Module categories with heart

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Preliminaries

- A – artin algebra (over a fixed commutative artin ring R)
- $\operatorname{mod} A$ – category of finite generated right A-modules
- $\operatorname{mod} A \supseteq \operatorname{ind} A$ – full subcategory of indecomposable modules
- rad_A – Jacobson radical of $\operatorname{mod} A$ (generated by irreducible homomorphisms between modules in $\operatorname{ind} A$)
- $\operatorname{rad}_A^\infty$ – intersection of all powers rad_A^i, $i \geq 1$, of rad_A
- Γ_A – Auslander-Reiten quiver of A
- $\mathscr{C}$ – family of connected components of Γ_A
- $\mathscr{C}$ is **sincere** if every simple module in $\operatorname{mod} A$ occurs as a composition factor of a module in $\mathscr{C}$
- $\mathscr{C}$ is **generalized standard** if $\operatorname{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$

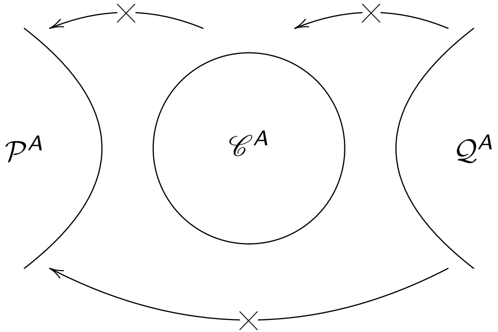
Preliminaries

A family $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ of components in Γ_A is said to be **separating** (in $\operatorname{mod} A$) if the components in Γ_A split into three disjoint classes $\mathcal{P}^A$, $\mathscr{C}^A = \mathscr{C}$ and $\mathcal{Q}^A$ such that:

1. $\mathscr{C}^A$ is sincere and generalized standard;
2. $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathscr{C}^A) = 0$, $\operatorname{Hom}_A(\mathscr{C}^A, \mathcal{P}^A) = 0$;
3. any morphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\operatorname{mod} A$ factors through $\operatorname{add}(\mathscr{C}^A)$.

Then $\mathcal{P}^A$ and $\mathcal{Q}^A$ are uniquely determined by $\mathscr{C}^A$.

Preliminaries



(We allow $\mathcal{P}^A$ or $\mathcal{Q}^A$ to be empty)

If $\mathscr{C}^A$ is generalized standard then components in $\mathscr{C}^A$ are pairwise orthogonal and almost periodic.

Preliminaries

A family $\mathcal{H} = (\mathcal{H}_i)_{i \in I}$ of components in Γ_A is said to be a **heart** of mod A if the following conditions are satisfied:

1. $\mathcal{H}$ is sincere and generalized standard;
2. $\mathcal{H}$ has no external short paths in ind A.

A short path $X \to Y \to Z$ in ind A is **external** for $\mathcal{H}$ if $X, Z \in \mathcal{H}$ but $Y \notin \mathcal{H}$.

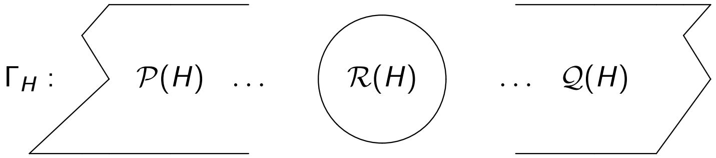
Note that if $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ is a separating family of components in Γ_A then $\mathscr{C}$ is a heart of mod A.

PROBLEM

Describe module categories mod A having a heart $\mathcal{H}$.

Tilted algebras

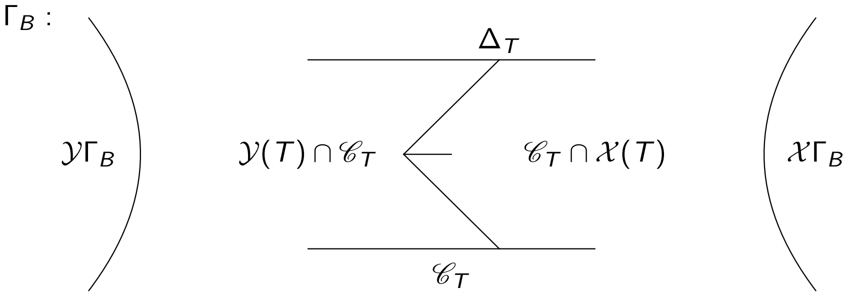
H – hereditary (connected) algebra, Q_H – valued quiver of H



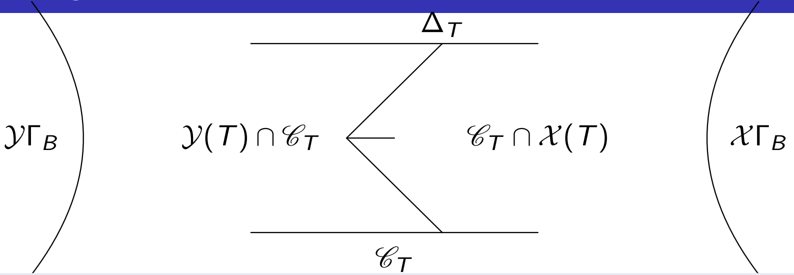
- $\mathcal{P}(H)$ and $\mathcal{Q}(H)$ are separating components and hearts of mod H
- Q_H Euclidean quiver $\Longrightarrow$ $\mathcal{R}(H)$ is an infinite family of pairwise orthogonal stable tubes and is a separating family in mod H
- Q_H Euclidean quiver $\Longrightarrow$ any stable tube in $\mathcal{R}(H)$ is a heart of mod H

Tilted algebras

- $T \in \operatorname{mod} H$ – tilting module: $\operatorname{Ext}^1_H(T, T) = 0$,
  $T = T_1 \oplus \ldots \oplus T_n$, $T_1, \ldots, T_n \in \operatorname{ind} H$,
  $T_i \ncong T_j$ for $i \neq j$, $n = \operatorname{rank} K_0(H)$
- B – tilted algebra: $B = \operatorname{End}_H(T)$



Tilted algebras



- $\mathscr{C}_T$ – **connecting component** of Γ_B with the section Δ_T given by the images $\mathrm{Hom}_H(T, I)$ of injectives modules I in ind H via the functor $\mathrm{Hom}_H(T, -) : \mathrm{mod}\, H \to \mathrm{mod}\, B$
- $(\mathcal{X}(T), \mathcal{Y}(T))$ – torsion pair in mod B
- $\mathcal{X}(T) = \{ M \in \mathrm{mod}\, B \mid M \otimes_B T = 0 \}$ – torsion part
- $\mathcal{Y}(T) = \{ M \in \mathrm{mod}\, B \mid \mathrm{Tor}_1^B(M, T) = 0 \}$ – torsion-free part

$\mathscr{C}_T$ is a separating component (hence a heart) in mod H with a section Δ_T (Happel–Ringel)

Generalized double tilted algebras

$\mathscr{C}$ – component of Γ_A

$\mathscr{C}$ – **almost acyclic** if all but finitely many modules in $\mathscr{C}$ do not lie on oriented cycles in $\mathscr{C}$

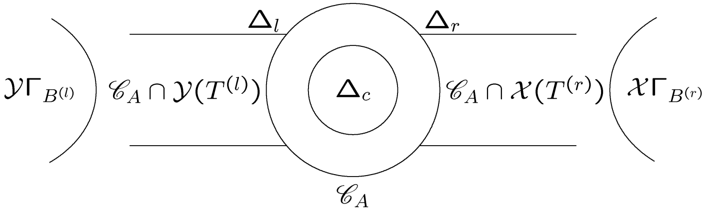
$\mathscr{C}$ is an almost acyclic $\Longleftrightarrow$ $\mathscr{C}$ admits a multisection (Reiten–Skowroński)

Theorem (Reiten-Skowroński)

Let A be an algebra. TFAE

1. Γ_A *admits an almost acyclic separating component.*
2. *A is a* **generalized double tilted algebra***.*

Generalized double tilted algebras



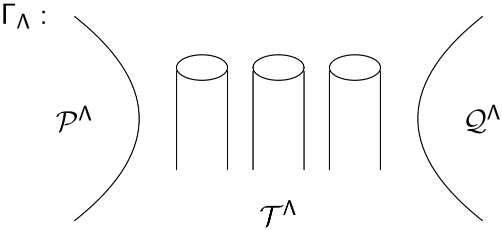
$\mathscr{C}_A$ – connecting component
$B^{(l)} = \mathrm{End}_{H^{(l)}}(T^{(l)})$, $B^{(r)} = \mathrm{End}_{H^{(r)}}(T^{(r)})$ – tilted algebras

Note that generalized double tilted algebras contain:

- all algebras of finite representation type
- tilted algebras
- double tilted algebras
  ($\mathrm{gl\,dim} = 3$ and for any $X \in \mathrm{ind}\, A$ $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$)

Concealed canonical algebras

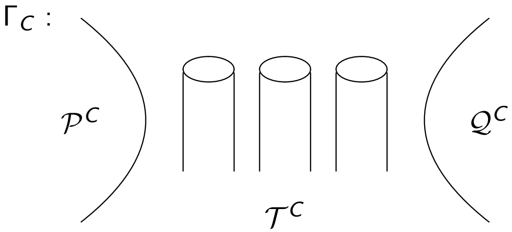
Λ – canonical algebra (Ringel)



$\mathcal{T}^\Lambda$ – separating family of stable tubes

- T – tilting Λ-module from the additive category $\mathrm{add}(\mathcal{P}^\Lambda)$ of $\mathcal{P}^\Lambda$
- C – **concealed canonical algebra** (of type Λ) : $C = \mathrm{End}_\Lambda(T)$

Concealed canonical algebras



$\mathcal{T}^C = \mathrm{Hom}_\Lambda(T, \mathcal{T}^\Lambda)$ – separating family of stable tubes

Theorem (Lenzing–de la Peña)

Let A be an algebra. TFAE

1. Γ_A *admits a separating family of stable tubes.*
2. A *is a concealed canonical algebra.*

A is a concealed canonical algebra $\iff$ mod A admits a heart formed by a stable tube

Quasitilted algebras

A – **quasitilted**: $\operatorname{gl\,dim} A \leq 2$ and for any $X \in \operatorname{ind} A$ we have $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$

Theorem (Happel–Reiten)

Let A be a quasitilted algebra. Then A is either a tilted algebra or a quasitilted algebra of canonical type.

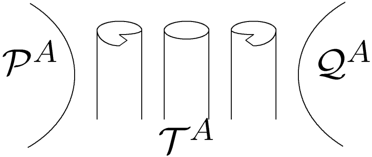
Theorem (Lenzing–Skowroński)

Let A be an algebra. TFAE

1. *A is a quasitilted algebra of canonical type.*
2. *A is a semiregular branch enlargement of a concealed canonical algebra.*
3. *Γ_A admits a separating family of ray and coray tubes.*

Quasitilted algebras

Γ_A :



$\mathcal{T}^A$ – separating family (hence a heart) of ray and coray tubes in mod A

Generalized multicoil algebras

$\mathscr{C}$ – component of Γ_A

- $\mathscr{C}$ is **almost cyclic** if all but finitely many modules in $\mathscr{C}$ lie on oriented cycles contained entirely in $\mathscr{C}$
- $\mathscr{C}$ is **coherent** if the following two conditions are satisfied:
  1. For each projective module P in $\mathscr{C}$ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to \cdots$ in $\mathscr{C}$,
  2. For each injective module I in $\mathscr{C}$ there is an infinite sectional path $\cdots \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ in $\mathscr{C}$.

$\mathscr{C}$ is almost cyclic and coherent $\Longleftrightarrow$ $\mathscr{C}$ is a **generalized multicoil** (obtained from a finite family of stable tubes by an iterated application of admissible operations (ad 1)-(ad 5) and their duals (ad 1^*)-(ad 5^*)) (Malicki–Skowroński)

Generalized multicoil algebras

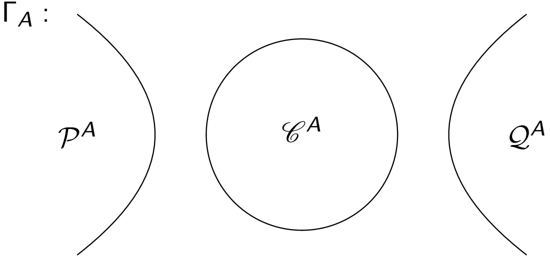
Theorem (Malicki–Skowroński)

Let A be an algebra. TFAE

1. *Γ_A admits a separating family of almost cyclic coherent components.*
2. *A is a generalized multicoil algebra.*

A – **generalized multicoil algebra**: obtained from a finite family of concealed canonical algebras $C_1, \ldots, C_m$ by an iterated application of admissible operations (ad 1)-(ad 5) and their duals, using the separating families $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ of stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$

Generalized multicoil algebras



$\mathscr{C}^A$ – separating family (hence a heart) of generalized multicoils in mod A

- $\operatorname{gl\,dim} A \leq 3$
- $\operatorname{pd}_A X \leq 1$ for any module X in $\mathcal{P}^A$
- $\operatorname{id}_A X \leq 1$ for any module X in $\mathcal{Q}^A$
- $\operatorname{pd}_A X \leq 2$ and $\operatorname{id}_A X \leq 2$ for any module X in $\mathscr{C}^A$
- $\mathcal{P}^A = \mathcal{P}^{A^{(l)}}$ and $\mathcal{Q}^A = \mathcal{Q}^{A^{(r)}}$
- $A^{(l)}$ /resp. $A^{(r)}$/ – quasitilted quotient algebra of A having a separating family of coray /resp. ray/ tubes in $\Gamma_{A^{(l)}}$ /resp. $\Gamma_{A^{(l)}}$/

Main results

Theorem (Jaworska–M–Skowroński)

Let A be an algebra. TFAE

1. $\operatorname{mod} A$ *admits a heart.*
2. Γ_A *admits a separating family of components.*

$\operatorname{mod} A$ admits a heart $\mathcal{H} \implies$

- $\operatorname{mod} A$ admits a heart consisting of a finite number of components of $\mathcal{H}$
- $\mathcal{H}$ can be completed to a separating family $\mathscr{C}$ of components in Γ_A by a family (possibly empty) of stable tubes

Main results

Theorem (Jaworska–M–Skowroński)

Let A be an algebra with a separating family $\mathscr{C}^A$ of components in Γ_A, and $\Gamma_A = \mathcal{P}^A \cup \mathscr{C}^A \cup \mathcal{Q}^A$ the associated decomposition of Γ_A. Then there exist quotient algebras $B_1, \ldots, B_m, B_{m+1}, \ldots, B_{m+p}$ of A such that the following statements hold.

- *$B_1, \ldots, B_m$ are generalized double tilted algebras such that all but finitely many acyclic modules in $\mathscr{C}^A$ belong to the connecting components $\mathscr{C}_{B_1}, \ldots, \mathscr{C}_{B_m}$ of $\Gamma_{B_1}, \ldots, \Gamma_{B_m}$.*
- *$B_{m+1}, \ldots, B_{m+p}$ are generalized multicoil algebras such that all but finitely many cyclic modules in $\mathscr{C}^A$ belong to the cyclic parts ${}_c\mathscr{C}^{B_{m+1}}, \ldots, {}_c\mathscr{C}^{B_{m+p}}$ of the separating families $\mathscr{C}^{B_{m+1}}, \ldots, \mathscr{C}^{B_{m+p}}$ of generalized multicoils of $\Gamma_{B_{m+1}}, \ldots, \Gamma_{B_{m+p}}$.*
- $$\mathcal{P}^A = \Big(\bigcup_{i=1}^{m} \mathcal{Y}\Gamma_{B_i^{(l)}}\Big) \cup \Big(\bigcup_{i=m+1}^{m+p} \mathcal{P}^{B_i^{(l)}}\Big).$$

Main results

Theorem (continued ...)

- $$\mathcal{Q}^A = \Big(\bigcup_{j=1}^{n} \mathcal{X}\Gamma_{B_j^{(r)}}\Big) \cup \Big(\bigcup_{j=n+1}^{n+p} \mathcal{Q}^{B_j^{(r)}}\Big).$$

- $$\operatorname{ind} A = \bigcup_{i=1}^{m+p} \operatorname{ind} B_i.$$

$A^{(l)} = B_1^{(l)} \times \cdots \times B_m^{(l)} \times B_{m+1}^{(l)} \times \cdots \times B_{m+p}^{(l)}$ – **left quasitilted algebra** of A

$A^{(r)} = B_1^{(r)} \times \cdots \times B_m^{(r)} \times B_{m+1}^{(r)} \times \cdots \times B_{m+p}^{(r)}$ – **right quasitilted algebra** of A

$\mathcal{P}^A = \mathcal{P}^{A^{(l)}}$ and $\mathcal{Q}^A = \mathcal{Q}^{A^{(r)}}$

We note that in general $A^{(l)}$ or $A^{(r)}$ are products of many tilted algebras and quasitilted algebras of canonical type.