Derived equivalence classification of generalized multifold extensions of piecewise hereditary algebras of tree type

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- $\mathbb{k}$: algebraically closed field

- A : basic finite-dimensional $\mathbb{k}$-algebra

- $D = \mathrm{Hom}_{\mathbb{k}}(-, \mathbb{k})$

Definition

The *repetitive category* of A is a $\Bbbk$-category

$$\widehat{A} := \begin{pmatrix} \ddots & & & & \\ \ddots & A^{[-1]} & & & \\ & DA^{[-1]} & A^{[0]} & & \\ & & DA^{[0]} & A^{[1]} & \\ & & & \ddots & \ddots \end{pmatrix} = \bigoplus_{i \in \mathbb{Z}} (A^{[i]} \oplus DA^{[i]}).$$

Definition

Let $\phi \in \mathrm{Aut}(\widehat{A})$ and $n \in \mathbb{Z}$. If there exists $i \in \mathbb{Z}$ such that $\phi(A^{[i]}) = A^{[i+n]}$, then ϕ is said to have a *jump* n.

$$\left(\begin{tikzcd}
\ddots & & & & \\
\ddots & A^{[i]} \arrow[ddrr, bend left, "\phi"] & & & \\
 & DA^{[i]} \arrow[ddrr, bend right, "\phi"'] & \ddots & & \\
 & & \ddots & A^{[i+n]} & \\
 & & & DA^{[i+n]} & \\
 & & & \ddots & \ddots
\end{tikzcd}\right)$$

Definition

Let $n \in \mathbb{Z}$. A *generalized n-fold extension* of A is a category (or an algebra) of the form $\hat{A}/\langle\phi\rangle$, where $\phi \in \mathrm{Aut}(\hat{A})$ having a jump n.

Definition

A is called *piecewise hereditary* if A is derived equivalent to a hereditary algebra H. Moreover if the ordinary quiver Q of H is an oriented tree, then A is said to be of tree type.

Definition

Let $n \in \mathbb{Z}$. A *twisted n-fold extension* of A is a category (or an algebra) of the form $T^n_\psi(A) := \hat{A}/\langle\hat{\psi}\nu_A^n\rangle$, where $\hat{\psi}$ is an automorphism of $\hat{A}$ induced by $\psi \in \mathrm{Aut}(A)$ and ν_A is the Nakayama automorphism of $\hat{A}$.

Theorem[Asashiba]

Let Λ, Λ' be twisted multifold extensions of piecewise hereditary algebras of tree type. Then the following are equivalent:

(i) Λ and Λ' are derived equivalent.

(ii) Λ and Λ' are stably equivalent.

(iii) $\mathrm{type}(\Lambda) = \mathrm{type}(\Lambda')$.

Remark For each $\psi \in \mathrm{Aut}(A)$, the automorphism $\hat{\psi}$ of $\hat{A}$ has a jump 0 and ν_A has a jump 1, therefore $\hat{\psi}\nu_A^n$ has a jump n. Hence $\hat{A}/\langle\hat{\psi}\nu_A^n\rangle$ is a generalized n-fold extension of A.

Proposition

Let A be a piecewise hereditary algebra of tree type and $\phi \in \mathrm{Aut}(\hat{A})$ having a jump n $(\in \mathbb{Z})$. Then there is some $\phi_Q \in \mathrm{Aut}(\widehat{\Bbbk Q})$ having a jump n such that $\hat{A}/\langle\phi\rangle$ and $\widehat{\Bbbk Q}/\langle\phi_Q\rangle$ are derived equivalent.

Proposition

Let Q be an oriented tree, $\phi, \psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$. If the actions of ϕ and ψ coincide on the objects, then $\widehat{\Bbbk Q}/\langle\phi\rangle$ is isomorphic to $\widehat{\Bbbk Q}/\langle\psi\rangle$.

Theorem

Let A be a piecewise hereditary algebra of tree type and $\phi \in \mathrm{Aut}(\hat{A})$ having a jump n $(\in \mathbb{Z})$. Then $\hat{A}/\langle\phi\rangle$ is derived equivalent to $T^n_{\phi_0}(A) = \hat{A}/\langle\hat{\phi}_0\nu^n_A\rangle$ where $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu_A^{-n}\phi\mathbb{1}^{[0]} \in \mathrm{Aut}(A)$.

<u>proof</u> By Proposition, there is $\psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ such that $\widehat{\Bbbk Q}/\langle\psi\rangle$ and $\hat{A}/\langle\phi\rangle$ are derived equivalent for $\hat{A}/\langle\phi\rangle$. On the other hand for $\hat{A}/\langle\hat{\phi}_0\nu^n_A\rangle$, there is $\psi_0 \in \mathrm{Aut}(\widehat{\Bbbk Q})$ such that $\widehat{\Bbbk Q}/\langle\psi_0\rangle$ and $\hat{A}/\langle\hat{\phi}_0\nu^n_A\rangle$ are derived equivalent. Note that an automorphism of repetitive category has the same action on the objects in each $i \in \mathbb{Z}$. Hence ϕ and $\hat{\phi}_0\nu^n_A$ coinside on the objects, $\widehat{\Bbbk Q}/\langle\psi\rangle$ is isomorphic to $\widehat{\Bbbk Q}/\langle\psi_0\rangle$ because ψ and ψ_0 coinside on the objects.

Theorem

Let Λ, Λ' be generalized multifold extensions of piecewise hereditary algebras of tree type. Then the following are equivalent:

(i) Λ and Λ' are derived equivalent.

(ii) Λ and Λ' are stably equivalent.

(iii) $\mathrm{type}(\Lambda) = \mathrm{type}(\Lambda')$.

The type $\mathrm{type}(\Lambda)$ of $\Lambda = \widehat{A}/\langle\phi\rangle$ is a triple $(\bar{Q}, n, \bar{\pi}(\phi_0))$ where

- Q is the type of A,
- n is a jump of ϕ,
- $\bar{\pi}(\phi_0)$ is the third ingredient of the type of the twisted multifold extension $T^n_{\phi_0}(A)$ of A.