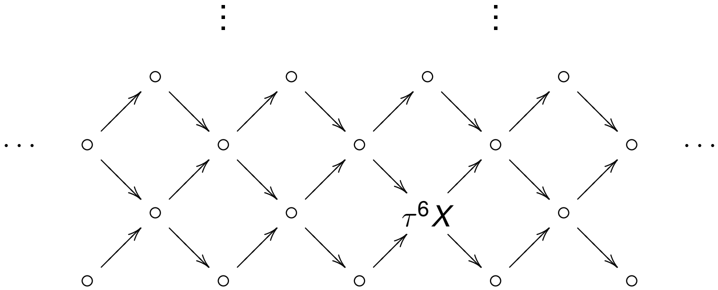

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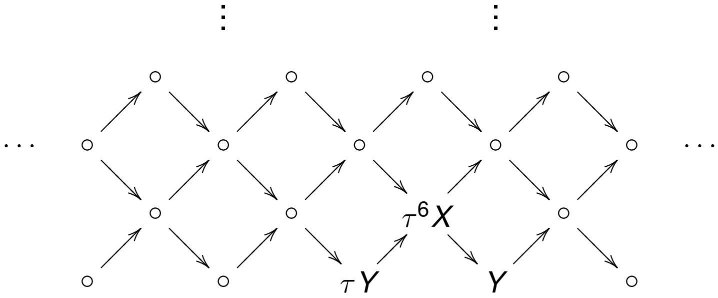

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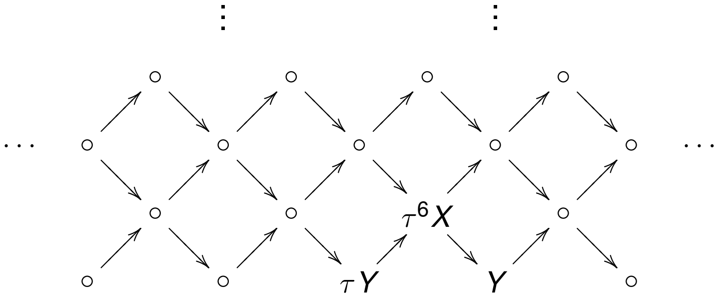

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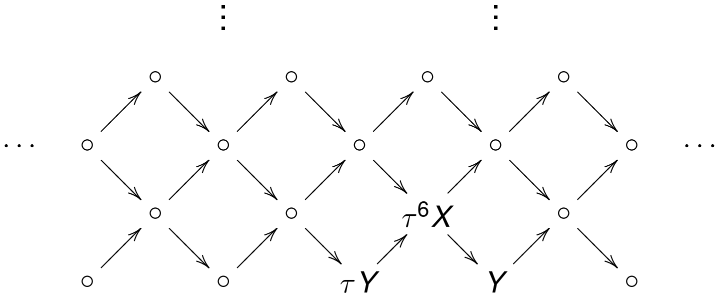

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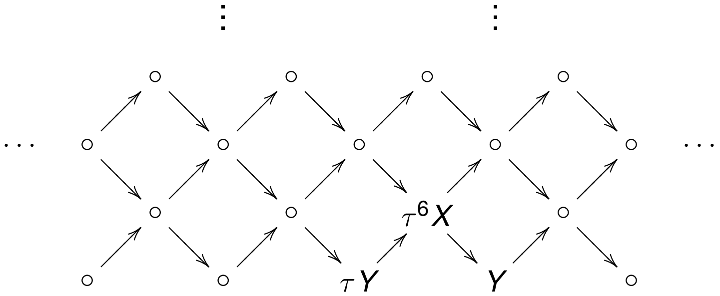

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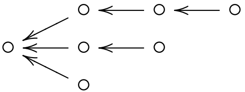
The support of X is the subquiver

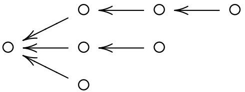


Kerner's second question

The representation X has dimension vector

$$\underline{\dim}\, X = \begin{pmatrix} \;\;221000 \\ 321\;\;\;\;\;\; \\ \;2\;\;\;\;\;\;\;\; \end{pmatrix}.$$

The support of X is the subquiver



which is of type E_7.

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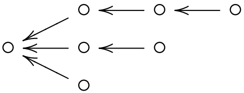


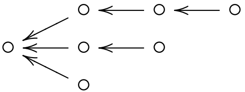
which is of type E_7. Kerner has proved that each exceptional component has a non-sincere representation with tame or representation-finite support.

Kerner's second question

The representation X has dimension vector

$$\underline{\dim}\, X = \left(\begin{array}{l} 221000 \\ 321 \\ 2 \end{array} \right).$$

The support of X is the subquiver



which is of type E_7. Kerner has proved that each exceptional component has a non-sincere representation with tame or representation-finite support.

Question (Kerner)

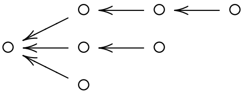
Does the representation-finite case occur?

Kerner's second question

The representation X has dimension vector

$$\underline{\dim}\, X = \left(\begin{smallmatrix} & 221000 \\ 321 & \\ 2 & \end{smallmatrix} \right).$$

The support of X is the subquiver



which is of type E_7. Kerner has proved that each exceptional component has a non-sincere representation with tame or representation-finite support.

Question (Kerner)

Does the representation-finite case occur?

Yes.