Cell 2-representations of fiat 2-categories

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2-categories - Definition

Let $\Bbbk$ be an algebraically closed field. All categories are assumed to be (locally) small.

Definition

A 2-category is a category enriched over the monoidal category **Cat** of small categories.

I.e. a 2-category $\mathscr{C}$ consists of

- a set $\mathscr{C}$ of objects
- $\forall \mathtt{i}, \mathtt{j} \in \mathscr{C}$: a small category $\mathscr{C}(\mathtt{i}, \mathtt{j})$
  (objects are 1-morphisms; morphisms are 2-morphisms)
- functorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$
- $\forall \mathtt{i} \in \mathscr{C}$: identity 1-morphisms $\mathbb{1}_{\mathtt{i}}$
- natural (strict) axioms.

2-categories - Examples

2-category	objects	1-morphisms	2-morphisms
Cat	small categories	functors	natural transformations
$\mathfrak{A}_{\Bbbk}$	small fully additive $\Bbbk$-linear categories	additive $\Bbbk$-linear functors	natural transformations
$\mathfrak{A}_{\Bbbk}^{f}$	$\mathcal{C} \in \mathfrak{A}_{\Bbbk}$ with finitely many indecomposables/ $\cong$ and $\dim \mathcal{C}(X, Y) < \infty$ $\forall X, Y \in \mathcal{C}$	additive $\Bbbk$-linear functors	natural transformations
$\mathfrak{R}_{\Bbbk}$	$\mathcal{C} \in \mathfrak{A}_{\Bbbk}$ with $\mathcal{C} \sim A\text{-mod}$ for f.dim. assoc. $\Bbbk$-algebra A	additive $\Bbbk$-linear right exact functors	natural transformations

Fiat 2-categories - Definition

Definition

A 2-category $\mathscr{C}$ is called fiat, if

- $|\mathscr{C}| < \infty$
- $\forall \mathtt{i}, \mathtt{j} \in \mathscr{C} : \quad \mathscr{C}(\mathtt{i}, \mathtt{j}) \in \mathfrak{A}_{\Bbbk}^{f}$
- composition is biadditive and $\Bbbk$-linear
- $\forall \mathtt{i} \in \mathscr{C} : \quad \mathbb{1}_{\mathtt{i}}$ is indecomposable
- $\mathscr{C}$ has an object-preserving (weak) involution $*$
- $\mathscr{C}$ has adjunctions $\mathrm{F} \circ \mathrm{F}^* \to \mathbb{1}_{\mathtt{j}}$ and $\mathbb{1}_{\mathtt{i}} \to \mathrm{F}^* \circ \mathrm{F}$ (for $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$).

From now on, let $\mathscr{C}$ be a fiat 2-category.

Fiat 2-categories - Example

Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ for connected, basic, pairwise non-isomorphic, weakly symmetric, finite dimensional associative $\mathbb{k}$-algebras A_i.

Define a 2-category $\mathscr{C}_A$ which has

- objects $\mathtt{1}, \dots, \mathtt{k}$ where $\mathtt{i}$ is identified with a small version of $A_i\text{-mod}$;
- 1-morphisms isomorphic to identity functors on $\mathtt{i}$ or to tensoring with projective $A_j \otimes_{\mathbb{k}} A_i$-bimodules;
- 2-morphisms all natural transformations of such functors.

Then $\mathscr{C}_A$ is fiat.

2-representations

Definition

A finitary (resp. abelian) 2-representation $\mathbf{M}$ of $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to $\mathfrak{A}^f_{\Bbbk}$ (resp. $\mathfrak{R}_{\Bbbk}$).
Finitary (resp. abelian) 2-representations together with 2-natural transformations and modifications form again a 2-category, denoted by $\mathscr{C}\text{-afmod}$ (resp. $\mathscr{C}\text{-mod}$).

Definition

For $\mathtt{i} \in \mathscr{C}$ the principal 2-representation $\mathbb{P}_{\mathtt{i}}$ is given by $\mathbb{P}_{\mathtt{i}}(\mathtt{j}) := \mathscr{C}(\mathtt{i}, \mathtt{j})$ with the natural left action of $\mathscr{C}$.

Theorem (Yoneda Lemma)

For $\mathbf{M} \in \mathscr{C}\text{-afmod}$, *we have* $\quad \mathrm{Hom}_{\mathscr{C}\text{-afmod}}(\mathbb{P}_{\mathtt{i}}, \mathbf{M}) \cong \mathbf{M}(\mathtt{i})$.

2-ideals

Definition

A left (right, two-sided) ideal $\mathscr{I}$ of $\mathscr{C}$ consists of

- the same objects and 1-morphisms as $\mathscr{C}$;
- for each pair $\mathtt{i}, \mathtt{j}$, an ideal $\mathscr{I}(\mathtt{i}, \mathtt{j}) \subset \mathscr{C}(\mathtt{i}, \mathtt{j})$ such that horizontal composition preserves $\mathscr{I}$, i.e. $\mathscr{I}(\mathtt{j}, \mathtt{k}) \times \mathscr{I}(\mathtt{i}, \mathtt{j}) \to \mathscr{I}(\mathtt{i}, \mathtt{k}) \subset \mathscr{C}(\mathtt{i}, \mathtt{k})$.

For example, $\mathbb{P}_{\mathtt{k}}$ can be viewed as a left 2-ideal $\mathscr{I}_{\mathtt{k}}$ by setting

$$\mathscr{I}_{\mathtt{k}}(\mathtt{i}, \mathtt{j}) := \begin{cases} \mathscr{C}(\mathtt{k}, \mathtt{j}), & \mathtt{i} = \mathtt{k}; \\ 0, & \text{else.} \end{cases}$$

Cells

Definition

Define preorders on 1-morphisms in $\mathscr{C}$, saying that

- $\mathrm{F} \leq_L \mathrm{G}$ if G appears as a summand in $\mathrm{H} \circ \mathrm{F}$ for some H
- $\mathrm{F} \leq_R \mathrm{G}$ if G appears as a summand in $\mathrm{F} \circ \mathrm{H}$ for some H
- $\mathrm{F} \leq_{LR} \mathrm{G}$ if G appears as a summand in $\mathrm{H} \circ \mathrm{F} \circ \mathrm{K}$ for some H, K.

Equivalence classes under the preorders $\leq_L$ $(\leq_R, \leq_{LR})$ are called left (right, two-sided) cells respectively.

Cell 2-representations

Theorem

Let $\mathcal{J}$ be a two-sided cell in $\mathscr{C}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. Then there is a unique $\mathtt{i} \in \mathscr{C}$ and a unique maximal left ideal $\mathscr{J}_{\mathcal{L}}$ contained in $\mathscr{I}_{\mathtt{i}}$ such that it does not contain the identity 2-morphism id_{F} for any $\mathrm{F} \in \mathcal{L}$.

The (additive) cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}$ associated to $\mathcal{L}$ is defined as the additive closure of 1-morphisms in $\mathcal{L}$ inside $\mathbb{P}_{\mathtt{i}}/\mathscr{J}_{\mathcal{L}}$.

Theorem

Any non-trivial two-sided 2-ideal in $\mathscr{C}/\mathrm{Ker}\mathbf{C}_{\mathcal{L}}$ contains id_{F} for all $F \in \mathcal{L}$.

Strong simplicity

Definition

An abelian 2-representation $\mathbf{M}$ is generated by $M \in \mathbf{M}(\mathtt{i})$ if, for any $\mathtt{j} \in \mathscr{C}$, we can obtain any indecomposable projective in $\mathbf{M}(\mathtt{j})$ by applying 1-morphisms from $\mathscr{C}$ to M, and if 2-morphisms in $\mathscr{C}$ surject onto morphisms between projectives.

An abelian 2-representation $\mathbf{M}$ is called strongly simple if it is generated by any simple object in any $\mathbf{M}(\mathtt{j})$.

Let $\overline{\mathbf{C}}_{\mathcal{L}}$ be the abelianisation of $\mathbf{C}_{\mathcal{L}}$.

Cell 2-representations

Theorem

Let $\mathcal{J}$ be a two-sided cell in $\mathscr{C}$ and $\mathcal{L}$ a left cell in $\mathcal{J}$. Assume $\mathcal{J}$ has the following properties:

- *different left cells inside $\mathcal{J}$ are not comparable w.r.t. the left order;*
- *for any left cell $\mathcal{L}$ and right cell $\mathcal{R}$ in $\mathcal{J}$ we have $|\mathcal{L} \cap \mathcal{R}| = 1$;*
- *the function $F \mapsto m_F$, where $F^* \circ F = m_F H$ is constant on right cells of $\mathcal{J}$.*

Then we have

1. *The abelian 2-representation $\overline{\mathbf{C}}_{\mathcal{L}}$ is strongly simple.*
2. *We have* $\mathrm{End}_{\mathscr{C}\text{-mod}}(\overline{\mathbf{C}}_{\mathcal{L}}) \sim \mathbb{k}\text{-mod}.$